Strategic equilibrium in auctions

UCLA Auction House http://games.jriley.sscnet.ucla.edu/

38 pages

A. Sealed high bid auction model

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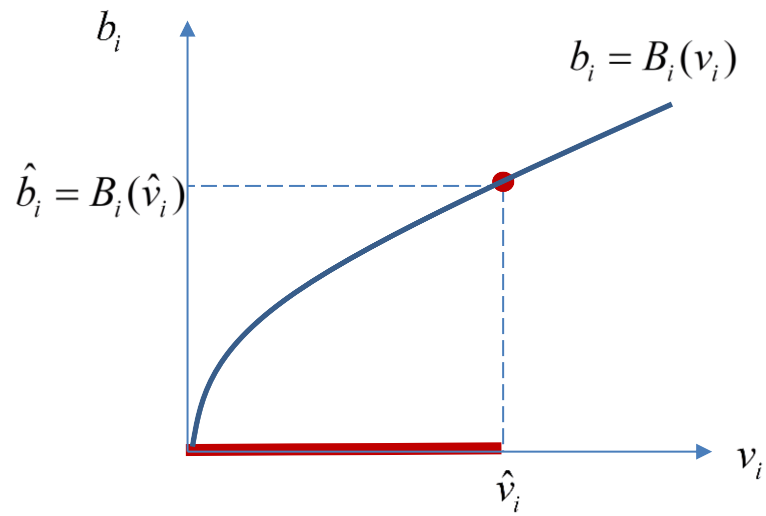
The values: The values lie on an interval $[0, \beta]$.

Strategies

With private information a player's action depends upon his private information. In the sealed high-bid auction, a player's private information is the value v_i that he places on the item for sale. His bid is then some mapping $b_i = B_i(v_i)$ from every possible value v_i into a non-negative bid.

This mapping is the player's bidding strategy.

Buyers with higher values have more to lose by not winning. So it is natural to assume that buyers with higher values will bid more so that $B_i(v_i)$ is a strictly increasing function.



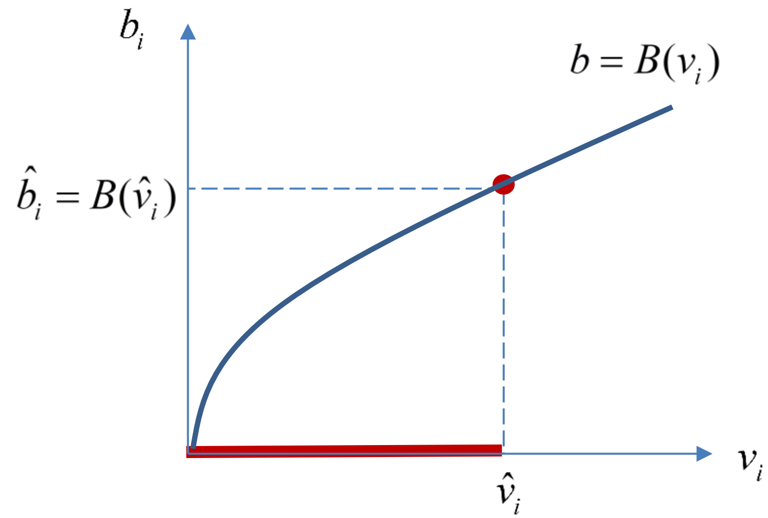
Equilibrium strategies

Since we assume that each buyer's value

is a draw from the same distribution it is

natural to assume that the equilibrium is

symmetric. $B_i(v_i) = B(v_i)$



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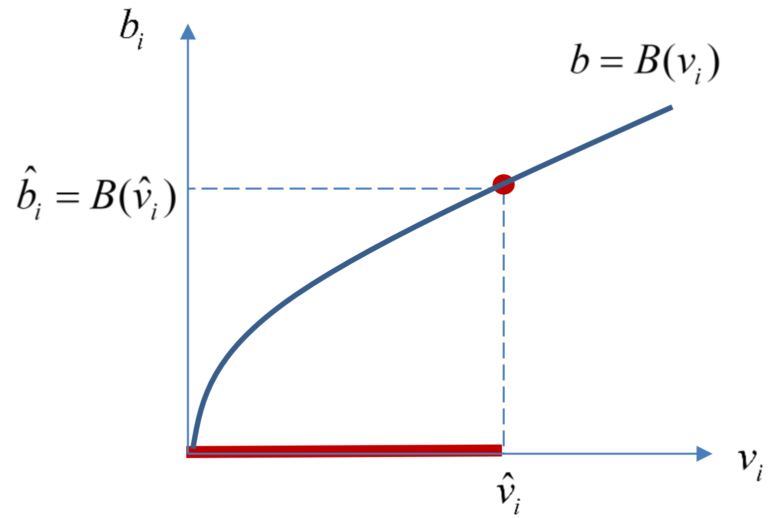

In a symmetric equilibrium, what is buyer 1's win probability if he has value $\hat{v}_1$?

Key observation:

Two buyers. Buyer 1 wins if buyer 2's bid is lower. Buyer 2 bids lower if and only if his value is lower.

So buyer 1's equilibrium win probability is $w(v_1) = \Pr\{B(v_2) \le B(v_1\} = \Pr\{v_2 \le v_1\} = F(v_1)$

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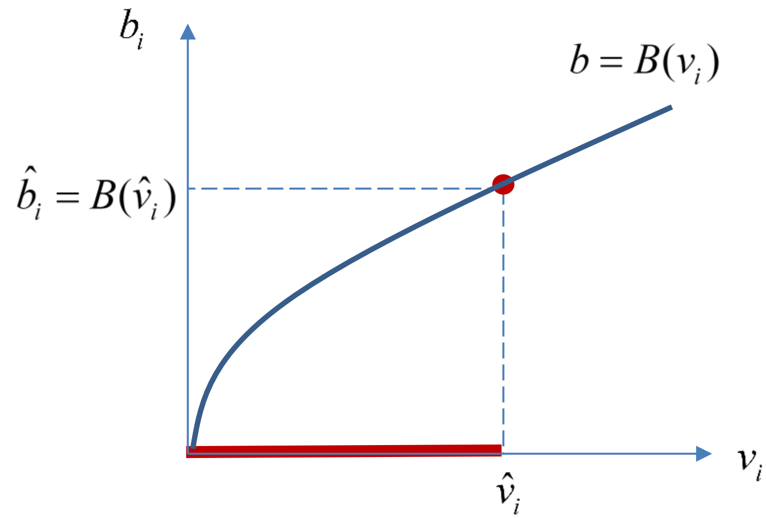

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Three buyers. Buyer 1 wins if and only if the other two buyers' bids are both lower. This is true only if their values are both lower. So buyer 1's equilibrium win probability is

$$w(v_1) = \Pr\{v_2 \leq v_1\} \times \Pr\{v_3 \leq v_1\} = F(v_1)^2$$

Equilibrium Strategies

Bayesian Nash Equilibrium (BNE) strategies: With private information, strategies that are mutual best response strategies are called Bayesian Nash Equilibrium strategies.

Remark: With uncertainty about values, a strategy is a description of an action conditional upon a participants value. So conditional probabilities play a role.

Symmetric BNE of the sealed high bid auction

If all other buyers other then buyer i use the bidding strategy $b_j = B(v_j)$ then buyer i's best response is to use the same strategy, i.e. $b_i = B(v_i)$.

i.e. equilibrium strategies are best responses.

An example: Two buyers with values uniformly distributed on $[0,100]$.

For the uniform distribution values are equally likely.

Therefore $\Pr\{\tilde{v}_i \le 25\} = \frac{25}{100}$, $\Pr\{\tilde{v}_i \le 50\} = \frac{50}{100}$, $\Pr\{\tilde{v}_i \le 80\} = \frac{80}{100}$.....

Thus the c.d.f. is $F(v_i) = \Pr(\tilde{v}_i \le v_i) = \frac{v_i}{100}$

For any guess as to the equilibrium strategy,

We can check to see if the guess is correct.

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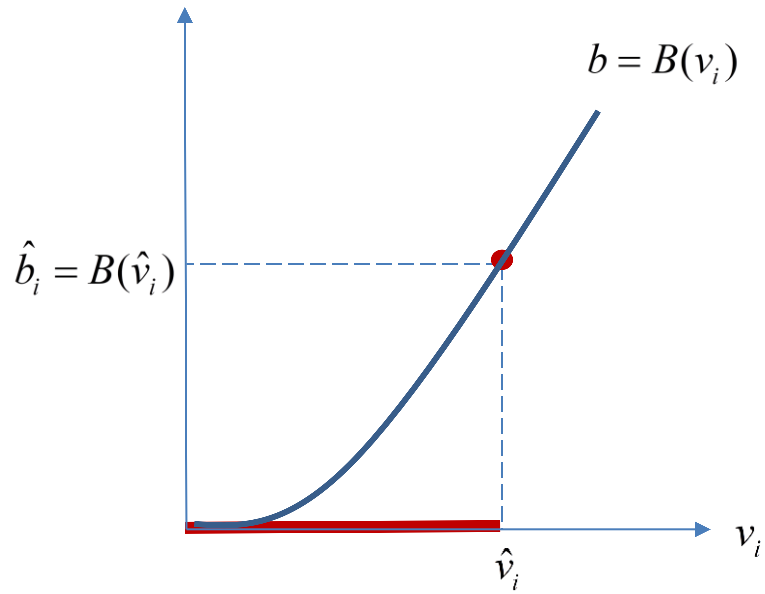

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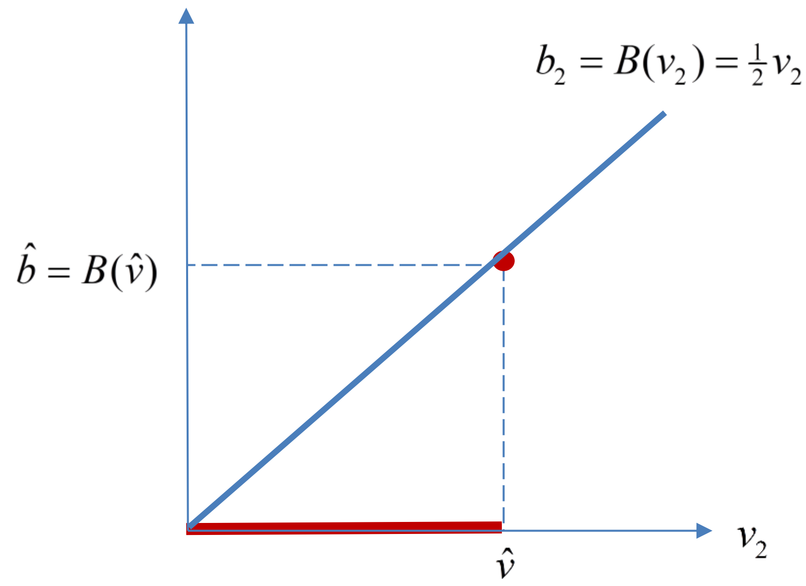
There are two buyers.

Suppose that buyer 2 bids according to the strategy

$$B(v_2) = \tfrac{1}{2} v_2 .$$



Proposed equilibrium strategy

We need to show that buyer 1's best response is to bid $b_1 = \frac{1}{2} v_1$.

Then these strategies are mutual best responses.

Solving for buyer 1's best response when his value is v_1

If buyer 1 bids $\hat{b} = 20$ he has the high bid if

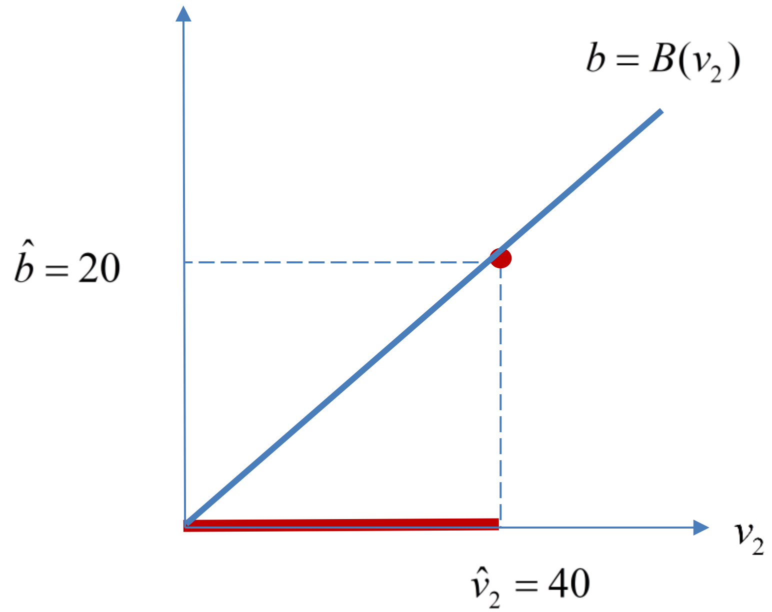
$B(v_2) = \frac{1}{2} v_2 \le \hat{b} = 20$, i.e. if $v_2 \le 40$.

Buyer 1's win probability is therefore

$$\hat{w} = \Pr\{\hat{v}_2 \le 40\} = \frac{40}{100} \quad .$$

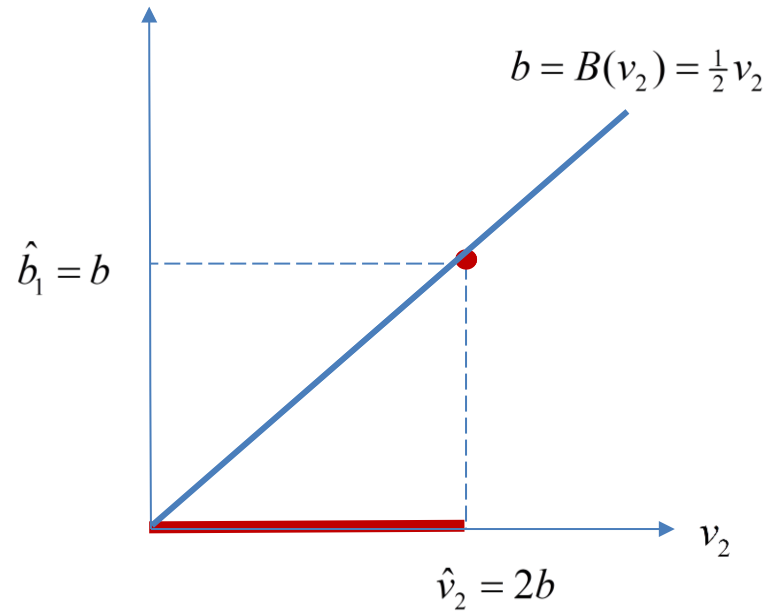
Buyer 1's expected payoff is therefore

$$U_1(v_1, b) = (v_1 - \hat{b}) w(\hat{b}) = (v_1 - 20)\frac{40}{100}$$



If buyer 1 bids b he has the high bid if $B(v_2) = \frac{1}{2}v_2 \leq b$,

i.e. $v_2 \leq 2b$.



Buyer 1's win probability is therefore

$$w(b) = \Pr\{v_2 \leq 2b\} = F(2b) = \frac{2b}{100} \quad .$$

Buyer 1's expected payoff is therefore

$$U_1(v_1, b) = (v_1 - b)w(b) = (v_1 - b)\frac{2b}{100} = \frac{2}{100}(v_1 b - b^2)$$

$$\frac{\partial U_1}{\partial b}(v_1, b) = \frac{2}{100}(v_1 - 2b)$$

$$= 0 \quad \text{for a maximum.}$$

If buyer 2 uses the strategy $b_2 = \frac{1}{2}v_2$ we have therefore shown that buyer 1's expected gain is maximized if $b_1 = \frac{1}{2}v_1$.

Thus the strategies are mutual best responses.

Group Exercise: Three buyers with values uniformly distributed

(a) Show that if buyer 2 and buyer 3 bid according to $b_j = \frac{1}{2}v_j$, then buyer 1's best response is to bid $b_1 > \frac{1}{2}v_1$ when his value is v_1

(b) Show that for some $\alpha > \frac{1}{2}$, $b_j = B(v_j) = \alpha v_j$ is the equilibrium bidding strategy

(c) What is the equilibrium bidding strategy with 4 buyers?

Answer to (b)

The probability that buyer 1 wins with a bid of b is the joint probability that $b_2 \le b$ and $b_3 \le b$, ie.

$$w_1(b) = \Pr\{b_2 \le b\} \times \Pr\{b_3 \le b\}$$

$$= \Pr\{\alpha v_2 \le b\} \times \Pr\{\alpha v_3 \le b\}$$

$$= \Pr\{v_2 \le \frac{b}{\alpha}\} \times \Pr\{v_3 \le \frac{b}{\alpha}\} = F^2(\frac{b}{\alpha}) = (\frac{b}{\alpha})^2$$

$$U_1(v_1, b) = (v_1 - b)w(b) = (v_1 - b)(\frac{b}{\alpha})^2 = \frac{1}{\alpha^2}(v_1 b^2 - b^3)$$

$\frac{\partial U_1}{\partial b} = \frac{1}{\alpha^2}(2v_1 b - 3b^2) = 0$ for a maximum.

Therefore buyer 1's best response is $B_1(v_1) = \frac{2}{3}v_1$.

Note that this is true if $\alpha = \frac{2}{3}$. Thus if the other buyers bid $B_j(\theta_j) = \frac{2}{3}\theta_j$, then buyer 1's best response is to do so as well.

The problem with this approach is that it requires an inspired guess.

B. Sealed high bid auctions: A general approach - - Characterize equilibrium payoffs

$$U(v) = w(v)(v - B(v))$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{equilibrium}\\ \text{win probability}\end{pmatrix} \times \begin{pmatrix}\text{net gain}\\ \text{if buyer wins}\end{pmatrix}$$

Rewrite as follows

$$U(v) = w(v)v - w(v)B(v)$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{expected}\\ \text{gross gain}\end{pmatrix} - \begin{pmatrix}\text{expected}\\ \text{buyer payment}\end{pmatrix}$$

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Rewrite as follows

$$U(v) = w(v)v - w(v)B(v) \qquad (*)$$

$$\begin{pmatrix} \text{equilibrium} \\ \text{payoff} \end{pmatrix} = \begin{pmatrix} \text{expected} \\ \text{gross gain} \end{pmatrix} - \begin{pmatrix} \text{expected} \\ \text{buyer payment} \end{pmatrix}$$

We solve (i) for the equilibrium win probability and hence the expected gross gain, (ii) for the equilibrium payoff.

We can then solve for the equilibrium bid function by appealing to (*).

Remark: We can write the buyer's equilibrium payoff as

$$U(v) = w(v)v - w(v)B(v)$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{expected}\\ \text{gross gain}\end{pmatrix} - \begin{pmatrix}\text{expected}\\ \text{buyer payment}\end{pmatrix}$$

Since the payment by the buyer is the revenue that the seller receives from the buyer,

$$U(v) = w(v)v - R(v)$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{expected}\\ \text{gross gain}\end{pmatrix} - \begin{pmatrix}\text{expected}\\ \text{seller revenue}\end{pmatrix}$$

(This will be useful when we consider other auctions.)

Solution:

Step 1: Obtain an expression for buyer 1's equilibrium win probability

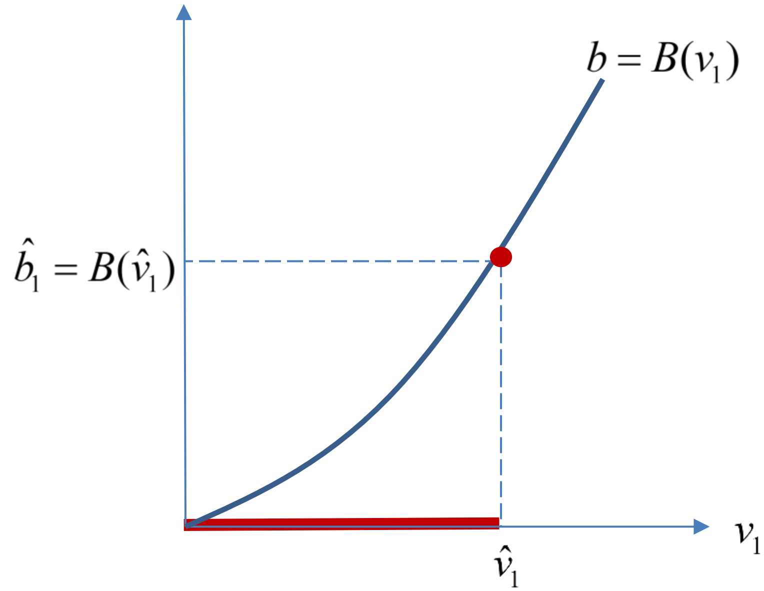
Class Exercise:

Two buyers:

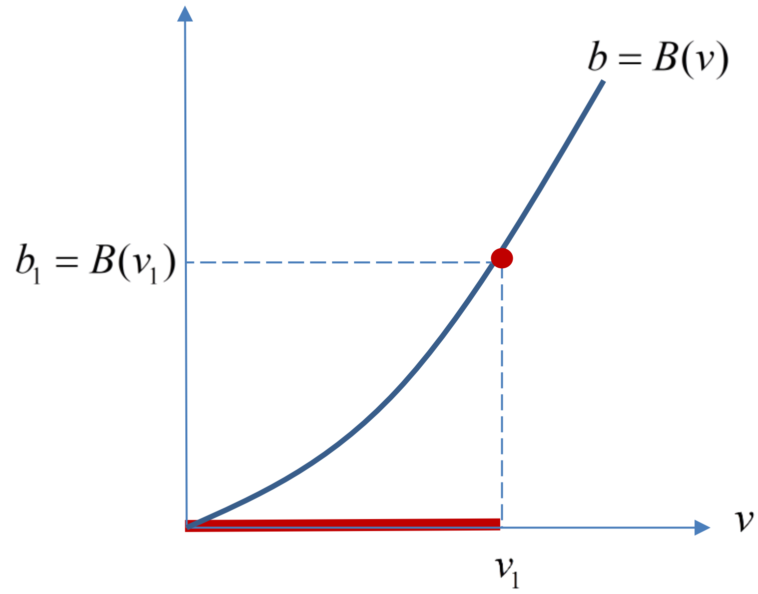
Why is the equilibrium probability $w(v_1) = F(v_1)$?

Three buyers:

Why is the equilibrium probability $w(v_1) = F(v_1)^2$?



Equilibrium win probability with I buyers



If buyer 1 with a value of v_1 makes an

equilibrium bid of $B(v_1)$, then

she has the high bid if

$$b_j = B(v_j) \le B(v_1) \text{ , for } j = 2,...,I$$

equivalently,

$$v_j \le v_1, \text{ for } j = 2,...,I$$

The joint probability of this event is

$$w(v_1) = \Pr\{\tilde{v}_2 \le v_1\} \times \Pr\{\tilde{v}_3 \le v_1\} \times ... \times \Pr\{\tilde{v}_I \le v_1\} = F(v_1)^{I-1}$$

Step 2: Suppose that buyer 1 is naïve (stupid!)

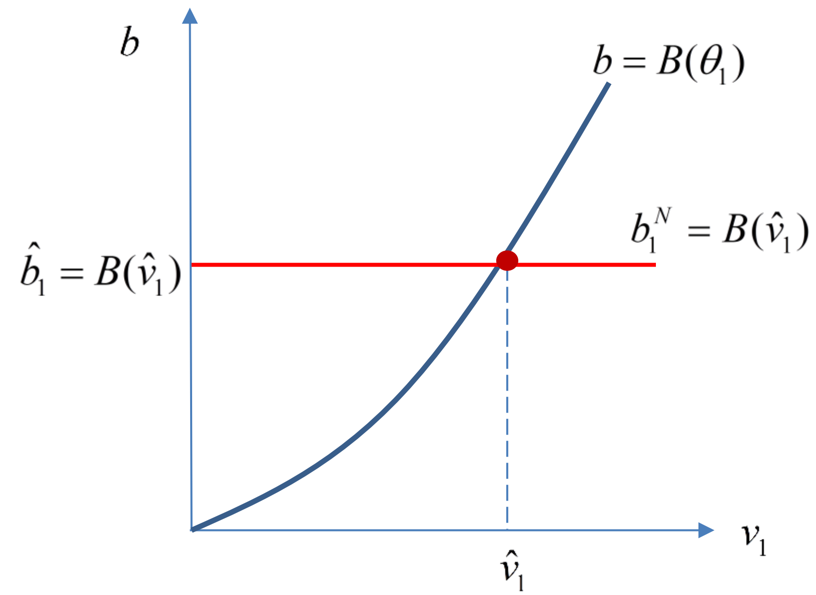
Suppose buyer 1 uses the strategy

$B_N(v_1) = B(\hat{v}_1) \equiv \hat{b}_1$ regardless of her value

Since $B(v_1)$ is her best response strategy,

The naïve strategy is a best response

If and only if $v_1 = \hat{v}_1$.



(Note that, regardless of her value, buyer 1 bids $\hat{b}_1$. Then her win probability is $w(\hat{v}_1)$.)

Let $U(v_1)$ be her equilibrium payoff.

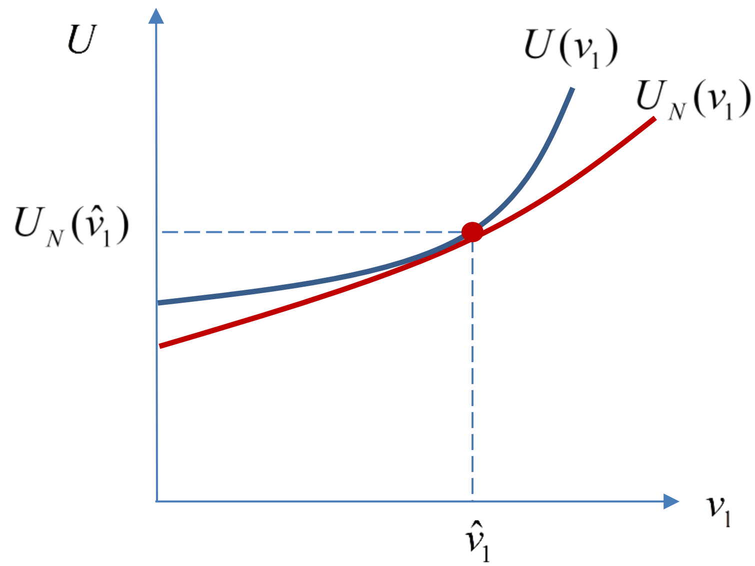
Let $U_N(v_1)$ be her payoff if she is naive.

Then

$U_1(\hat{v}_1)=U_N(\hat{v}_1)$

and

$U_1(v_1)\geq U_N(v_1)$ for all $v_1 \neq \hat{v}_1$



Key conclusion:

The graphs of the two functions must be tangential* at $v_1=\hat{v}_1$

$U'(\hat{v}_1)=U_N{}'(\hat{v}_1)$

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* As we shall next show, the graph of $U_N(v_1)$ is actually a line.

The naïve buyer 1 always bids $\hat{b}_1$

so her win probability is always $w(\hat{v}_1)$.

Therefore

$$U_N(v_1) = w(\hat{v}_1)(v_1 - \hat{b}_1) = w(\hat{v}_1)v_1 - w(\hat{v}_1)\hat{b}_1$$

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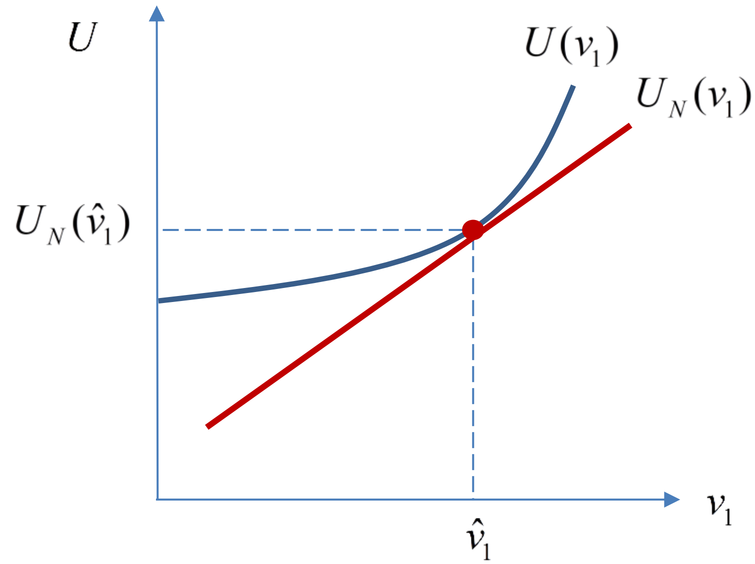


But the naïve buyer always bids $\hat{b}_1$

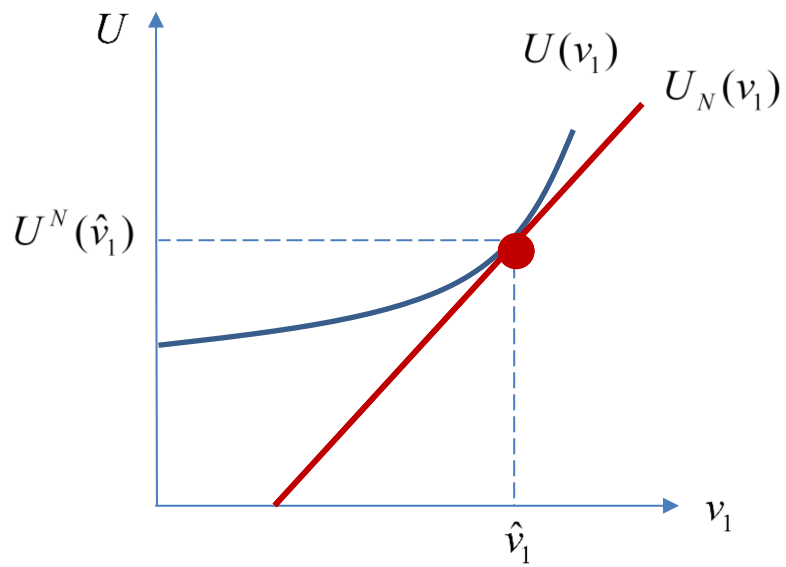
so her win probability is always $w(\hat{v}_1)$.

Therefore

$$U_N(v_1) = w(\hat{v}_1)(v_1 - \hat{b}_1) = w(\hat{v}_1)v_1 - w(\hat{v}_1)\hat{b}_1$$

$$= w(\hat{v}_1)v_1 - R(\hat{v}_1)$$

This is a line of slope $w(\hat{v}_1)$.



Therefore

$$U'(\hat{v}_1) = U_N{}'(\hat{v}_1) = w(\hat{v}_1) .$$

Since we can make the same argument for any $\hat{v}_1$, we have proved the following result.

Proposition: Marginal equilibrium payoff

The rate at which the equilibrium payoff rises with the buyer's value is equal to the equilibrium win probability $U'(v) = w(v)$

The lowest value participant has an equilibrium payoff of zero since other buyers have higher values with probability 1. Thus they make higher bids with probability 1.

We can then integrate $U'(v)$ to obtain the equilibrium payoff $U(v)$.

$$U(v_i) = \int_0^{v_i} w(v)dv \quad \text{where} \quad w(v) = F(v)^{I-1}$$

Also

$$U(v_i) = w(v_i)(v_i - B(v_i))$$

$$\begin{pmatrix} \text{equilibrium} \\ \text{payoff} \end{pmatrix} = \begin{pmatrix} \text{equilibrium} \\ \text{win probability} \end{pmatrix} \times \begin{pmatrix} \text{net gain} \\ \text{if buyer wins} \end{pmatrix}$$

Then we can solve for the equilibrium bid function.

Example: Uniform distribution with 3 buyers

Proposition: $U'(v) = w(v) = F(v)$ where $U(v) = w(v)(v - B(v))$

Suppose that buyers' values are uniformly distributed on $[0,1]$ (in millions of dollars) so $F(v) = v$

$U'(v) = F(v)^2 = v^2$.

Then

$U(v) = \frac{1}{3}v^3 + k$

But the lowest type wins with zero probability so $U(0) = 0$. Therefore

$U(v) = \frac{1}{3}v^3$

Also

$U(v) = w(v)(v - B(v)) = v^2(v - B(v))$.

Therefore $v^2(v - B(v)) = \frac{1}{3}v^3$ and so $v - B(v) = \frac{1}{3}v$.

Therefore $B(v) = \frac{2}{3}v$.

C. Other sealed bid auctions:

We will now generalize the above argument.

In the sealed high bid auction

$$U(v) = w(v)(v - B(v))$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{equilibrium}\\ \text{win probability}\end{pmatrix} \times \begin{pmatrix}\text{net gain}\\ \text{if buyer wins}\end{pmatrix}$$

This can be rewritten as follows

$$U(v) = w(v)v - w(v)B(v)$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{expected}\\ \text{gross gain}\end{pmatrix} - \begin{pmatrix}\text{expected}\\ \text{buyer payment}\end{pmatrix}$$

Since the payment by the buyer is the revenue of the seller we can also write

$$U(v) = w(v)v - R(v)$$

$$\begin{pmatrix}\text{equilibrium}\\ \text{payoff}\end{pmatrix} = \begin{pmatrix}\text{expected}\\ \text{gross gain}\end{pmatrix} - \begin{pmatrix}\text{expected}\\ \text{seller revenue}\end{pmatrix}$$

Consider some other auction in which the equilibrium bidding strategies are strictly increasing

Example 1: Sealed second-bid auction

Example 2: All pay auction

Buyers submit **non-refundable** bids. The high bidder is the winner.

Then the equilibrium win probability is $w(v_i) = F(v_i)^{I-1}$, just as in the sealed high-bid auction.

Let $R(v_i)$ be the equilibrium expected payment by buyer i .

Then buyer i's equilibrium payoff is

$$U(v_i) = w(v_i)v_i - R(v_i)$$

$$\begin{pmatrix} \text{equilibrium} \\ \text{payoff} \end{pmatrix} = \begin{pmatrix} \text{expected} \\ \text{gross gain} \end{pmatrix} - \begin{pmatrix} \text{expected} \\ \text{seller revenue} \end{pmatrix}$$

We can argue step 2 in the same way.

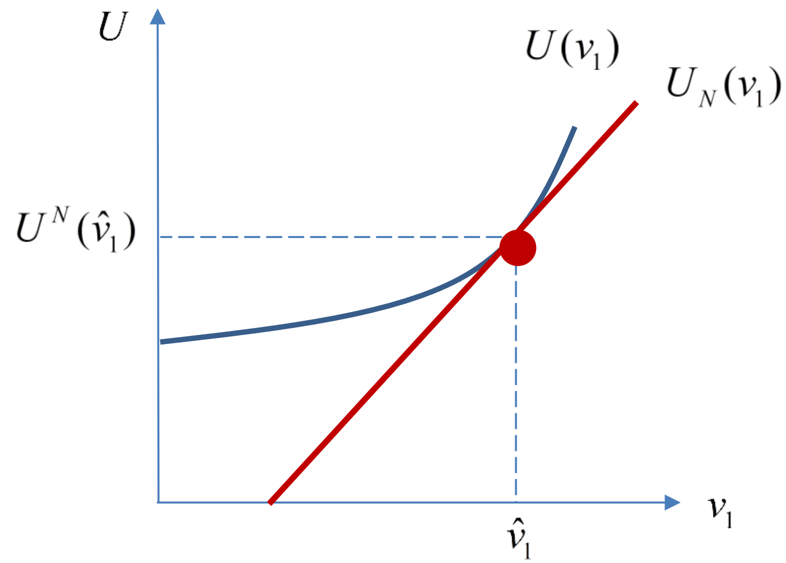
Equilibrium payoff:

$$U(v_i) = w(v_i)v_i - R(v_i)$$

Step 2: Suppose that buyer 1 is naïve (stupid!)

Suppose buyer 1 uses the strategy $B_N(v_1) = B(\hat{v}_1)$ regardless of her value. Then her expected payment is $R(\hat{v}_1)$, regardless of her value

$$U_N(v_1) = w(\hat{v}_1)v_1 - R(\hat{v}_1)$$



Since $B(v_1)$ is her best response strategy,

the naïve strategy is a best response

If and only if $v_1 = \hat{v}_1$.

$$U_N(v_1) = w(\hat{v}_1)v_1 - R(\hat{v}_1)$$

Is a line with slope $w(\hat{v}_1)$.

Therefore $U'(\hat{v}_1) = U_N{}'(\hat{v}_1) = w(\hat{v}_1)$.

Since we can make the same argument for any $\hat{v}_1$, we have proved the following result.

Proposition: The incremental equilibrium payoff for a buyer with a higher value is equal to the equilibrium win probability $U'(\theta) = w(\theta)$

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Proposition: The incremental equilibrium payoff for a buyer with a higher value is equal to the equilibrium win probability $U'(\theta) = w(\theta)$

A buyer with the lowest value has a zero probability of winning so $U(0) = 0$.

Therefore

Proposition: Buyer equivalence Theorem

The equilibrium payoff for a buyer with value v_i is $U(v_i) = \int_0^{v_i} w(v)dv$

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Proposition: Buyer equivalence Theorem

The equilibrium payoff for a buyer with value v_i is $U(v_i) = \int_0^{v_i} w(v)dv$

Since $U(v_i) = w(v_i)v_i - R(v_i)$

Proposition: Payment equivalence Theorem

The equilibrium expected payment by a buyer, $R(v_i)$ is the same as in the sealed high-bid auction.

Revenue equivalence: Since the expected payment is the expected revenue of the seller, the expected seller revenue is the same.

D. Reserve price in the sealed high-bid auction

Proposition: $U'(v) = w(v)$ where $U(v) = w(v)(v - B(v))$

Analysis:

The seller sets a "reserve price " (i.e. minimum bid of v_0). The probability that a buyer's value is less than v is $F(v)$. What is the equilibrium payoff?

A buyer has a value $v_i > v_0$ has a strictly positive payoff if he enters the auction and make a bid satisfying

$v_0 \leq B(v_i) < v_i$.

*

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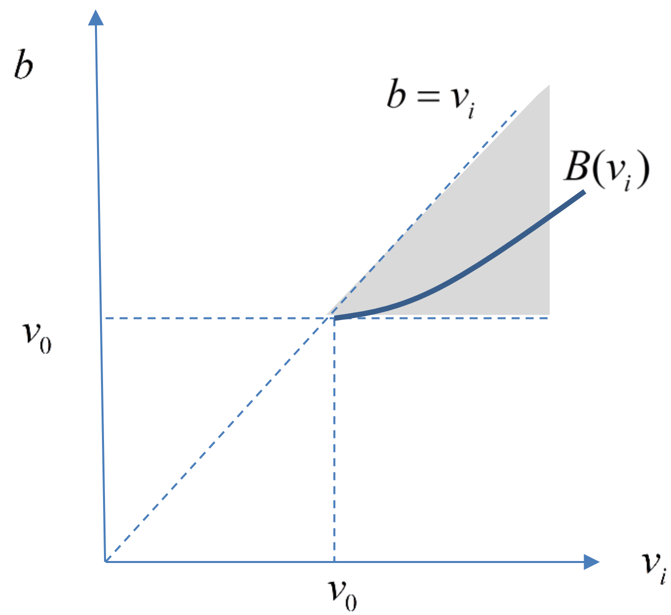

This is depicted opposite.

It follows that $v_0-B(v_0)=0$.

Hence $U(v_0)=w(v_0)(v_0-B(v_0))=0$

Therefore $U(v_i)=U(v_i)-U(v_0)=\int_{v_0}^{v_i} U'(v)dv=\int_{v_0}^{v_i} w(v)dv$

E. Open descending price auction (descending clock auction)

Clock starts ticking down. When a buyer raises his hand the clock stops and the buyer pays the price on the clock.

Proposition: The equilibrium bidding strategy is to stop the clock when the asking price is the equilibrium bid in the sealed high-bid auction

E. Exercises

The first two exercises consider the following model discussed on page 8.

If the total output is $q_1 + q_2 + ... + q_n$ then the market clearing price is $p(q_1 + ... + q_n)$.

Suppose $p(\sum_{i=1}^{n} q_i) = 60 - \sum_{i=1}^{n} q_i$, $C_i(q_i) = 12q_i$, i=1,...,n

Exercise 1: Nash equilibrium with more than two firms

What is the equilibrium price if there are (i) three identical firms (ii) 5 identical firms

Exercise 2: Nash equilibrium with a large number of firms

Show that the equilibrium price approaches 12 as the number of firms grows large.

Exercise 3: Equilibrium with increasing marginal cost

Suppose $p(q) = 60 - (q_1 + q_2)$, $C_1(q_1) = 50 + q_1^{\ 2}$ and $C_2(q_2) = 50 + q_2^{\ 2}$.

(a) Solve for the Nash Equilibrium outputs and price.

(b) What would the firms do if they could collude and so maximize the total profit of the two firms?

(c) Might the answer change if $C_2(q_2) = 100 + q_2^{\ 2}$

Exercise 5: Pricing game

If a firm cannot change its capacity quickly then the quantity setting model makes sense. But what if capacity can be easily changed. Then a firm can lower a price and still guarantee delivery, or raise a price and sell off unused capacity. $C_f(q_f) = c_f q_f$ where $c_1 = 4$ and $c_2 = 7$. Demands are

$$q_1 = 60 - 20p_1 + 10p_2 \,,\; q_2 = 150 + 10p_1 - 20p_2 .$$

Then the profit of firm f is $\pi_f(p_f, q_f(p)) = p_f q_f(p) - C_f(q_f(p)) = (p_f - c_f) q_f(p)$

(a) For each firm solve for the best response function $p_1 = b_1(q_2)$ and $p_2 = b_2(p_1)$.

(b) Depict these in a neat figure. What are the equilibrium prices in this game?

(c) If firm 2 produces nothing what will firm 1 do? Plot a price vector $p^0 = (p_1^0, p_2^0)$ indicating the monopoly outcome with only firm 1 producing.

Hint: The price of commodity 2 must be chosen so that demand for firm 2's output is zero.

(d) Starting from this price pair, examine the adjustment process proposed by Cournot.

(e) Compare the equilibrium profits with those in the quantity setting game.

6. Sealed high-bid auction with a reserve price (minimum price).

Each of two buyers has a value that is uniformly distributed on $[0,1]$.

The seller sets a reserve price of θ.

(a) Explain why $U'(v)=v$ for all $v\geq\theta$.

(b) Explain why $U(v)=0$ for all $v\leq\theta$.

(c) Hence solve for $U(v)$.

(d) Use this result to solve for the equilibrium bid function.