Choice under uncertainty

A. Introduction to choice under uncertainty
B. Risk aversion
C. Favorable gambles
D. Measures of risk aversion
E. Insurance
F. Small favorable gambles (may not have time)
G. Portfolio choice
H. Efficient risk sharing

59 slides
A. Introduction to choice under uncertainty (two states)

Let $X$ be a set of possible outcomes ("states of the world").

An element of $X$ might be a consumption vector, health status, inches of rainfall etc.

Initially, simply think of each element of $X$ as a consumption bundle. Let $\bar{x}$ be the most preferred element of $X$ and let $\underline{x}$ be the least preferred element.

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Consumption prospects

Suppose that there are only two states of the world. $X = \{x_1, x_2\}$ Let $\pi_1$ be the probability that the state is $x_1$ so that $\pi_2 = 1 - \pi_1$ is the probability that the state is $x_2$.

We write this "consumption prospect" as follows:

$$(x; \pi) = (x_1, x_2; \pi_1, \pi_2)$$
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**Consumption prospects**

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We write this “consumption prospect” as follows:

$$(x; \pi) = (x_1, x_2; \pi_1, \pi_2)$$

If we make the usual assumptions about preferences, but now on prospects, it follows that preferences over prospects can be represented by a continuous utility function

$$U(x_1, x_2, \pi_1, \pi_2).$$
Prospect or “Lottery”

\[ L = (x_1, x_2, \ldots, x_S; \pi_1, \ldots, \pi_S) \]

(outcomes; probabilities)

Consider two prospects or “lotteries”, \( L_A \) and \( L_B \)

\[ L_A = (x_1, x_2, \ldots, x_S; \pi^A_1, \ldots, \pi^A_S) \quad L_B = (c_1, c_2, \ldots, c_S; \pi^B_1, \ldots, \pi^B_S) \]

**Independence Axiom (axiom of complex gambles)**

Suppose that a consumer is indifferent between these two prospects (we write \( L_A \sim L_B \)).

Then for any probabilities \( \pi_1 \) and \( \pi_2 \) summing to 1 and any other lottery \( L_C \)

\[ (L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_C; \pi_1, \pi_2) \]

**Tree representation**

\[ \begin{array}{c}
\pi_1 \\
L_A \\
\pi_1 \\
\pi_2 \\
L_C \\
\end{array} 
\begin{array}{c}
\pi_1 \\
L_B \\
\pi_1 \\
\pi_2 \\
L_C \\
\end{array} \sim \begin{array}{c}
\pi_1 \\
L_C \\
\pi_1 \\
\pi_2 \\
L_C \\
\end{array} \]
This axiom can be generalized as follows:

Suppose that a consumer is indifferent between the prospects \( L_A \) and \( L_B \) and is also indifferent between the two prospects \( L_C \) and \( L_D \), i.e. \( L_A \sim L_B \) and \( L_C \sim L_D \).

Then for any probabilities \( \pi_1 \) and \( \pi_2 \) summing to 1,

\[
(L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_D; \pi_1, \pi_2)
\]

Tree representation

We wish to show that if \( L_A \sim L_B \) and \( L_C \sim L_D \) then

\[
\begin{array}{c}
\pi_1 \\
\pi_2
\end{array}
\begin{array}{c}
L_A \\
L_C
\end{array}
\sim
\begin{array}{c}
\pi_1 \\
\pi_2
\end{array}
\begin{array}{c}
L_B \\
L_D
\end{array}
\]
Proof: $L_A \sim L_B$ and $L_C \sim L_D$

Step 1: By the Independence Axiom, since $L_A \sim L_B$
Proof: \( L_A \sim L_B \) and \( L_C \sim L_D \)

Step 1: By the Independence Axiom, since \( L_A \sim L_B \)

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
L_A & \quad L_C
\end{align*}
\]

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
L_B & \quad L_C
\end{align*}
\]

Step 2: By the Independence Axiom, since \( L_C \sim L_D \)

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
L_A & \quad L_C
\end{align*}
\]

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
L_B & \quad L_C
\end{align*}
\]

\[
\begin{align*}
\pi_1 & \quad \pi_2 \\
L_B & \quad L_D
\end{align*}
\]
**Expected utility**

Consider some very good outcome \( \bar{x} \) and very bad outcome \( x \) and outcomes \( x_1 \) and \( x_2 \) satisfying

\( x < x_1 < \bar{x} \) and \( x < x_2 < \bar{x} \)

**Reference lottery**

\[ L_R(v) = (\bar{w}, w, v, 1-v) \] so \( v \) is the probability of the very good outcome.

\[ L_R(0) < x_1 < L_R(1) \] and \( L_R(0) < x_2 < L_R(1) \)

Then for some probabilities \( v(x_1) \) and \( v(x_2) \)

\[ x_1 \sim L_R(v(x_1)) = (\bar{x}, x; v(x_1), 1-v(x_1)) \] and \( x_2 \sim L_R(v(x_2)) = (\bar{x}, x; v(x_2), 1-v(x_2)) \)

Then by the independence axiom

\[(x_1, x_2; \pi_1, \pi_2) \sim (L_R(v(x_1)), L_R(v(x_2)); \pi_1, \pi_2) \]

**Definition: Expectation of \( v(x) \)**

\[ \mathbb{E}[v(x)] = \pi_1 v(x_1) + \pi_2 v(x_2) \]
Note that in the big tree there are only two outcomes, \( \bar{x} \) and \( x \). The probability of the very good outcome is

\[
\pi_1 v(x_1) + \pi_2 v(x_2) = \mathbb{E}[v(x)]
\]

The probability of the very bad outcome is \( 1 - \mathbb{E}[v(x)] \). Therefore
We showed that

$$\pi_1 \sim \pi_2 \quad \Leftrightarrow \quad \pi_1 \sim \pi_2$$

Thus the expected win probability in the reference lottery is a representation of preferences over prospects.

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An example:

A consumer with wealth $\hat{w}$ is offered a “fair gamble”. With probability $\frac{1}{2}$ his wealth will be $\hat{w} + x$ and with probability $\frac{1}{2}$ his wealth will be $\hat{w} - x$. If he rejects the gamble his wealth remains $\hat{w}$. Note that this is equivalent to a prospect with $x = 0$.

In prospect notation the two alternatives are

$$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}, \frac{1}{2}, \frac{1}{2})$$

and

$$(w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x, \frac{1}{2}, \frac{1}{2}).$$

These are depicted in the figure assuming $x > 0$.

Expected utility

$$U(w_1, w_2, \pi_1, \pi_2) = E[v] = \pi_1 v(w_1) + \pi_2 v(w_2)$$

Class discussion

MRS if $v(w)$ is a concave function
Convex preferences

The two prospects are depicted opposite.

The level set for \( U(w_1, w_2; \frac{1}{2}, \frac{1}{2}) \) through the riskless prospect \( N \) is depicted.

Note that the superlevel set
\[
U(w_1, w_2; \frac{1}{2}, \frac{1}{2}) \geq U(\hat{w}, \hat{w}; \frac{1}{2}, \frac{1}{2})
\]
is a convex set.
Convex preferences

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The level set for $U(w_1, w_2; 1/2, 1/2)$ through the riskless prospect $N$ is depicted.

Note that the superlevel set
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is a convex set.

This is the set of acceptable gambles for the consumer.

As depicted the consumer strictly prefers the riskless prospect $N$ to the risky prospect $R$.

Most individuals, when offered such a gamble (say over $\$5$) will not take this gamble.
B. Risk aversion

Class Discussion: Which alternative would you choose?

\[ N: (w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}; \pi_1, \pi_2) \quad R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where } \pi_1 = \frac{50}{100} \]

What if the gamble were “favorable” rather than “fair”

\[ R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where (i) } \pi_1 = \frac{55}{100} \text{ (ii) } \pi_1 = \frac{60}{100} \text{ (iii) } \pi_1 = \frac{75}{100} \]

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\[ R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where (i) } \pi_1 = \frac{55}{100} \quad \text{(ii) } \pi_1 = \frac{60}{100} \quad \text{(iii) } \pi_1 = \frac{75}{100} \]

What is the smallest integer \( n \) such that you would gamble if \( \pi_1 = \frac{n}{100} \)?

Preference elicitation

In an attempt to elicit your preferences write down your number \( n \) (and your first name) on a piece of paper. The two participants with the lowest number \( n \) will be given the riskless opportunity.

Let the three lowest integers be \( n_1, n_2, n_3 \). The win probability will not be \( \frac{n_1}{100} \) or \( \frac{n_2}{100} \). Both will get the higher win probability \( \frac{n_3}{100} \).
B. Risk preferences

\[ U(x, \pi) = \pi_1 v(x_1) + \pi_2 v(x_2) \] or \[ U(x, \pi) = \mathbb{E}[v] \]

Risk preferring consumer

Consider the two wealth levels \( x_1 \) and \( x_2 > x_1 \).

\[ v(\pi_1 x_1 + \pi_2 x_2) < \pi_1 v(x_1) + \pi_2 v(x_2) \]

If \( v(x) \) is convex, then the slope of \( v(x) \) is strictly increasing as shown in the top figure.

Consumer prefers risk
\[ U(x, \pi) = \pi_1 v(x_1) + \pi_2 v(x_2) \]

**Risk averse consumer**

\[ v(\pi_1 x_1 + \pi_2 x_2) > \pi_1 v(x_1) + \pi_2 v(x_2) \]

In the lower figure \( u(x) \) is strictly concave so that

\[ v(\pi_1 x_1 + \pi_2 x_2) > \pi_1 v(x_1) + \pi_2 v(x_2) = \mathbb{E}[v]. \]

In practice consumers exhibit aversion to such a risk.

Thus we will (almost) always assume that the expected utility function \( v(x) \) is a strictly increasing strictly concave function.

**Class Discussion:**

If consumers are risk averse why do they go to Las Vegas?
C. Favorable gambles: Improving the odds to make the gamble just acceptable.

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x, \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\).

Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

****
C. Favorable gambles: Improving the odds to make the gamble just acceptable.

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Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

For small \(x\) we can use the quadratic approximation of the utility function

\[
v_a(\hat{w} + x) = v(\hat{w}) + v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2
\]

**Class Exercise:** Confirm that the value and the first two derivatives of \(v(\hat{w} + x)\) and \(v_a(\hat{w} + x)\) are equal at \(x = 0\).

***
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For small \(x\) we can use the quadratic approximation of the utility function

**Quadratic approximation of his utility**

As long as \(x\) is small we can approximate his utility as a quadratic. Suppose \(u(w + x) = \ln(w + x)\).

Define \(\bar{u}(x) = \ln(w + x)\).

Then (i) \(\bar{u}(0) = \ln w\) (ii) \(\bar{u}'(0) = \frac{1}{w}\) and (iii) \(\bar{u}''(0) = -\frac{1}{w^2}\)

Consider the quadratic function

\[
q(x) = \ln w + \left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2.
\]  

(3.1)

If you check you will find that \(\bar{u}(x)\) and \(q(x)\) have the same, value, first derivative and second derivative at \(x = 0\). We then use this quadratic approximation to compute the gambler’s (approximated) expected gain.
With probability \( \frac{1}{2} + \alpha \) his payoff is \( q(x) \) and with probability \( \frac{1}{2} - \alpha \) his payoff is \( q(-x) \). Therefore his expected payoff is

\[
\mathbb{E}[q(x)] = \left( \frac{1}{2} + \alpha \right) q(x) + \left( \frac{1}{2} - \alpha \right) q(-x)
\]

Substituting from (3.1)

\[
\mathbb{E}[q(x)] = \left( \frac{1}{2} + \alpha \right) \left[ \ln w + \left( \frac{1}{w} \right) x - \frac{1}{2} \left( \frac{1}{w^2} \right) x^2 \right]
\]

\[
+ \left( \frac{1}{2} - \alpha \right) \left[ \ln w + \left( \frac{1}{w} \right)(-x) - \frac{1}{2} \left( \frac{1}{w^2} \right)(-x)^2 \right].
\]
With probability $\frac{1}{2} + \alpha$ his payoff is $q(x)$ and with probability $\frac{1}{2} - \alpha$ his payoff is $q(-x)$. Therefore his expected payoff is

$$\mathbb{E}[q(x)] = \left(\frac{1}{2} + \alpha\right)q(x) + \left(\frac{1}{2} - \alpha\right)q(-x)$$

Substituting from (3.1)

$$\mathbb{E}[q(x)] = \left(\frac{1}{2} + \alpha\right)[\ln w + \left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2]$$

$$+ \left(\frac{1}{2} - \alpha\right)[\ln w + \left(\frac{1}{w}\right)(-x) - \frac{1}{2}\left(\frac{1}{w^2}\right)(-x)^2].$$

Collecting terms,

$$\mathbb{E}[q(x)] = \ln w + 2\alpha\left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2.$$ 

If the gambler rejects the opportunity his utility is $\ln w$. Thus his expected gain is

$$\mathbb{E}[q(x)] - \ln w = 2\alpha\left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2 = \frac{2x}{w}[\alpha - \frac{1}{4}\left(\frac{1}{w}\right)x].$$

Thus the gambler should take the small gamble if and only if $\alpha > \frac{1}{4}\left(\frac{1}{w}\right)x$. 
The general case: quadratic approximation of his utility

\[ q(x) = v(\hat{\omega}) + v'(\hat{\omega})x + \frac{1}{2} v''(\hat{\omega})x^2 \]

Class Exercise: Confirm that the value and the first two derivatives of \( v(\hat{\omega} + x) \) and \( q(x) \) are equal at \( x = 0 \).

The expected value utility of the risky alternative is

\[ (\frac{1}{2} + \alpha)v(\hat{\omega} + x) + (\frac{1}{2} - \alpha)v(\hat{\omega} - x) \]

\[ \approx (\frac{1}{2} + \alpha)q(x) + (\frac{1}{2} - \alpha)q(-x) \]

**
C. Favorable gamble: Improving the odds to make the gamble just acceptable.

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\).

Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

**Quadratic approximation of his utility**

\[
q(x) = v(\hat{w}) + v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2
\]

**Class Exercise:** Confirm that the value and the first two derivatives of \(v(\hat{w} + x)\) and \(q(x)\) are equal at \(x = 0\).

The expected value utility of the risky alternative is

\[
\left(\frac{1}{2} + \alpha\right)v(\hat{w} + x) + \left(\frac{1}{2} - \alpha\right)v(\hat{w} - x)
\]

\[
\approx \left(\frac{1}{2} + \alpha\right)q(x) + \left(\frac{1}{2} - \alpha\right)q(-x)
\]

\[
= \left(\frac{1}{2} + \alpha\right)[v(\hat{w}) + v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2] + \left(\frac{1}{2} - \alpha\right)[v(\hat{w}) + v'(\hat{w})(-x) + \frac{1}{2} v''(\hat{w})(-x)^2]
\]

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Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

For small \(x\) we can use the quadratic approximation of the utility function

\[ v_a(\hat{w} + x) = v(\hat{w}) + v'(\hat{w})x + \frac{1}{2}v''(\hat{w})x^2 \]

Note that the value and the first two derivatives of \(v(\hat{w} + x)\) and \(v_a(\hat{w} - x)\) are equal at \(x = 0\).

The expected utility of the risky alternative is

\[ (\frac{1}{2} + \alpha)v(\hat{w} + x) + (\frac{1}{2} - \alpha)v(\hat{w} - x) \]

\[ \approx (\frac{1}{2} + \alpha)v_a(\hat{w} + x) + (\frac{1}{2} - \alpha)v_a(\hat{w} - x) \]

\[ = (\frac{1}{2} + \alpha)[v(\hat{w}) + v'(\hat{w})x + \frac{1}{2}v''(\hat{w})x^2] + (\frac{1}{2} - \alpha)[v(\hat{w}) + v'(\hat{w})(-x) + \frac{1}{2}v''(\hat{w})(-x)^2] \]

\[ = v(\hat{w}) + 2\alpha v'(\hat{w})x + \frac{1}{2}v''(\hat{w})x^2 \]

The utility of the riskless alternative is \(v(\hat{w})\). Therefore the consumer is indifferent if

\[ 2\alpha v'(\hat{w})x + \frac{1}{2}v''(\hat{w})x^2 = 0 \]

i.e. \(\alpha = -v''(\hat{w}) \frac{x}{v'(\hat{w})} \frac{1}{4}\).
D. Measure of risk aversion

Absolute aversion to risk

The bigger is $ARA(w) \equiv -\frac{v''(w)}{v'(w)}$ the bigger is $\alpha = \left(\frac{-v''(w)}{v'(w)}\right) \frac{x}{4} = ARA(w)\frac{x}{4}$.

Thus an individual with a higher $ARA(w)$ requires the odds of a favorable outcome to be moved more. Thus $ARA(w)$ is a measure of an individual’s aversion to risk.

$ARA(w) \equiv$ degree of absolute risk aversion
Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w}(1 + \beta), \hat{w}(1 - \beta); \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\).

Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

Note that we can rewrite the risky alternative as follows:

\((w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\) where \(x = \beta\hat{w}\).
Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: $$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}(1 + \beta), \hat{w}(1 - \beta); \frac{1}{2} + \alpha, \frac{1}{2} - \alpha).$$

Choose $\alpha$ so that the consumer is indifferent between gambling and not gambling.

Note that we can rewrite the risky alternative as follows:

$$(w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)$$

where $x = \beta \hat{w}$.
Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w}(1 + \beta), \hat{w}(1 - \beta); \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\).

Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

Note that we can rewrite the risky alternative as follows:

\[(w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\text{ where } x = \beta\hat{w}.

From our earlier argument,

\[
\alpha = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right) \frac{x}{4} = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right) \frac{\beta\hat{w}}{4} = \left(-\frac{\hat{w}v''(\hat{w})}{v'(\hat{w})}\right) \frac{\beta}{4}.
\]

Relative aversion to risk

The bigger is \(\text{RRA}(w) \equiv -\frac{wv''(w)}{v'(w)}\) the bigger is \(\alpha = \left(-\frac{wv''(w)}{v'(w)}\right) \frac{\beta}{4} = \text{RRA}(w) \frac{\beta}{4}.

Thus an individual with a higher \(\text{RRA}(w)\) requires the odds of a favorable outcome to be moved more. Thus \(\text{RRA}(w)\) is a measure of an individual’s aversion to risk.

\[\text{RRA}(w) \equiv \text{degree of relative risk aversion}\]
Remark on estimates of relative risk aversion

\[ RRA(w) \equiv -\frac{w v''(w)}{v'(w)} \]. Typical estimate between 1 and 2

Remark on estimates of absolute risk aversion

\[ ARA(w) \equiv -\frac{v''(w)}{v'(w)} = \frac{1}{w} RRA(w) \]

Thus ARA is very small for anyone with significant life-time wealth
E. Insurance

A consumer with a wealth \( \hat{w} \) has a financial loss of \( L \) with probability \( \pi_1 \). We shall call this outcome the “loss state” and label it state 1. With probability \( \pi_2 = 1 - \pi_1 \) the consumer is in the “no loss state” and label it state 2.

With no exchange the consumer’s state contingent wealth is

\[
(x_1, x_2) = (\hat{w} - L, \hat{w})
\]

This consumer wishes to exchange wealth in state 2 for wealth in state 1. Suppose there is a market in which such an exchange can take place. For each dollar of coverage in the loss state, the consumer must pay

\[
\rho = \frac{p_1}{p_2} \text{ dollars in the no loss state.}
\]
Suppose that the consumer purchases $q$ units.

\[ x_1 = \hat{w} - L + q \]

\[ \times p_1 \quad p_1 x_1 = p_1 (\hat{w} - L) + p_1 q \]

\[ x_2 = \hat{w} - \frac{p_1}{p_2} q \]

\[ \times p_2 \quad p_2 x_2 = p_2 \hat{w} - p_1 q \]

Adding these equations,

\[ p_1 x_1 + p_2 x_2 = p_1 (\hat{w} - L) + p_2 \hat{w} \]

*
Suppose that the consumer purchases $q$ units.

\[ x_1 = \hat{w} - L + q \]

\[ \times p_1 \quad p_1 x_1 = p_1 (\hat{w} - L) + p_1 q \]

\[ x_2 = \hat{w} - \frac{p_1}{p_2} q \]

\[ \times p_2 \quad p_2 x_2 = p_2 \hat{w} - p_1 q \]

Adding these equations,

\[ p_1 x_1 + p_2 x_2 = p_1 (\hat{w} - L) + p_2 \hat{w} \]

The consumer’s expected utility is

\[ U = \pi_1 v(x_1) + \pi_2 v(x_2) \]

We have argued that the consumer’s choices are constrained to satisfy the following budget constraint

\[ p_1 x_1 + p_2 x_2 = p_1 (\hat{w} - L) + p_2 \hat{w}. \]

This is the line depicted in the figure.

**Group Exercise:** What must be the price ratio if the consumer purchase full coverage? (i.e. $\bar{x}_1 = \bar{x}_2$)
F. Small favorable gambles (skip):

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w} + y, \hat{w} - x; \frac{1}{2}, \frac{1}{2})\), where \(y = (1 + \gamma)x\) for some \(\gamma > 0\)

It is tempting to believe that a highly risk averse consumer would not accept such a favorable gamble unless \(\gamma\) is sufficiently large.

We show that this intuition is false.
Small favorable gambles:

Consider a small $x$ and small $y > x$ where $y = (1 + \gamma)x$. Since $x$ is small, we can then use the quadratic approximation of the utility function.

The expected value of the risky alternative is

$$\frac{1}{2} v(\hat{w} + y) + \frac{1}{2} v(\hat{w} - x)$$

$$\approx \frac{1}{2} v_a(\hat{w} + y) + \frac{1}{2} v_a(\hat{w} - x)$$

$$= \frac{1}{2} [v(\hat{w}) + v'(\hat{w}) y + \frac{1}{2} v''(\hat{w}) y^2] + \frac{1}{2} [v(\hat{w}) + v'(\hat{w})(-x) + \frac{1}{2} v''(\hat{w})(-x)^2]$$

$$= v(\hat{w}) + \frac{1}{2} v'(\hat{w})(y - x) + \frac{1}{4} v''(\hat{w})(y^2 + x^2)$$

*
Small favorable gambles:

Consider a small $x$ and small $y > x$ where $y = (1 + \gamma)x$. Since $x$ is small, we can then use the quadratic approximation of the utility function.

The expected value of the risky alternative is

$$\frac{1}{2}v(\hat{w} + y) + \frac{1}{2}v(\hat{w} - x)$$

$$\approx \frac{1}{2}v_{a}(\hat{w} + y) + \frac{1}{2}v_{a}(\hat{w} - x)$$

$$= \frac{1}{2}[v(\hat{w}) + v'(\hat{w})y + \frac{1}{2}v''(\hat{w})y^2] + \frac{1}{2}[v(\hat{w}) + v'(\hat{w})(-x) + \frac{1}{2}v''(\hat{w})(-x)^2]$$

$$= v(\hat{w}) + \frac{1}{2}v'(\hat{w})(y - x) + \frac{1}{4}v''(\hat{w})(y^2 + x^2)$$

The value of the riskless alternative is $v(\hat{w})$. Therefore the consumer is better off taking the risk if

$$\frac{1}{2}v'(\hat{w})(y - x) + \frac{1}{4}v''(\hat{w})(y^2 + x^2)$$

$$= \frac{1}{2}v'(\hat{w})[\gamma x - \frac{1}{2}ARA(\hat{w})((1 + \gamma)^2 x^2 + \gamma^2 x^2)]$$

$$= \frac{1}{2}xv'(\hat{w})[\gamma - \frac{1}{2} xARA(\hat{w})((1 + \gamma)^2 + \gamma^2)]$$

$$> 0 \text{ for any } x \text{ that is sufficiently small}$$
F. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1 + r_0$ be the return on each dollar invested in the riskless asset and let $1 + r$ be the return on the risky asset (a random variable.) If the investor spends $x$ on the risky asset (and so $\hat{W} - x$ on the riskless asset) her final wealth is

$$\hat{W} = (\hat{W} - x)(1 + r_0) + x(1 + r)$$
F. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1 + r_0$ be the return on each dollar invested in the risky asset and let $1 + \bar{r}$ be the return on the risky asset (a random variable.) If the investor spends $q$ on the risky asset (and so $\hat{W} - q$ on the riskless asset) her final wealth is

$$\hat{W} = (\hat{W} - q)(1 + r_0) + q(1 + \bar{r})$$

$$= \hat{W}(1 + r_0) + q(\bar{r} - r_0)$$

*
G. Portfolio choice

An investor with wealth $\hat{W}$ chooses how much to invest in a risky asset and how much in a riskless asset. Let $1+r_0$ be the return on each dollar invested in the risky asset and let $1+\bar{r}$ be the return on the risky asset (a random variable.) If the investor spends $q$ on the risky asset (and so $\hat{W} - q$ on the riskless asset) her final wealth is

$$\hat{W} = (\hat{W} - q)(1+r_0) + q(1+\bar{r})$$
$$= \hat{W}(1+r_0) + q(\bar{r} - r_0)$$
$$= \hat{W}(1+r_0) + q\theta \quad \text{where} \quad \theta \equiv \bar{r} - r_0 .$$

Class exercise:

What is the simplest possible model that we can use to analyze the investor’s decision?
Two state model

Wealth in state $s$, $s = 1, 2$

$$W_s = (\hat{W} - q)(1 + r_0) + q(1 + r_s)$$

$$= \hat{W}(1 + r_0) + q(r_s - r_0)$$

$$= \hat{W}(1 + r) + q\theta_s \text{ where } \theta_s \equiv r_s - r_0.$$
Two state model

Wealth in state \( s, \ s = 1,2 \)

\[
W_s = (\hat{W} - q)(1 + r_0) + q(1 + r_s)
\]

\[
= \hat{W}(1 + r_0) + q(r_s - r_0)
\]

\[
= \hat{W}(1 + r_0) + q\theta_s \quad \text{where} \quad \theta_s = r_s - r_0.
\]

\[q = 0\]

\[
W_s = \hat{W}(1 + r)
\]

\[q = \hat{W}\]

\[
W_s = \hat{W}(1 + r) + \hat{W}\theta_s
\]

*
Two state model

Wealth in state $s$, $s=1,2$

$$W_s = (\hat{W} - q)(1+r_0) + q(1+r_s)$$

$$= \hat{W}(1+r_0) + q(r_s - r_0)$$

$$= \hat{W}(1+r) + q\theta_s$$

where $\theta_1 = r_1 - r_0 < 0 < r_2 - r_0 = \theta_2$.

$q = 0$

$$W_s = \hat{W}(1+r)$$

$q = \hat{W}$

$$W_s = \hat{W}(1+r) + \hat{W}\theta_s$$

Expected utility of the investor

$$U(w, \pi) = \pi_1 u(W_1) + \pi_2 u(W_2)$$

When will the investor purchase some of the risky asset?
Two state model

\[ W_s = \hat{W}(1 + r) + q\theta_s \] where \( \hat{W} \).

The steepness of the boundary of the set of feasible outcomes is \( \frac{\theta_2}{\theta_1} \).

**
Two state model

\[ W_s = \hat{W}(1+r) + q\theta_s \]  
where \[ \checkmark \].

The steepness of the boundary of the set of feasible outcomes is \[ -\frac{\theta_2}{\theta_1} \]

\[ U(W, \pi) = \pi_1 \nu(W_1) + \pi_2 \nu(W_2) \]

The steepness of the level set through the no risk portfolio is

\[ MRS^N = \frac{\pi_1}{\pi_2} \]

*
Two state model

\[ W_s = \hat{W}(1+r) + q\theta_s \] where \( \hat{W} \).

The steepness of the boundary of the set of feasible outcomes is \( -\frac{\theta_2}{\theta_1} \)

\[ U(W, \pi) = \pi_1 v(W_1) + \pi_2 v(W_2) \]

The steepness of the level set through the no risk portfolio is

\[ MRS^N = \frac{\pi_1}{\pi_2} \]

Purchase some of the risky asset as long as \( -\frac{\theta_2}{\theta_1} > \frac{\pi_1}{\pi_2} \) i.e. \( \pi_1 \theta_1 + \pi_2 \theta_2 > 0 \).

The risky asset has a higher expected return
**Calculus approach**

\[
U(q) = \pi_1 v(W_1) + \pi_2 v(W_2) = \pi_1 v(W(1 + r_0) + \theta_1 q) + \pi_2 v(W(1 + r_0) + \theta_2 q)
\]

Where

\[
\theta_1 = r_1 - r_0 \quad \text{and} \quad \theta_2 = r_2 - r_0
\]

\[
U'(q) = \pi_1 \theta_1 v'(W(1 + r_0) + \theta_1 q) + \pi_2 \theta_2 v'(W(1 + r_0) + \theta_2 q)
\]

*
Calculus approach

\[ U(q) = \pi_1 v(W_1) + \pi_2 v(W_2) = \pi_1 v(W(1 + r_0) + \theta_1 q) + \pi_2 v(W(1 + r_0) + \theta_2 q) \]

Where

\( \theta_1 = r_1 - r_0 \) and \( \theta_2 = r_2 - r_0 \)

\[ U'(q) = \pi_1 \theta_1 v'(W(1 + r_0) + \theta_1 q) + \pi_2 \theta_2 v'(W(1 + r_0) + \theta_2 q) \]

Therefore

\[ U'(0) = \pi_1 \theta_1 v'(W(1 + r_0)) + \pi_2 \theta_2 v'(W(1 + r_0)) = (\pi_1 \theta_1 + \pi_2 \theta_2)v'(W(1 + r_0)) \]

Thus if \( q = 0 \), then the marginal gain to investing in the risky asset is strictly positive if and only if

\[ \pi_1 \theta_1 + \pi_2 \theta_2 > 0 \]

i.e.

\[ \pi_1 (r_1 - r_0) + \pi_2 (r_2 - r_0) = \pi_1 r_1 + \pi_2 r_2 - r_0 > 0 \]

i.e. the expected payoff is strictly greater for the risky asset

Class Exercise: Is this still true with more than two states
H. Sharing the risk on a South Pacific Island

Alex lives on the west end of the island and has 600 coconut palm trees. Bev lives on the East end and has 800 coconut palm trees. If the hurricane approaching the island makes landfall on the west end it will wipe out 400 of Alex’s palm trees. If instead the hurricane makes landfall on the East end of the island it will wipe out 400 of Bev’s coconut palms. The probability of each event is 0.5.

Let the West end landing be state 1 and let the East end landing be state 2. Then the risk facing Alex is \((200,600; \frac{1}{2}, \frac{1}{2})\) while the risk facing Bev is \((800,400; \frac{1}{2}, \frac{1}{2})\).

What should they do?

What would be the WE outcome if they could trade state contingent commodities?
Let $v^B(\cdot)$ be Bev’s utility function so that her expected utility is

$$U^B(x^B) = \pi_1 v^B(x^B_1) + \pi_2 v^B(x^B_2).$$

where $\pi_s$ is the probability of state $s$.

In state 1 Bev’s “endowment“ is $\omega^B_1 = 800$

In state 2 the endowment is $\omega^B_2 = 400$. 

*
Let $v^B(\cdot)$ be Bev’s utility function so that her expected utility is

$$U^B(x^B) = \pi_1 u^B(x_1^B) + \pi_2 u^B(x_2^B).$$

where $\pi_s$ is the probability of state $s$.

In state 1 Bev’s “endowment” is $\omega^B_1 = 800$

In state 2 the endowment is $\omega^B_2 = 400$.

The level set for $U^B(x^B)$ through the endowment point $\omega^B$ is depicted.

At a point $\hat{x}^B$ in the level set the steepness of the level set is

$$MRS^B(\hat{x}^B) = \frac{MU_1}{MU_2} = \frac{\partial U^B}{\partial x_1^B} = \frac{\pi_1 v^B_1(\hat{x}_1^B)}{\pi_2 v^B_2(\hat{x}_2^B)}.$$

Note that along the 45° line the MRS is the probability ratio $\frac{\pi_1}{\pi_2}$ (equal probabilities so ratio is 1).
The level set for Alex is also depicted. At each 45° line the steepness of the respective sets are both 1. Therefore

\[ MRS^B(\omega^B) > 1 > MRS^A(\omega^A) \]

Therefore there are gains to be made from trading state claims.

The consumers will reject any proposed exchange that does not lie in their shaded superlevel sets.
Equivalently, Bev will reject any proposed exchange that is in the shaded sublevel set. Since the total supply of coconut palms is 1000 in each state, the set of potentially acceptable trades must be the unshaded region in the red “Edgeworth Box” *
Equivalently, Bev will reject any proposed exchange that is in the shaded sublevel set. Since the total supply of coconut palms is 1000 in each state, the set of potentially acceptable trades must be the unshaded region in the red “Edgeworth Box”.

Note also that 

$$x^A + x^B = \omega$$

Thus if Bev’s allocation is \(\hat{x}^B\) then

Alex has the allocation \(\hat{x}^A = \omega - \hat{x}^B\).

We can then rotate the box \(180^\circ\) to analyze the choices of Alex.
The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$

Also added to the figure is the green level set for Alex’s utility function through $\omega^A$. **
The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$

Also added to the figure is the green level set for Alex’s utility function through $\omega^A$.

Any exchange must be preferred by both consumers over the no trade allocation (the endowments).

Such an exchange must lie in the lens shaped region to the right of Alex’s level set and to the left of Bev’s level set.

*
The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$

Also added to the figure is the green level set for Alex’s utility function through $\omega^A$.

Any exchange must be preferred by both consumers over the no trade allocation (the endowment).
Such an exchange must lie in the lens shaped region to the right of Alex’s level set and to the left of Bev’s level set.

**Pareto preferred allocations**

If the proposed allocation is weakly preferred by both consumers and strictly preferred by at least one of the two consumers the new allocation is said to be Pareto preferred.

In the figure both $\hat{x}^A$ and $\hat{x}^A$ (in the lens shaped region) are Pareto preferred to $\omega^A$ since Alex is strictly better off and Bev is no worse off.
Consider any allocation such as \( \hat{x}^A \).

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.
Consider any allocation such as $\hat{x}^A$

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.

**Pareto Efficient Allocations**

It follows that for an allocation $x^A$ and $x^B = \omega - x^A$

to be Pareto efficient (i.e. no Pareto improving allocations)

$$MRS^A(x^A) = MRS^B(x^B)$$

Along the 45° line $MRS^A(x^A) = \frac{\pi_1}{\pi_2} = MRS^B(x^B)$.

Thus the Pareto Efficient allocations are all the allocations along 45° degree line.

Pareto Efficient exchange eliminates all individual risk.
Walrasian Equilibrium?

Suppose that there are markets for state claims. Let \( p_s \) be the price that a consumer must pay for delivery of a unit in state \( s \), i.e. the price of “claim” in state \( s \).

A consumer’s endowment \( \omega = (\omega_1, \omega_2) \), thus has a market value of \( p \cdot \omega = p_1 \omega_1 + p_2 \omega_2 \). The consumer can then choose any outcome \((x_1, x_2)\) satisfying

\[
p \cdot x \leq p \cdot \omega
\]

Given a utility function \( u_h(x) \), the consumer chooses \( \bar{x}^h \) to solve

\[
\max \{ U_h(x^h, \pi) | p \cdot x \leq p \cdot \omega^h \}
\]

i.e.

\[
\max \{ \pi_1 v_h(x_1^h) + \pi_2 v_h(x_2^h) | p \cdot x^h \leq p \cdot \omega^h \}
\]

FOC:

\[
\frac{MU_1}{MU_2} = \frac{\pi_1 v_h'({\bar{x}}_1^h)}{\pi_2 v_h'({\bar{x}}_2^h)} = \frac{p_1}{p_2}
\]
We have seen that

\[ MRS_h(\bar{x}^h) = \frac{MU_1}{MU_2} = \frac{\pi_1 v_h'(\bar{x}_1^h)}{\pi_2 v_h'(\bar{x}_2^h)} = \frac{p_1}{p_2}. \]

Thus in the WE for Alex and Bev

\[ MRS_A(\bar{x}^A) = \frac{\pi_1 v_A'(\bar{x}_1^A)}{\pi_2 v_A'(\bar{x}_2^A)} = \frac{p_1}{p_2}, \quad \text{and} \quad MRS_B(\bar{x}^B) = \frac{\pi_1 v_B'(\bar{x}_1^B)}{\pi_2 v_B'(\bar{x}_2^B)} = \frac{p_1}{p_2}. \]

Class Question: What does the First Welfare Theorem tell us about the WE allocation?

Given this, what must be the WE price ratio.
Exercises (for the TA session)

1. Consumer choice

(a) If \( u(x_s) = \ln x_s \) what is the consumer’s degree of relative risk aversion?

(b) If there are two states, the consumer’s endowment is \( \omega \) and the state claims price vector is \( p \), solve for the expected utility maximizing consumption.

(c) Confirm that if \( \frac{p_1}{p_2} > \frac{\pi_1}{\pi_2} \) then the consumer will purchase more state 2 claims than state 1 claims.

2. Consumer choice

(a), (b), (c) as in Exercise 1 except that \( u(x_s) = x_s^{1/2} \).

(d) Try to compare the state claims consumption ratio in Exercise 1 with that in Exercise 2.

(e) Provide the intuition for your conclusion.
3. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

\[ U_h(x, \pi) = \pi_1 \ln x_1^h + \pi_2 \ln x_2^h. \]

The aggregate endowment is \( \omega = (\omega_1, \omega_2) \) where \( \omega_1 > \omega_2 \).

(a) Solve for the WE price ratio \( \frac{p_1}{p_2} \).

(b) Explain why \( \frac{p_1}{p_2} < \frac{\pi_1}{\pi_2} \).

4. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

\[ U_h(x, \pi) = \pi_1 (x_1^h)^{1/2} + \pi_2 (x_2^h)^{1/2}. \]

The aggregate endowment is \( \omega = (\omega_1, \omega_2) \) where \( \omega_1 > \omega_2 \).

(a) Solve for the WE price ratio \( \frac{p_1}{p_2} \).

(b) Compare the equilibrium price ratio and allocations in this and the previous exercise and provide some intuition.