58 slides

*Not examinable. Will be omitted from the lectures.

A1. Elasticity

Consider the figure opposite. A very useful measure of the sensitivity of y with respect to z is the proportional rate of change of y with respect to z.

This is called the "arc elasticity"

Arc elasticity = $\dfrac{\frac{\Delta y}{y}}{\frac{\Delta z}{z}} = \dfrac{z}{y}\dfrac{\Delta y}{\Delta z}$

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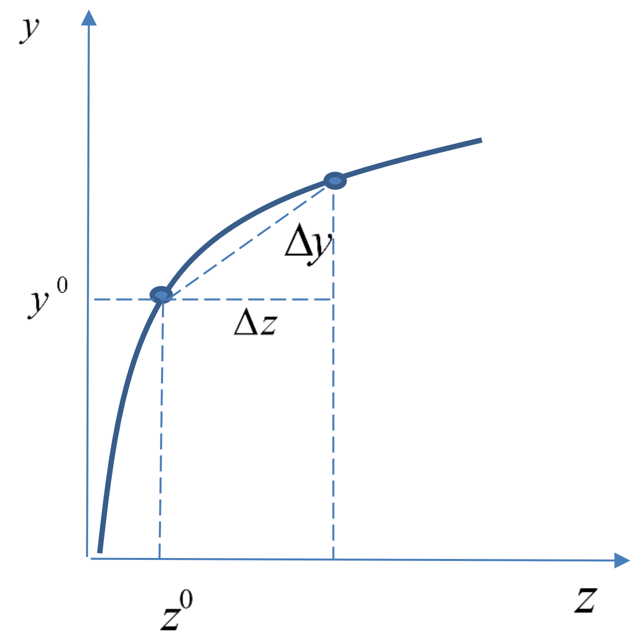

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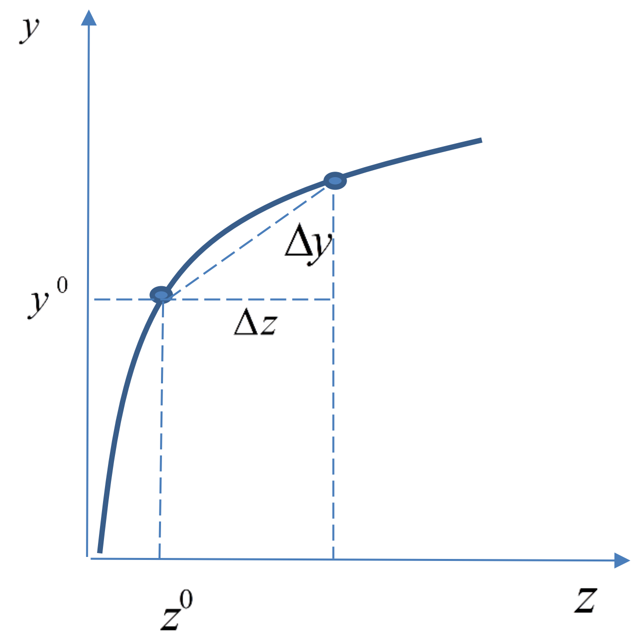

Arc elasticity

Consider two countries that measure both y and z in different units.

$Y = ay$ and $Z = bz$. Then $\Delta Y = a\Delta y$ and $\Delta Z = b\Delta z$

It follows that the arc elasticity is the same.

$$\frac{Z}{Y}\frac{\Delta Y}{\Delta Z} = \frac{(bz)}{(ay)}\frac{(a\Delta y)}{(b\Delta z)} = \frac{z}{y}\frac{\Delta y}{\Delta z}$$

(Point) Elasticity

In theoretical analysis it is helpful to take the limit

and define the (point) elasticity

Elasticity $=\mathcal{E}(y,z)=\lim\frac{z}{y}\frac{\Delta y}{\Delta z}=\frac{z}{y}\frac{dy}{dz}=\frac{zf'(z)}{f(z)}$

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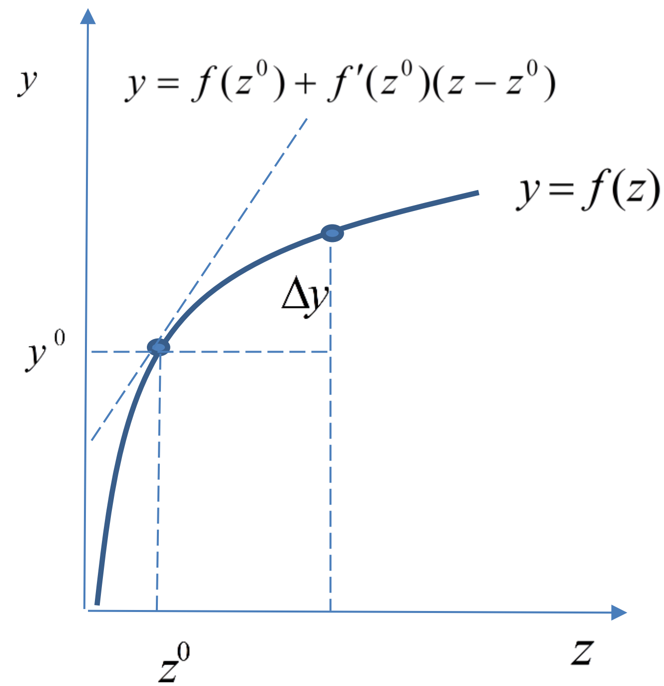

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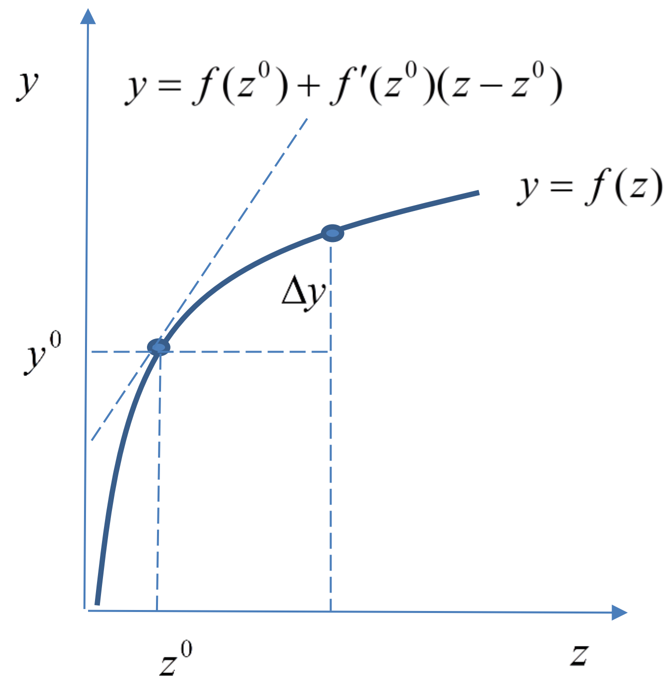


Point elasticity

Note that $\frac{d}{dz}\ln y = \frac{1}{y}\frac{dy}{dz}$. Therefore

$$\mathcal{E}(y,z) = \frac{z}{y}\frac{dy}{dz} = z\frac{d}{dz}\ln y.$$

Using this formula we can derive the following proposition

Elasticity of products and ratios

The elasticity of a product is the sum of the elasticities. $\mathcal{E}(xy,z) = \mathcal{E}(x,z) + \mathcal{E}(y,z)$

The elasticity of a ratio is the difference in elasticities $\mathcal{E}(\frac{x}{y},z) = \mathcal{E}(x,z) - \mathcal{E}(y,z)$

Derivation of the sum rule

$$\mathcal{E}(y,z)=\frac{z}{y}\frac{dy}{dz}=z\frac{d}{dz}\ln y$$

Consider the elasticity of a product.

$$\mathcal{E}(xy,z)=z\frac{d}{dz}\ln xy$$

$$=z\frac{d}{dz}[\ln x+\ln y]$$

$$=z\frac{d}{dz}\ln x+\frac{d}{dz}\ln y$$

$$=\mathcal{E}(x,z)+\mathcal{E}(y,z)$$

Group exercises: Group O: Linear demand $p=a-bq,\ q=\dfrac{a-p}{b}$

Group E: Log linear demand $q=ap^{-b},\ \ln q=\ln a-b\ln p$

A2. The Envelope Theorem

Consider the following constrained maximization problem with a parameter p in the function to be maximized. Let $\hat{x}$ be the solution when the parameter is $\hat{p}$. Let $\bar{x}(p)$ be the solution for all p . Let $F(p)$ be the maximized value.

$$F(p) = \underset{x}{Max}\{f(x,p) \mid g(x) \le b\}.$$

Simple Example: Profit maximization $F(p) = \underset{q}{Max}\{pq - C(q)\}$

To determine the rate at which $F(p) = f(\bar{x}(p), p)$ varies with p is appears that it is necessary to first solve for the maximizer $\bar{x}(p)$ and then substitute this into $f(x,p)$.

However this intuition is incorrect. The answer is much simpler. On the margin only the direct effect is a non-zero effect.

Envelope theorem

$$F(p) = \underset{x \geq 0}{Max}\{f(x,p) \mid g(x) \leq b\}\ .$$

$$\frac{dF}{dp} = \frac{\partial f}{\partial p}(\overline{x}(p), p)$$

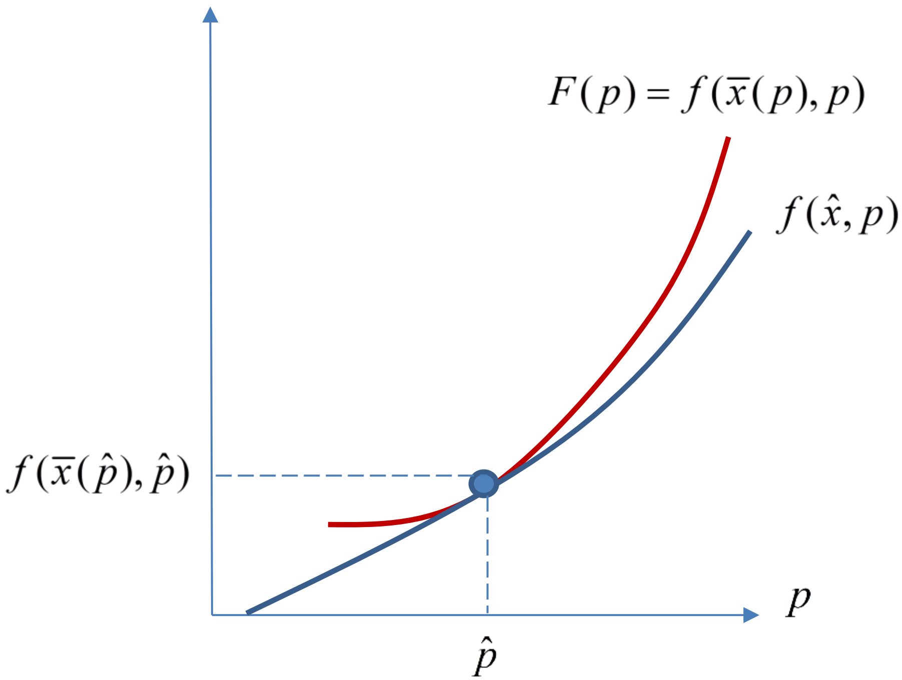
Informal proof: Let $\hat{x} = \overline{x}(\hat{p})$ be the solution when the price is $\hat{p}$.

Suppose that the decision-maker is naïve and does not change output as the parameter changes.

The naïve payoff is $f(\hat{x}, p) = f(\overline{x}(\hat{p}), \hat{p})$.

We compare $F(p)$ and $f(\hat{x}, p)$.

*



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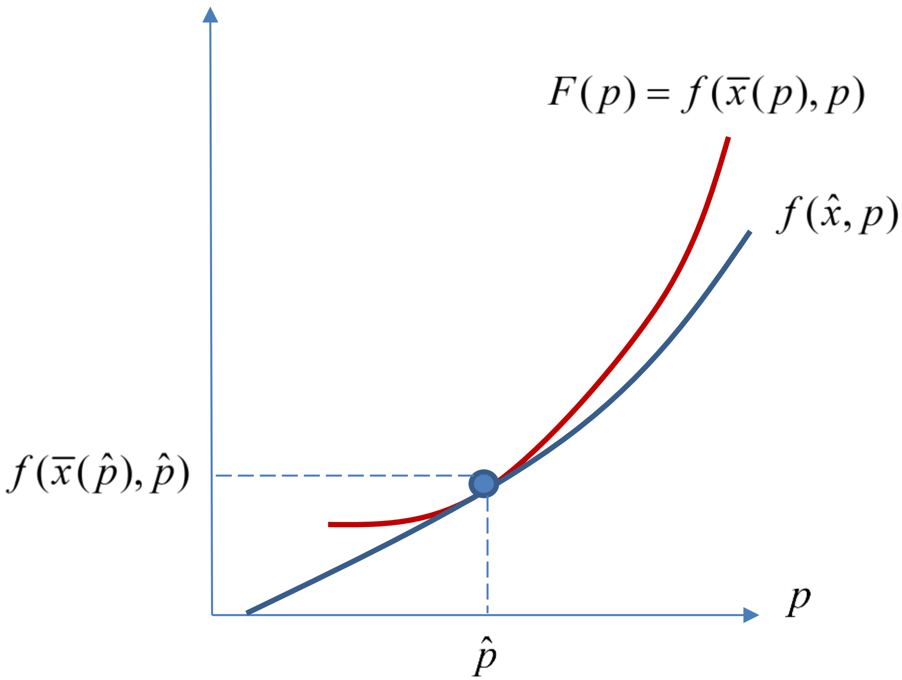
Note that

(i) $F(\hat{p}) = f(\bar{x}(\hat{p}), \hat{p})$

Since $\bar{x}(p)$ is optimal, for all p

(ii) $F(p) = f(\bar{x}(p), p) \ge f(\bar{x}(\hat{p}), p)$.

Assuming that the functions are differentiable, the graphs of the two functions must be as depicted. It follows that the graphs must be tangential at $\hat{p}$. i.e. $F'(p) = \frac{\partial f}{\partial p}(\hat{x}, p)$



Intuition for the simplest case with no constraint

$$F(p) = \underset{x}{Max}\{f(x,p)\}$$

There are two effects

1. Direct effect

Parameter change Δp

$f(\hat{x},\hat{p})$ rises to $f(x,\hat{p}+\Delta p)$

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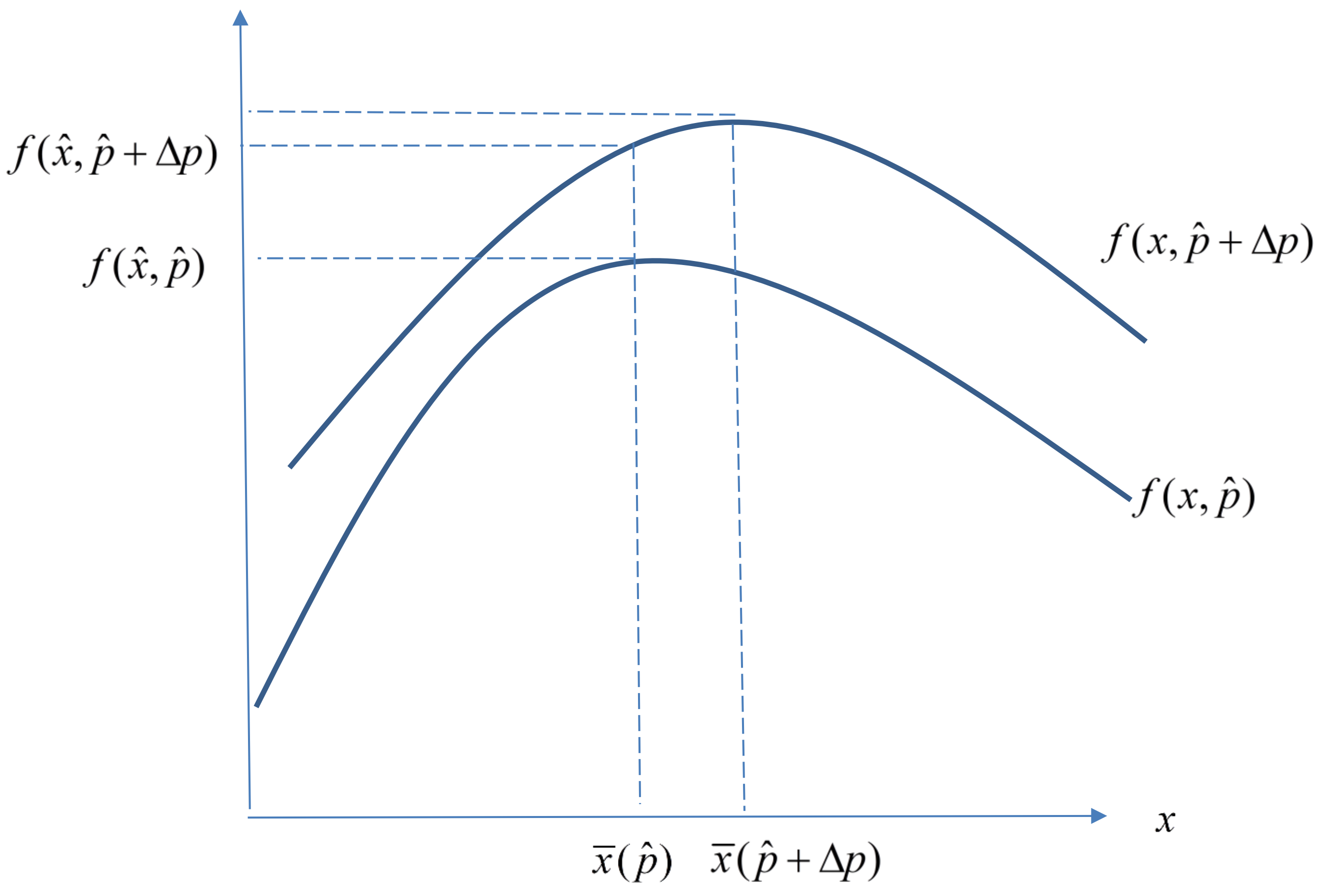

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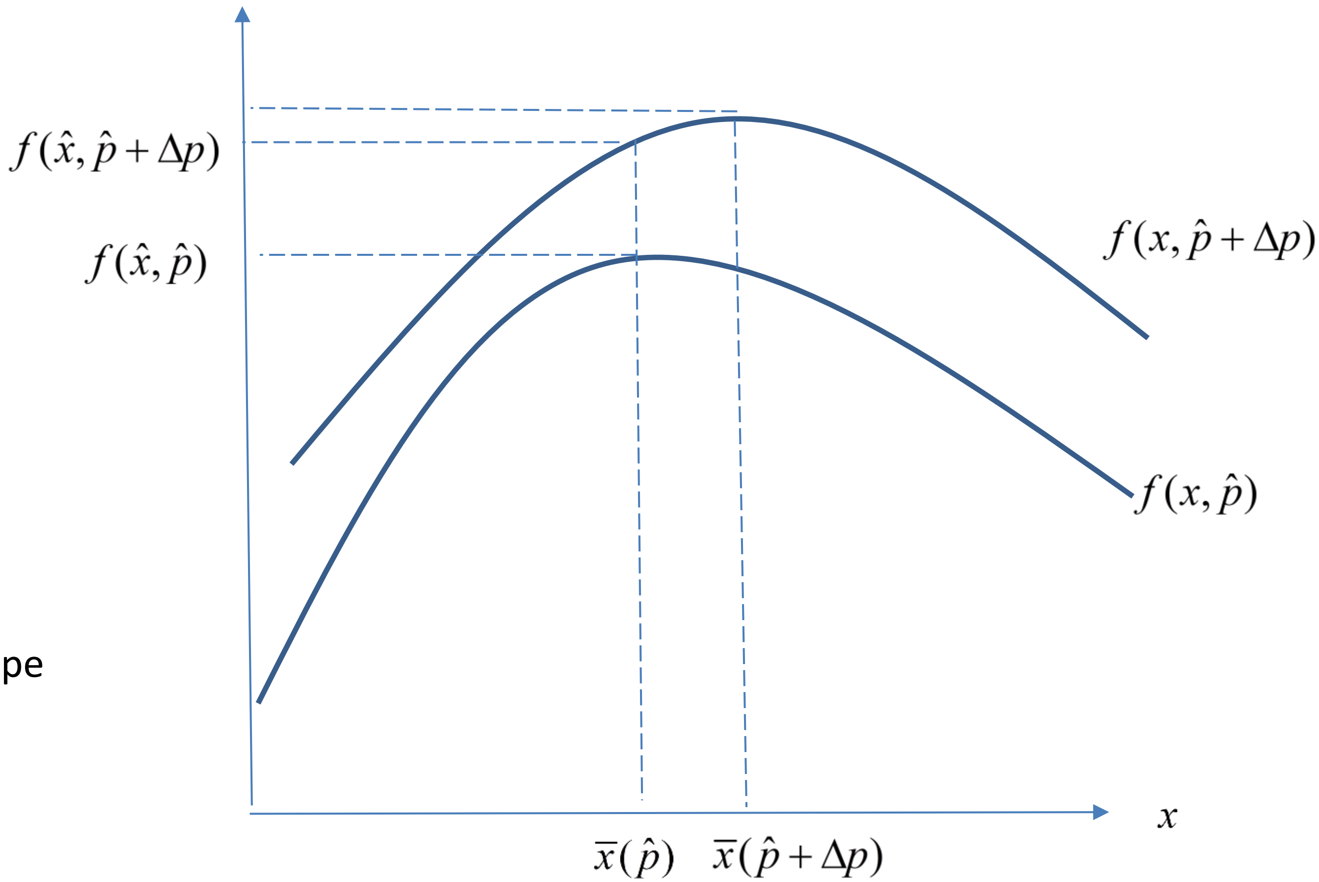

2. Indirect effect

Decision variable change Δx .

For small change in price the

The graph of $f(x,p)$ has a slope

which is close to zero.

This effect disappears in the limit. Only the direct effect is a "first order effect".

A more general result (Not examinable)

$F(p) = \underset{x \in X}{Max}\{f(x,p)\}$. Note that x is constrained to belong to some unspecified set.

Proposition: If $\bar{x}(p)$ is a continuous function then $F'(p) = \frac{\partial f}{\partial p}(\bar{x}(p), p)$

(i) Since $\bar{x}(p^0)$ is the optimizer when $p = p^0$, it follows that $F(p^0) = f(\bar{x}(p^0), p^0) \geq f(\bar{x}(p^1), p^0)$.

Therefore $F(p^1) - F(p^0) \equiv f(\bar{x}(p^1), p^1) - f(\bar{x}(p^0), p^0) \leq f(\bar{x}(p^1), p^1) - f(\bar{x}(p^1), p^0)$

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Together, these inequalities imply that

$$\frac{f(\bar{x}(p^0),p^1)-f(\bar{x}(p^0),p^0)}{p^1-p^0}\leq\frac{F(p^1)-F(p^0)}{p^1-p^0}\leq\frac{f(\bar{x}(p^1),p^1)-f(\bar{x}(p^1),p^0)}{p^1-p^0}.$$

Note that as $p^1\rightarrow p^0$, the lower and upper bounds both approach $\frac{\partial f}{\partial p}(\bar{x}(p^0),p^0)$. Thus the derivative

$$F'(p^0)=\frac{\partial f}{\partial p}(\bar{x}(p^0),p^0).$$

QED

B. Income and substitution effects on demand

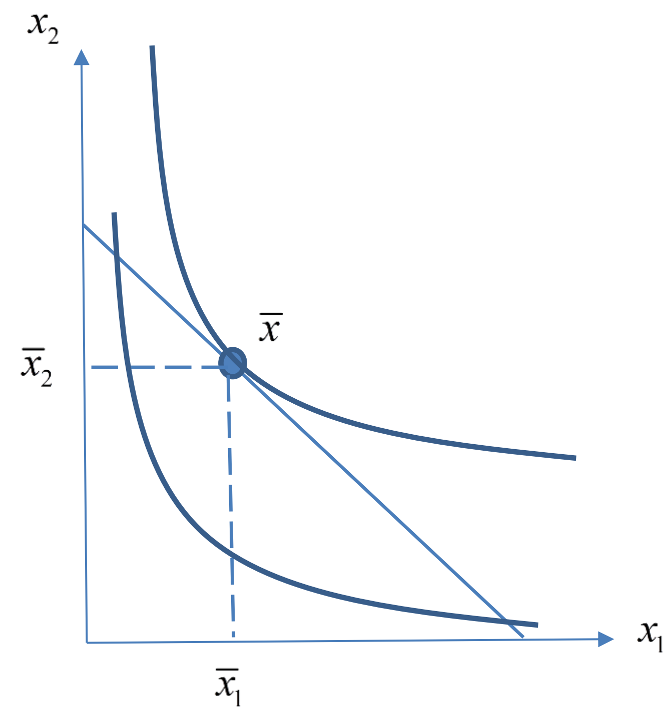
Decomposition of the effects of a price increase

with fixed income

A consumer with income I facing price vector $\bar{p}$

chooses $\bar{x}$ that solves

$$\underset{x\geq 0}{Max}\{U(x) \mid p\cdot x \leq I\}$$



Utility maximization

B. Income and substitution effects on demand

Decomposition of the effects of a price increase

A consumer with income I, facing price vector $\bar{p}$

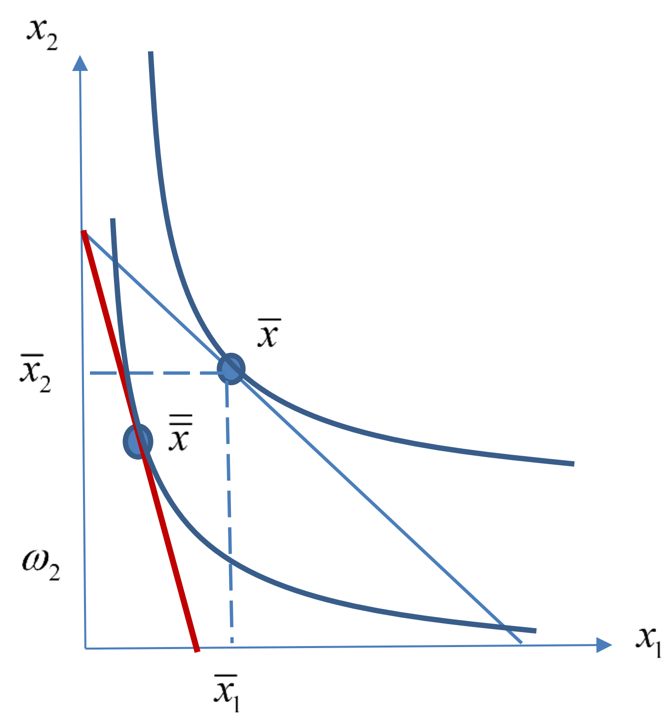
chooses $\bar{x}$ that solves

$$\underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq p \cdot \omega\}$$

The price of commodity 1 rises from $\bar{p}_1$ to $\bar{\bar{p}}_1$

so the consumer is worse off.

Consider the following thought experiment.



Price increase lowers utility

Step 1: Give the consumer enough income compensation so that she is no worse off, i.e. she ends up on the initial level set.

Step 2: Take away the compensation

Step 1: Compensated demand

The "substitution effect"

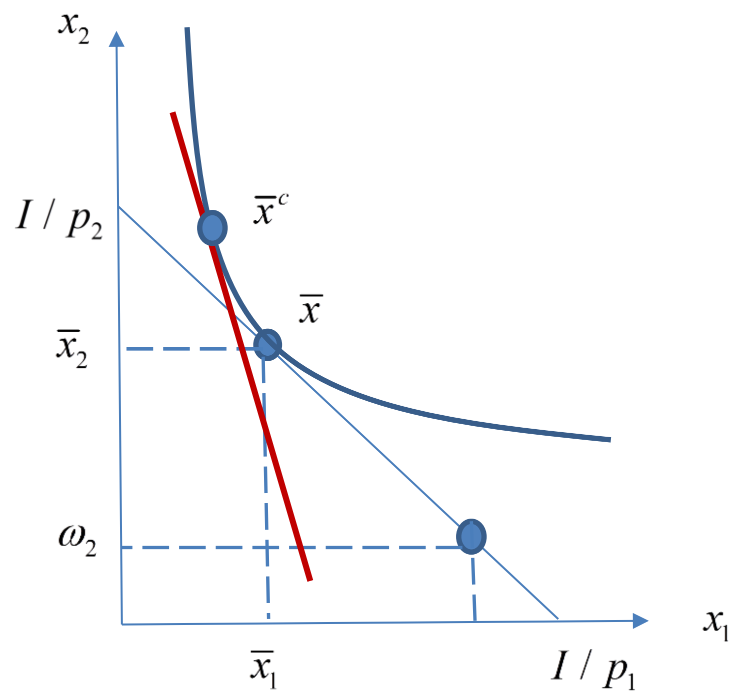
Suppose that the consumer is taxed just enough that her utility is unchanged.

This is called the compensated price effect.

The new optimum is $\bar{x}^c$.

Since the relative price of commodity 1 has risen, demand for commodity 1 falls and demand for commodity 2 rises.

The consumer has substituted away from the commodity that has become relatively more expensive.



Compensated effect of the price increase

Step 2: The compensation is taken away

The "income effect"

A commodity is called "normal" if more of it

Is consumed as income rises.

When the consumer pays back his compensation

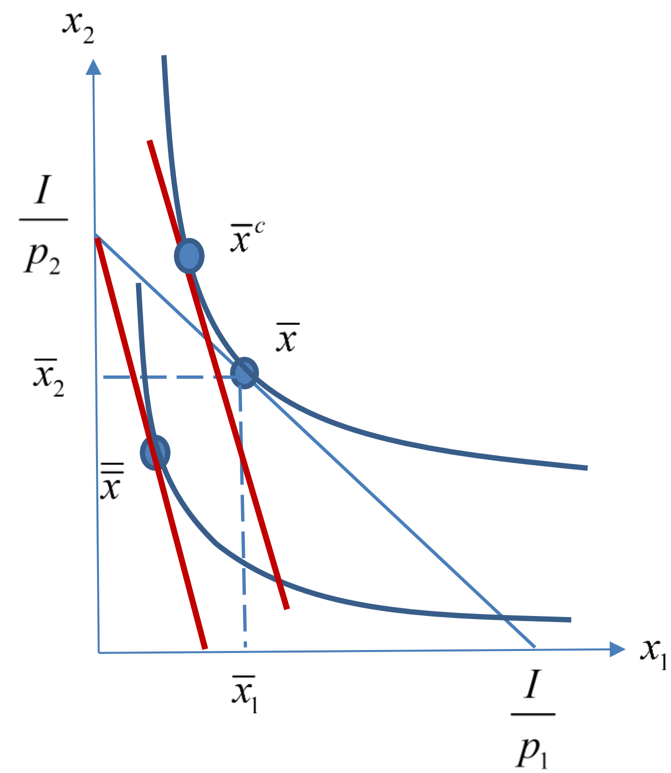
Demand for both goods falls if both are normal.

The total effect on demand for commodity 1.

The substitution effect and the income effect

are reinforcing. Both lead to lower demand

for commodity 1.



Income effect

Income and substitution effects on demand when the consumer has an endowment of commodities.

Decomposition of the effects of a price increase

A consumer with endowment ω facing price vector $\bar{p}$

chooses $\bar{x}$ that solves

$$\underset{x\geq 0}{Max}\{U(x) \mid p\cdot x \leq p\cdot \omega\}$$

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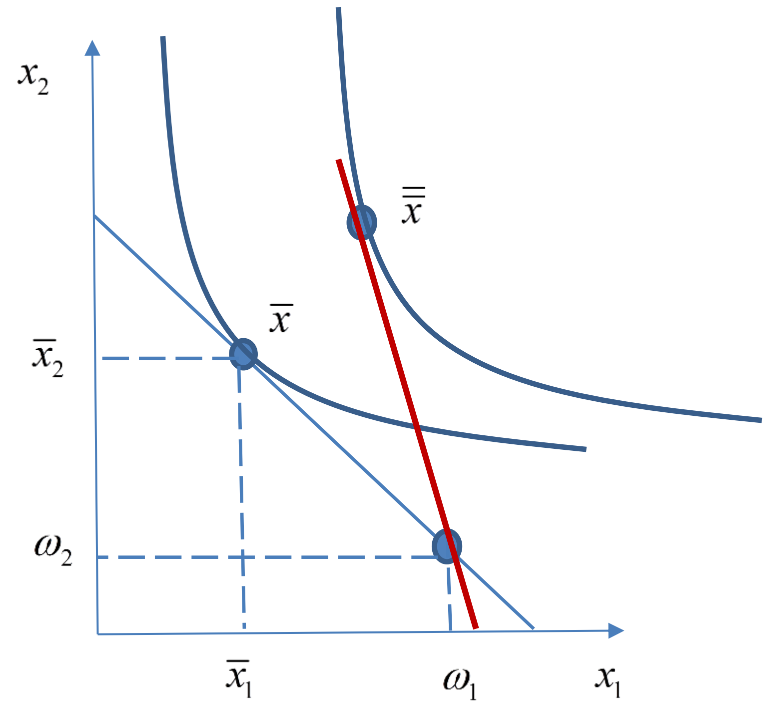


Price increase raises utility

Income and substitution effects on demand

Decomposition of the effects of a price increase

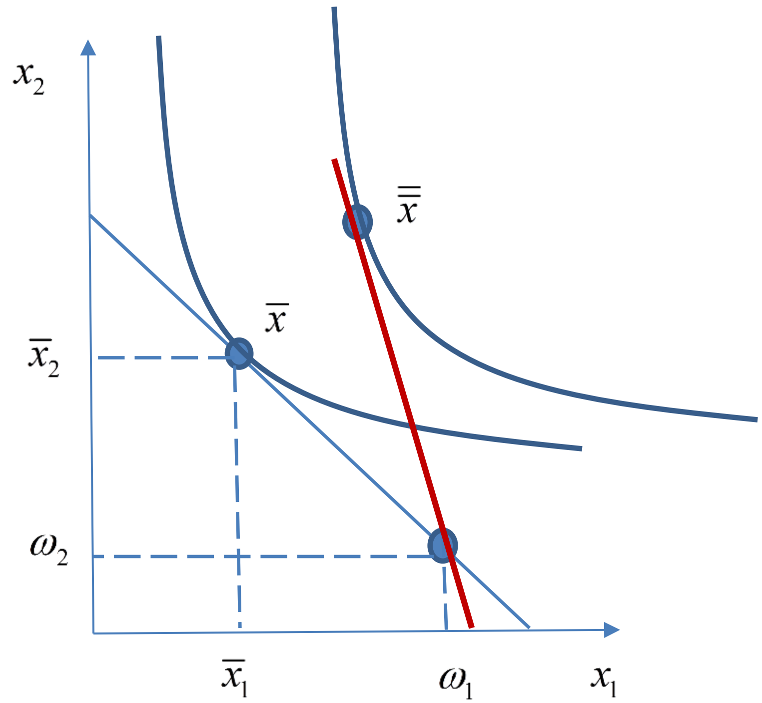
A consumer with endowment ω facing price vector $\overline{p}$

chooses $\overline{x}$ that solves

$$\underset{x\geq 0}{Max}\{U(x)\,|\,p\cdot x\leq p\cdot\omega\}$$

We consider an increase in the price of commodity 1

As depicted, the consumer's endowment of commodity 1 is so high that she is a net seller of this commodity. Therefore the price increase raises her utility.



Price increase raises utility

Substitution effect

Suppose that the consumer is taxed just enough that her utility is unchanged.

Note that the compensation is now negative.

The new optimum is $\bar{x}^c$.

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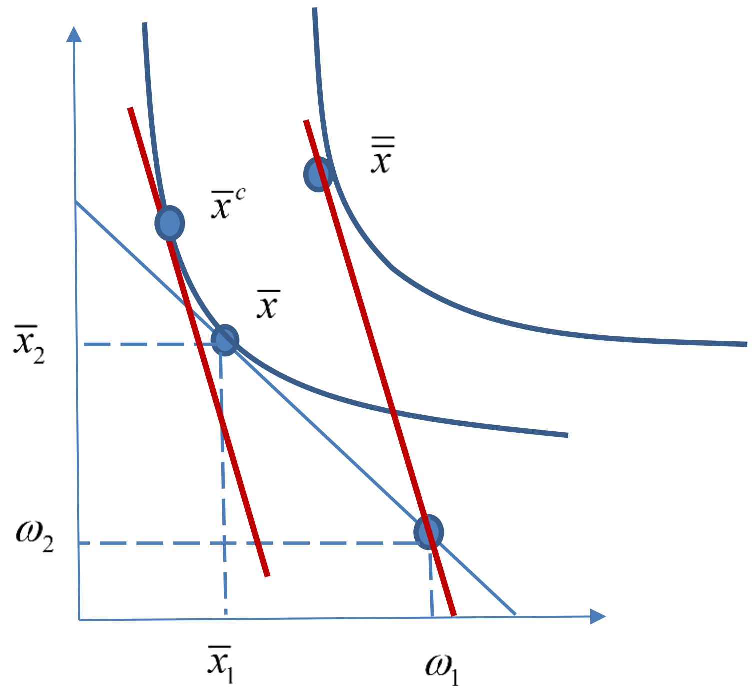

Compensated effect of the price increase

Income effect

Now give the tax back to the consumer.

A "normal good" is a commodity for which consumption rises with income. As depicted both commodities are normal goods so the income effect on the consumption of both commodities is positive.

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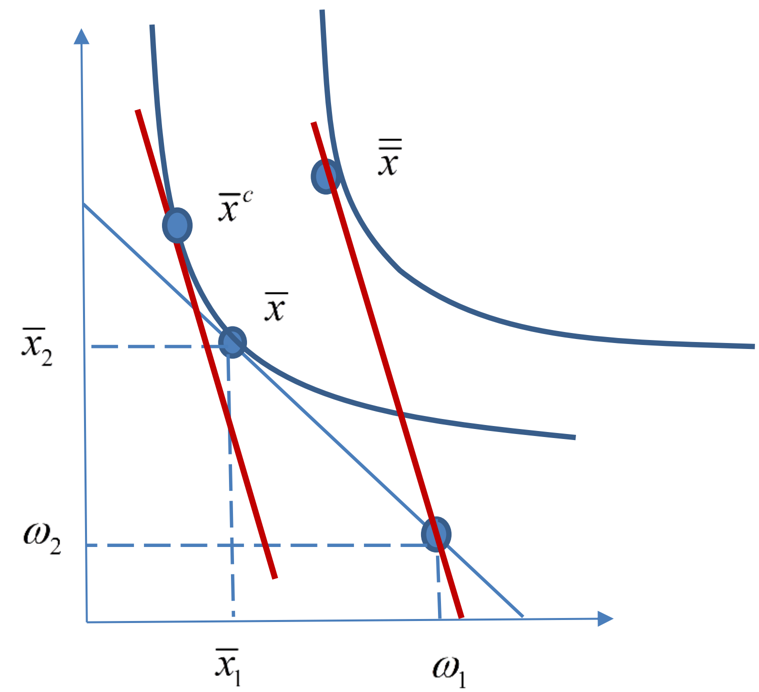


Compensated and income effects

Income effect

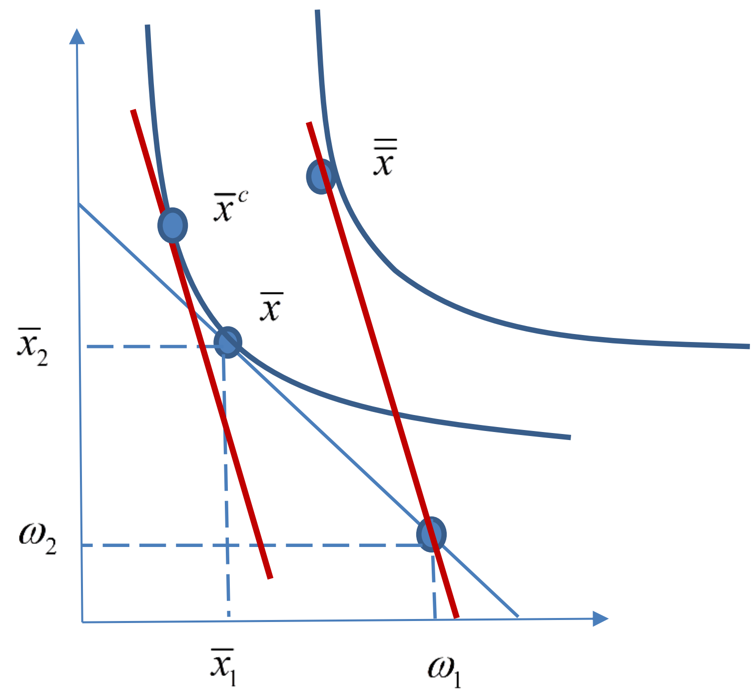
Now give the tax back to the consumer.

A "normal" commodity is a commodity for which consumption rises with income. As depicted both commodities are normal goods so the income effect on the consumption of both commodities is positive.

Total effect

Thus with normal commodities, the two effects Are opposing on the commodity for which the price has increased.

The two effects are reinforcing for the other commodity.



Compensated and income effects

C. Application: Labor supply

We now interpret commodity 1

as a consumer's leisure time. The

price of commodity 1 becomes the wage rate

If he does not work he has ω_1 units of leisure.

If he supplies z_1 units of labor, his income is

$p_1 z_1$ and so his budget constraint is

$$p_2 x_2 \leq w z_1 = p_1 z_1 \ . \qquad (*)$$

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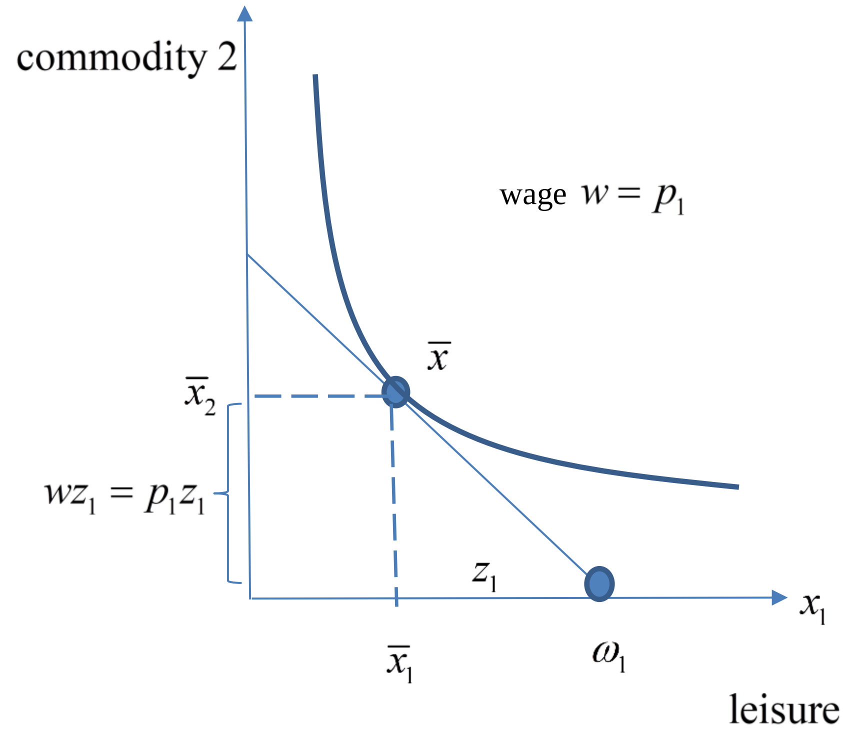

Labor supply

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$$p_2 x_2 \leq w z_1 = p_1 z_1 \,. \qquad (*)$$

Note that his hours of leisure have dropped to

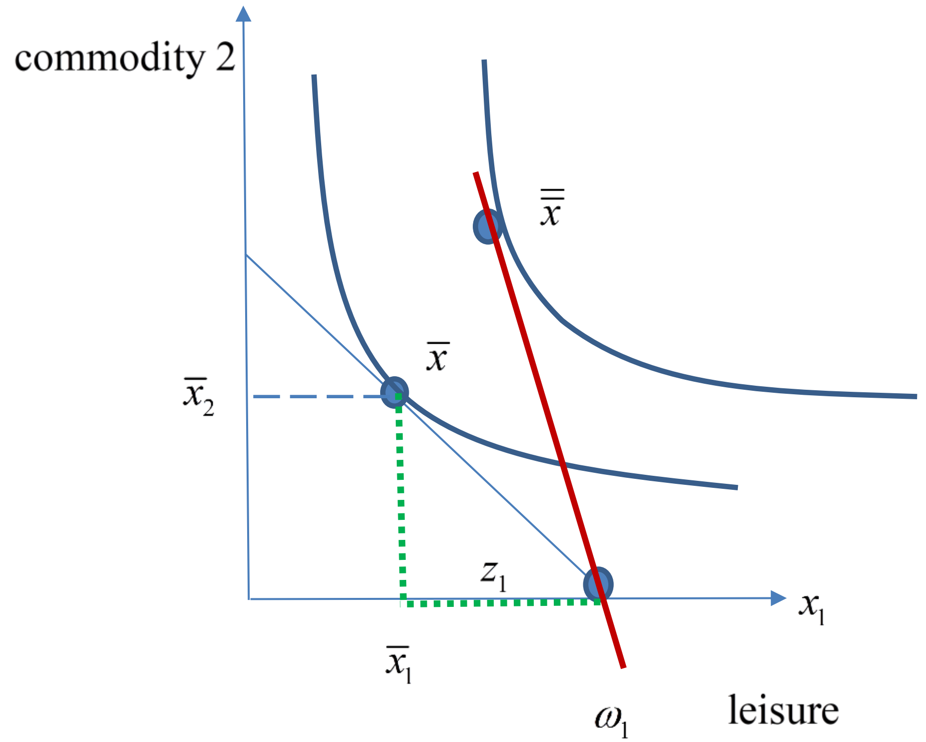
$$x_1 = \omega_1 - z_1$$

His expenditure o commodity 1 is therefore

$$p_1 x_1 = p_1 \omega_1 - p_1 z_1 \qquad (**)$$

Adding (*) and (**)

$$p_1 x_1 + p_2 x_2 \leq p_1 \omega_1$$



Labor supply

Budget constraint:

$$p_1 x_1 + p_2 x_2 \le p_1 \omega_1$$

The value of his consumption of leisure and the other commodity cannot exceed the value of his endowment.

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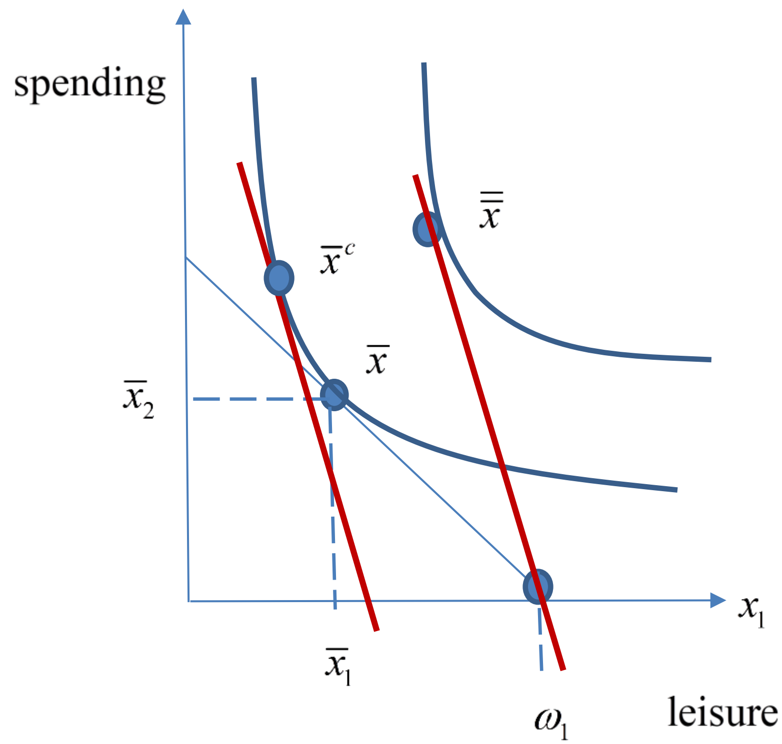

Compensated effect of the price increase

Budget constraint:

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Substitution effect

The substitution effect of a wage increase (increase in p_1) is a reduction in demand for commodity 1 and thus an increase in the labor supply if leisure is a normal commodity.

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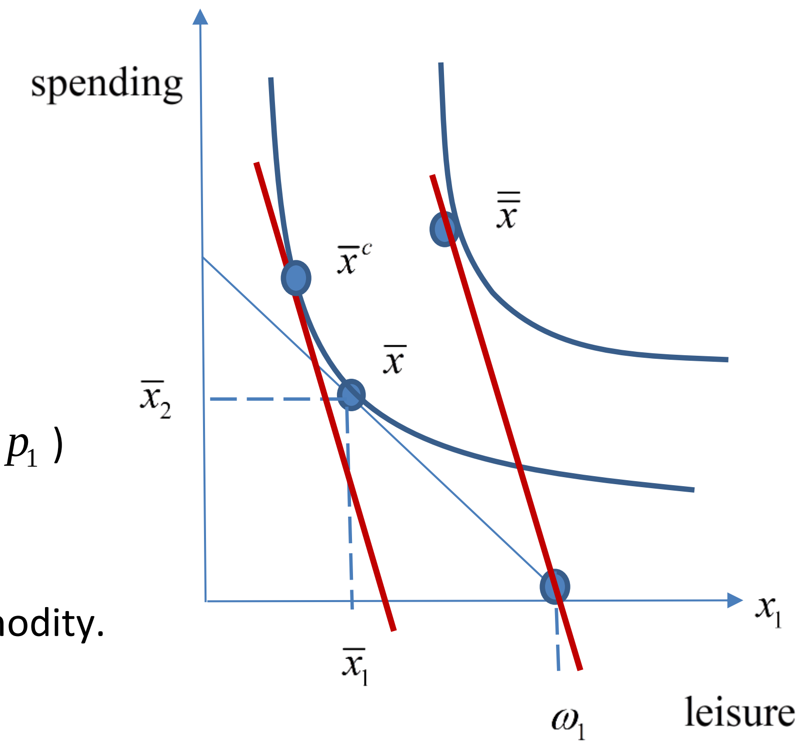

Compensated effect of the price increase

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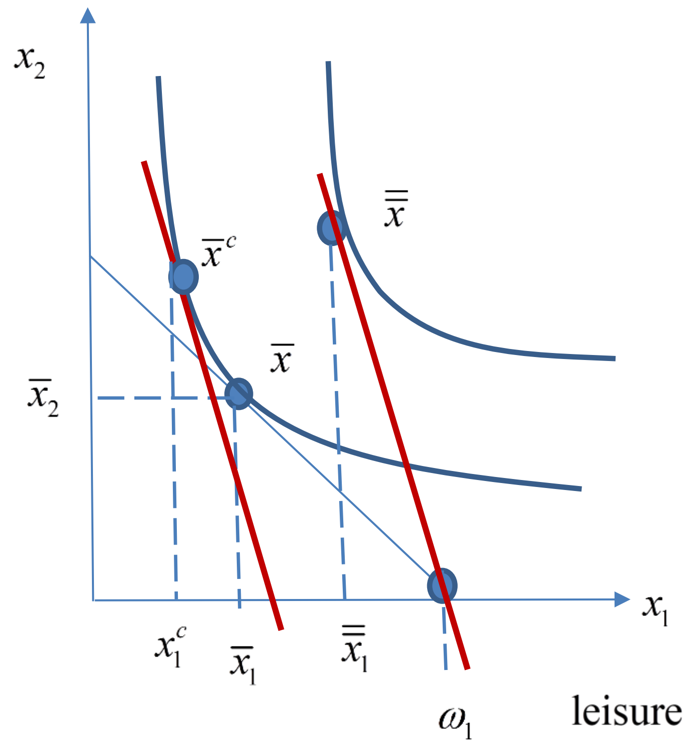
The substitution effect of a wage increase (increase in p_1) is a reduction in demand for commodity 1 and thus an increase in the labor supply if leisure is a normal commodity.

Income effect (normal commodities)

The income effect is to increase demand for leisure and thus reduce the supply of labor.

Thus the two effects are offsetting.

It is therefore not surprising that data analysis shows that wage effects on the aggregate supply of labor are small.



Compensated effect of the price increase

D. Determinants of demand with n commodities

Consider the standard consumer problem

$$\underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq I\}$$

To better understand the determinants of demand for a commodity (we label it commodity 1) it is helpful to think of the consumer as solving this problem in two steps.

**

D. Determinants of demand with n commodities

Consider the standard consumer problem

$$\underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq I\} \text{ where } x = (x_1, x_2, ..., x_n)$$

To better understand the determinants of demand for a commodity (we label it commodity 1) it is helpful to think of the consumer as solving this problem in two steps.

Step 1:

Suppose that the consumer must consume x_1 units of commodity 1 and is given y to spend on the other $n-1$ commodities. We write the vector of all the other commodities as $z = (x_2, ..., x_n)$ and define the price vector $r = (p_2, ..., p_n)$.

Then the consumer solves the following problem.

$$\underset{z \geq 0}{Max}\{U(x_1, z) \mid r \cdot z \leq y\}.$$

*

D. Determinants of demand with n commodities

Consider the standard consumer problem

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Then the consumer solves the following problem.

$$\underset{z \geq 0}{Max}\{U(x_1, z) \,|\, r \cdot z \leq y\} .$$

Let $\bar{z}(y)$ be a solution of this maximization problem. Then the maximized utility is

$$u(x_1, y) = U(x_1, \bar{z}(y))$$

Group Exercise:

$U(x) = x_1^{1/2} + bx_2^{1/4}x_3^{14}$,

Budget constraint $p_1x_1 + p_2x_2 + p_3x_3 \leq I$.

Step 1:

Fix x_1 and allocate y dollars to be spent on x_2 and x_3 .

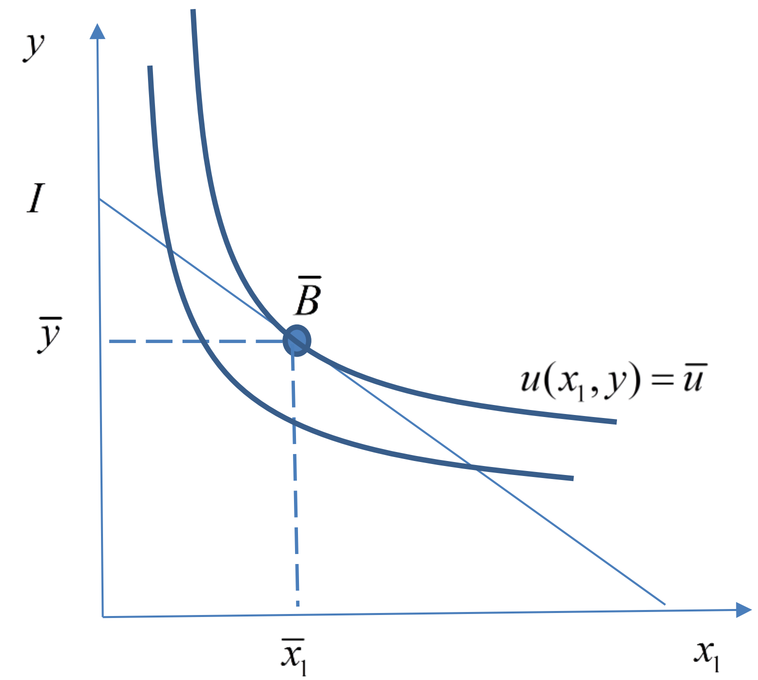
So maximize $bx_2^{1/4}x_3^{14}$ (a problem that we have seen a lot)

Step 2:

Choose $\bar{x}_1, \bar{y}$ that solves

$$\underset{x_1,y}{Max}\{u(x_1,y) \mid p_1x_1 + y \leq I\}.$$

It is this second step that we now consider.*



Aggregated utility function

*In the figure the superlevel sets of the aggregated utility function are convex. As long as the superlevel set of $U(x)$ are strictly convex, then so are the super level sets of $u(x_1, y)$. If you are interested in the proof, see the Technical Note at the end.

$(\bar{x}_1, \bar{y})$ solves

$$\underset{x_1, y}{Max}\{u(x_1, y) \mid p_1 x_1 + y \le I\}\ .$$

It is this second step that we now consider.

Decomposition of the effects of a price increase

A consumer with income I facing a price vector $\bar{p}$

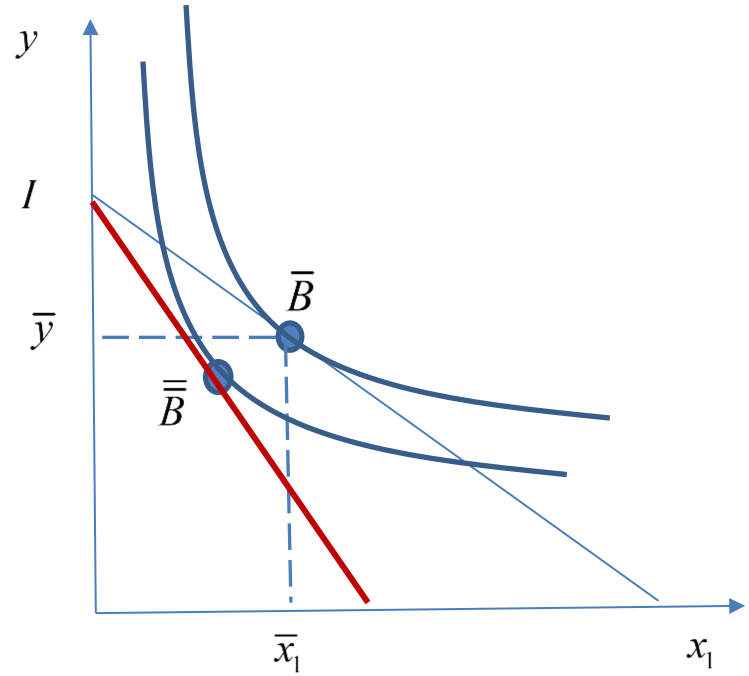
chooses $(\bar{x}_1, \bar{y})$ that solves

$$\underset{x_1, y}{Max}\{u(x, y) \mid p_1 x_1 + y \le I\}$$

We consider an increase in the price of x_1.

In the figure $\bar{B}$ is the maximizer at the initial

price $\bar{p}_1$ and $\bar{\bar{B}}$ is the maximizer after the price increase.



Price increase raises utility

Substitution effect

Suppose that the consumer is subsidized just enough

that her utility is unchanged.

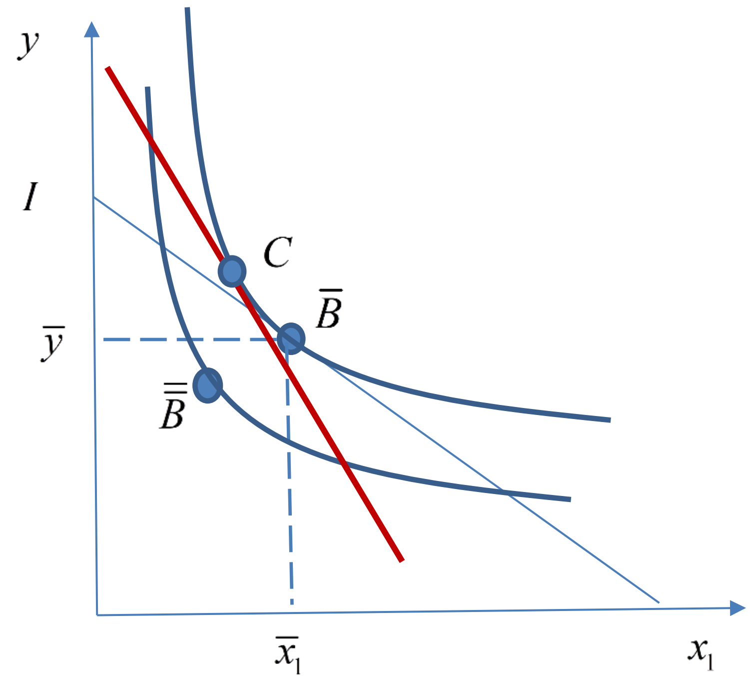
This is called the compensated price effect.

The new optimum is the point C .

Since the relative price of commodity 1 has

risen, demand for commodity 1 falls and spending

on other commodities rises .



Compensated price effect

Income effect (normal goods)

Now give the tax back to the consumer.

A “normal” commodity is a commodity for which consumption rises with income. As depicted commodity 1 is a normal commodity so the income effect is positive, offsetting the substitution effect.

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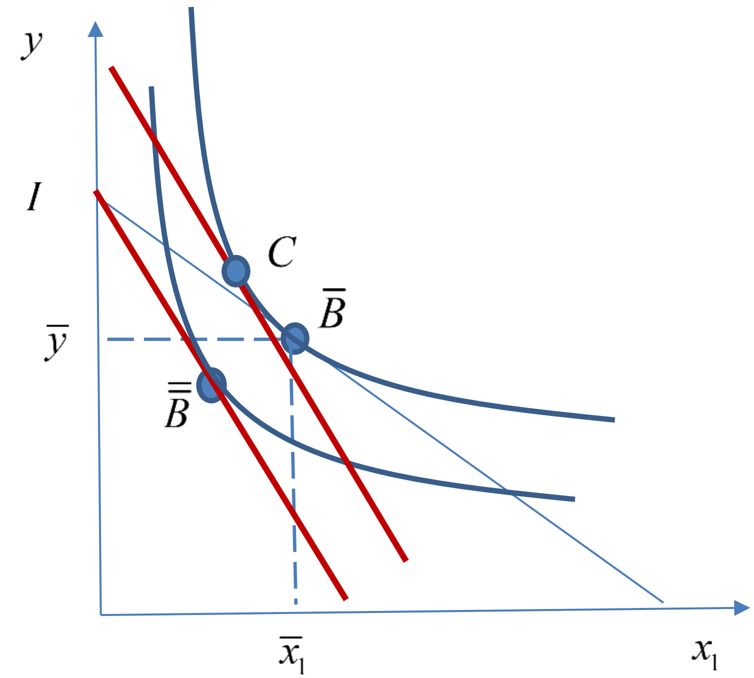

Price increase lowers utility

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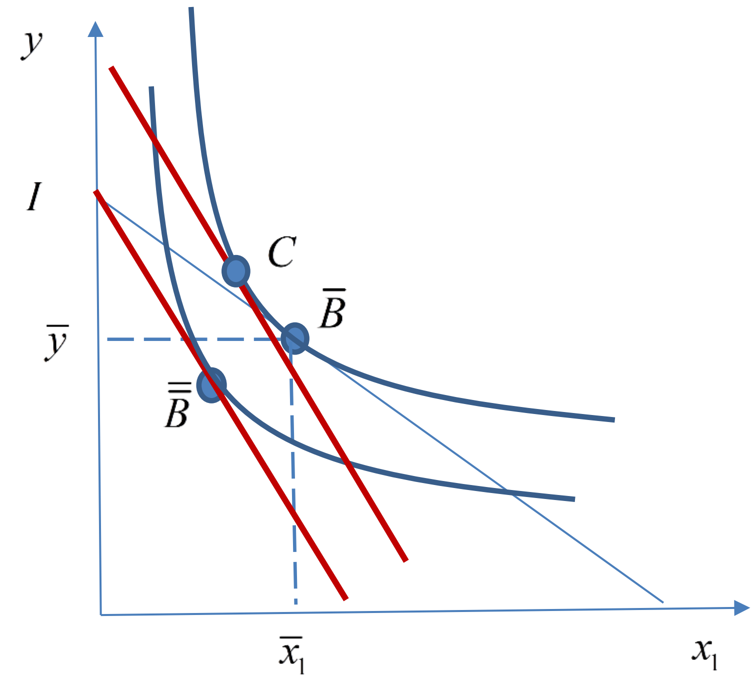


Price increase lowers utility

Theoretical possibility:

Giffen Good: demand increases with price

An example? The price of fuel oil rises sharply in New England. Enough people who were planning to winter in Florida can no longer afford to do so. They stay home and demand for fuel oil rises.

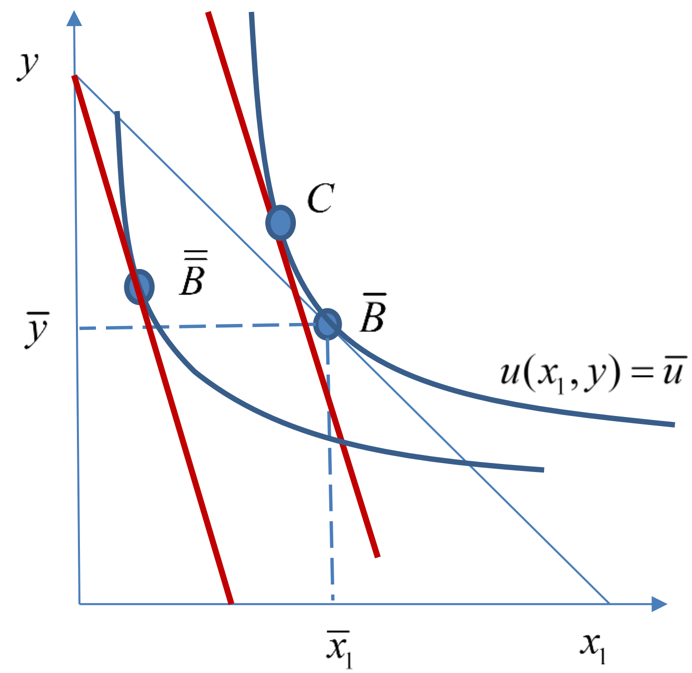
E. Income compensation and consumer surplus

Consider a consumer with income I . At the initial price $\bar{p}_1$ her choice is $(\bar{x}_1, \bar{y})$.

Let $M(p,\bar{u})$ be the income that the consumer requires to remain on her original indifference curve as p_1 rises. i.e.

$$M(p_1,\bar{u}) = \underset{x\geq 0}{Min}\{p_1x_1 + y \mid u(x_1,y) \geq \bar{u})\} .$$

**



Compensated effect of the price increase

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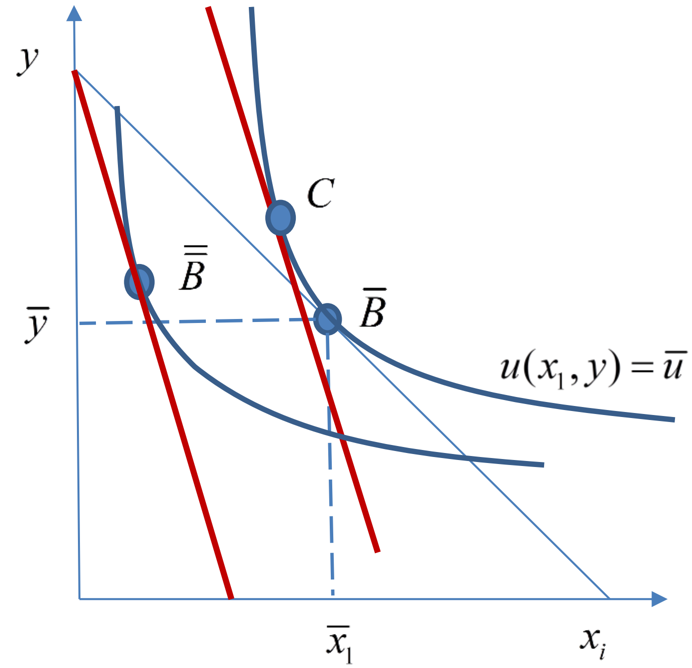
$$-M(p_1,\bar{u}) = \underset{x\geq 0}{Max}\{-p_1 x_1 - y \,|\, u(x_1,y) \geq \bar{u})\}$$

*



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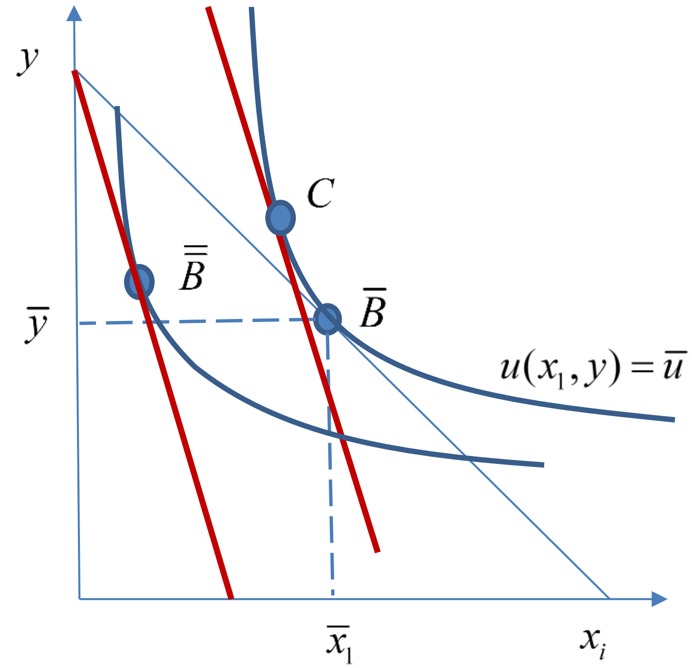
Appealing to the Envelope Theorem

$$-\frac{\partial M}{\partial p_1} = -x_1$$

Then the rate at which this income rises with p_1 is

$$\frac{\partial M}{\partial p_1} = x_1^c(p,\bar{u})$$

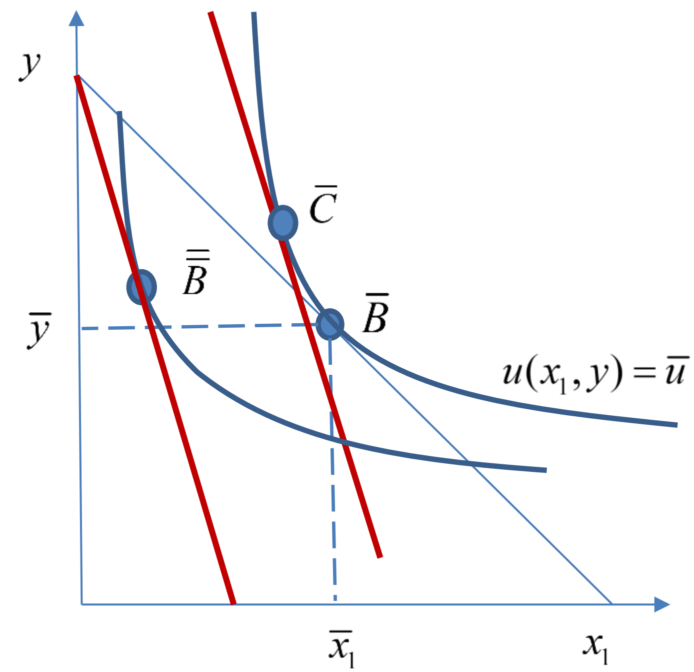


Compensated effect of the price increase

E. Income compensation and consumer surplus

In the Figure, with the additional income her consumption choice is $\bar{C}$. In the absence of the compensation her income is I and her consumption choice is $\bar{\bar{B}}$. Thus to be fully compensated for the price increase the consumer must be paid

$$M(p_1,\bar{u})-I \ .$$



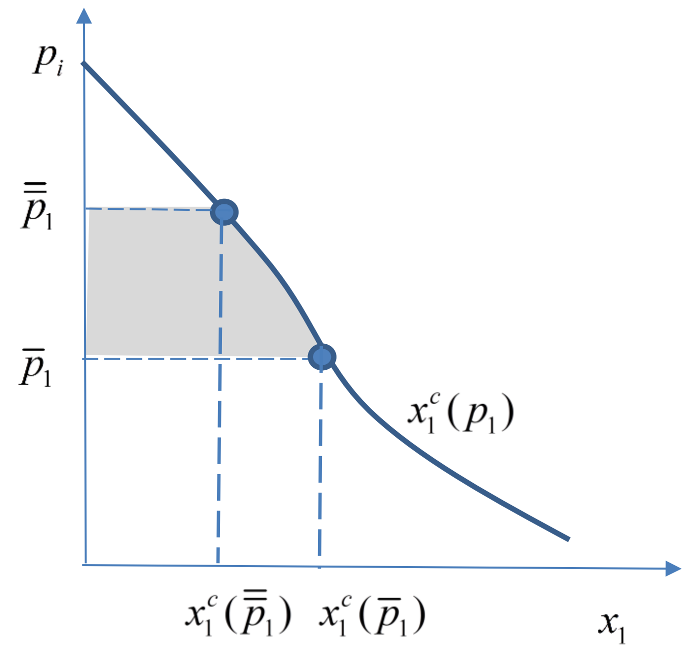
Compensated effect of the price increase

Compensating Variation in income

At the initial price vector no compensation is necessary so $M(\bar{p}_1,\bar{u})=I$. Then the compensating income change (called the compensating variation in income) is

$$CV = M(\bar{\bar{p}}_1,\bar{u}) - I = M(\bar{\bar{p}}_1,\bar{u}) - M(\bar{p}_1,\bar{u}) \ .$$

*



Compensated demand price function.

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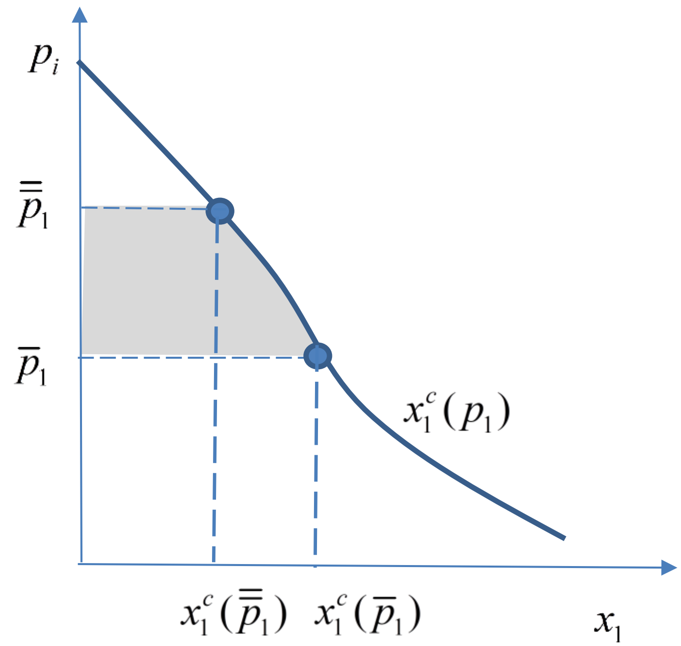
$$CV = M(\bar{\bar{p}}_1,\bar{u}) - I = M(\bar{\bar{p}}_1,\bar{u}) - M(\bar{p}_1,\bar{u}) \ .$$

Note that the definite integral $F(\bar{\bar{x}})-F(\bar{x})=\int_{\bar{x}}^{\bar{\bar{x}}} F'(x)dx$



Compensated demand price function.

Therefore

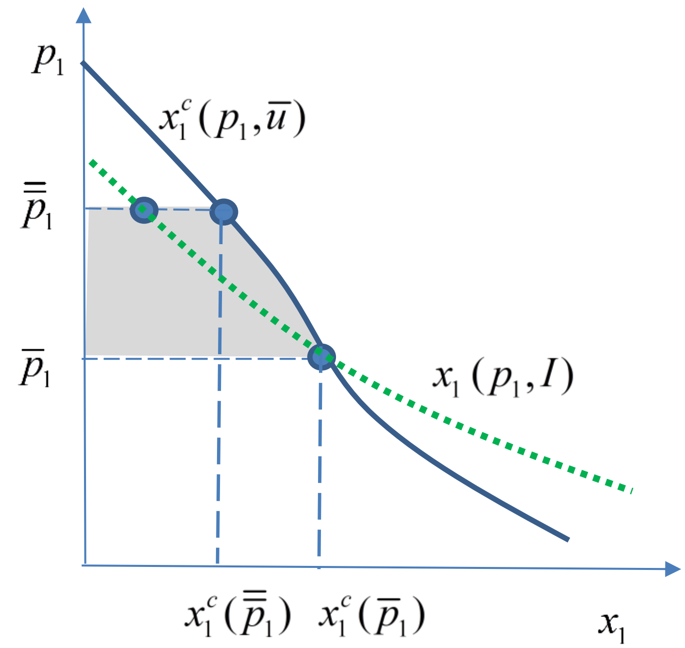
$$CV = M(\bar{\bar{p}}_1,\bar{u}) - M(\bar{p}_1,\bar{u}) = \int_{\bar{p}_1}^{\bar{\bar{p}}_1} \frac{\partial M}{\partial p_1} dp_1$$

By the Envelope Theorem, $\dfrac{\partial M}{\partial p_1} = x_1^c(p_1)$. Therefore $CV = \int_{\bar{p}_1}^{\bar{\bar{p}}_1} x_1^c(p_1)dp_1$

In the figure the compensating variation is the shaded area to the left of the compensated demand price function.

The dotted green ordinary demand function is $x_1(p_1, I)$ is also depicted. Assuming that commodity 1 is a normal commodity, income compensation raises demand when the price rises and lowers demand when the price falls.

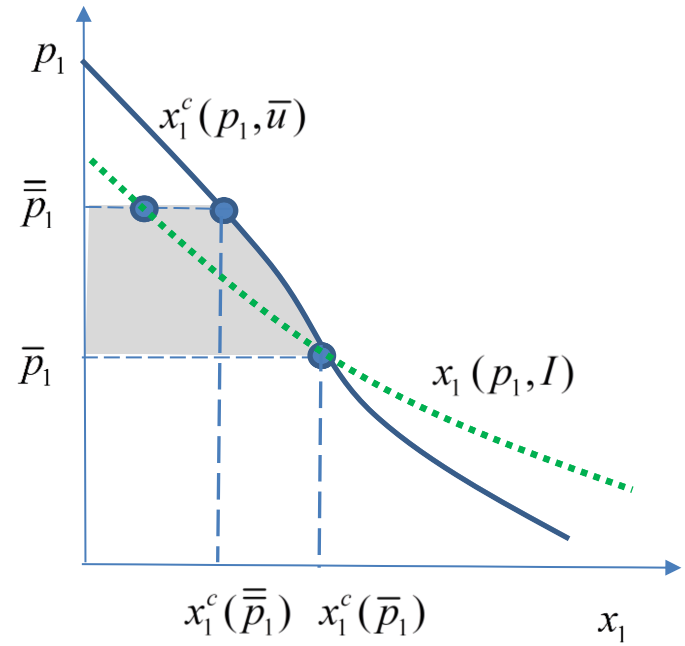
Thus the compensated demand price function is steeper.



Estimating the compensating variation

Note that the area to the left of the compensated demand curve is greater than the area to the left of the green ordinary demand price function for a normal good.

However, as Robert Willig showed, for most such calculations, the difference between the two areas is small.



In practice economists typically approximate the compensating variation by measuring the area to the left of the ordinary demand price function.

Remark on the use of survey data

Technical Notes (not examinable)

1. Equivalent variation in income (EV)

At the new higher price the consumer is worse off.

How much would he be willing to pay to have the initial price restored?

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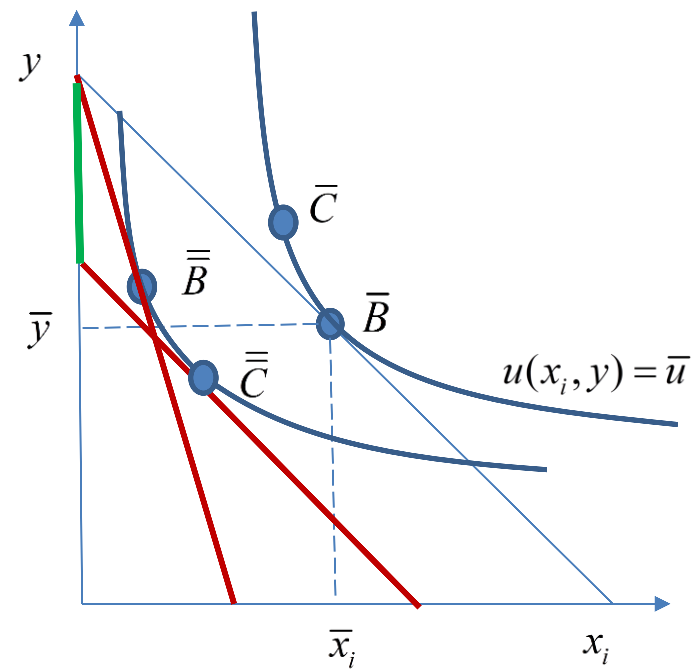


Equivalent variation in income (EV)

At the new higher price the consumer is worse off.

How much would he be willing to pay to have the initial price restored?

Arguing as above, he would be willing to pay.

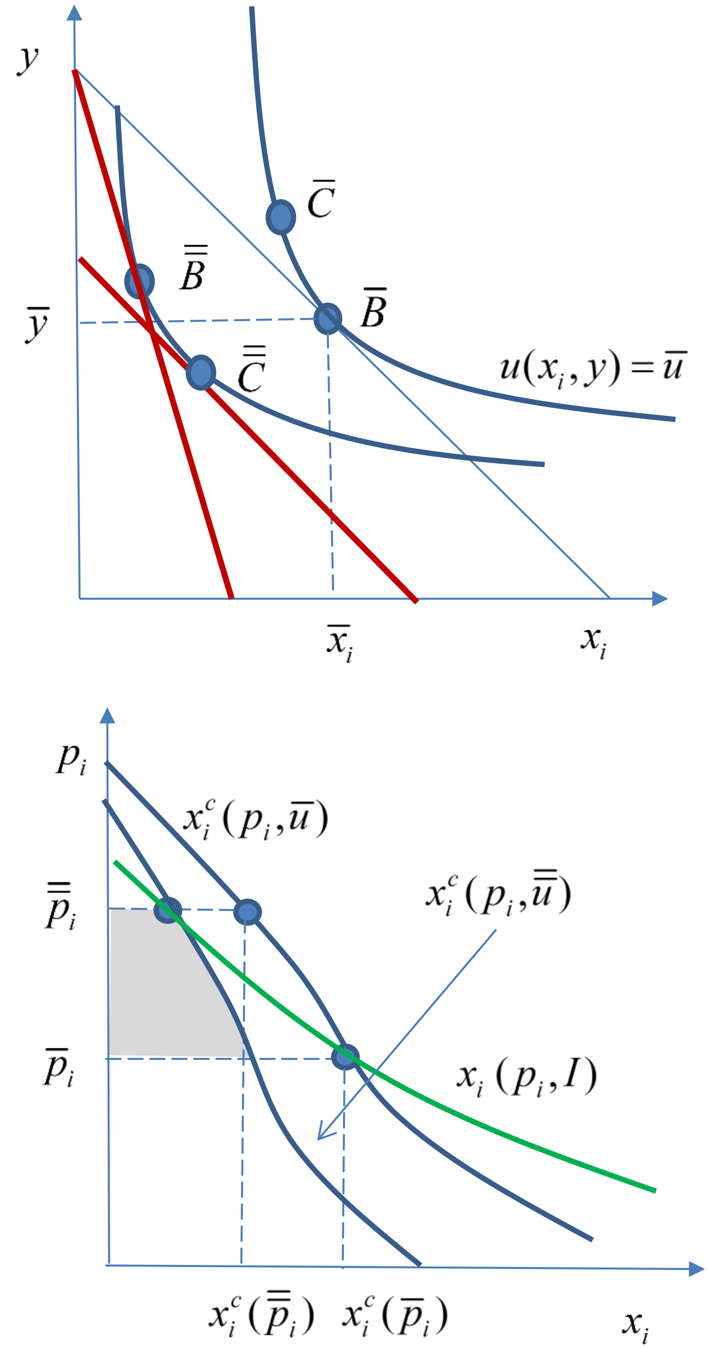
$$EV = I - M(\bar{p}_i, \bar{\bar{u}}) \text{ where } \bar{\bar{u}} = u(\bar{\bar{x}}_i, \bar{\bar{y}})$$

$$= M(\bar{\bar{p}}_i, \bar{\bar{u}}) - M(\bar{p}_i, \bar{\bar{u}}))$$

$$= \int_{\bar{p}_i}^{\bar{\bar{p}}_i} \frac{\partial M}{\partial p} dp = \int_{\bar{p}_i}^{\bar{\bar{p}}_i} x_i^c(p, \bar{\bar{u}}))dp_i$$

This differs from the compensating variation.

However, as noted above, in practice economists compute the area to the left of the green "ordinary" demand curve.



2. The mathematics of substitution and income effects

Definition: The elasticity of substitution for commodity i is the compensated elasticity of the ratio $\frac{y^c}{x_i^c}$ with respect to p_i.

$$\sigma_i = \varepsilon(\frac{y}{x_i}, p_i)$$

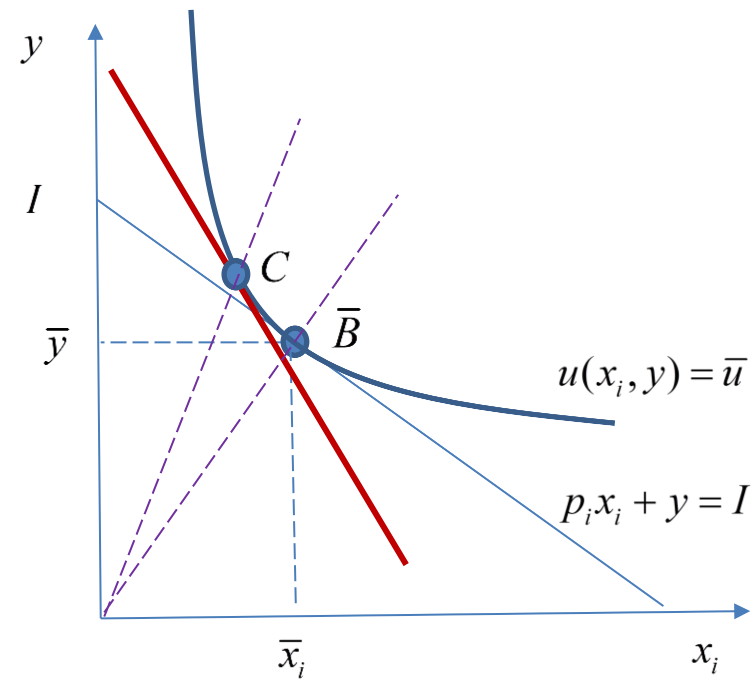
Around the level set as p_i increases

$$\frac{du}{dp_i} = \frac{\partial u}{\partial x_i}\frac{\partial x_i^c}{\partial p_i} + \frac{\partial u}{\partial y}\frac{\partial y^c}{\partial p_i}$$

$$= (\frac{1}{p_i}\frac{\partial u}{\partial x_i}) p_i \frac{\partial x_i^c}{\partial p_i} + (\frac{\partial u}{\partial y})\frac{\partial y^c}{\partial p_i} = 0 .$$

The consumer equates the marginal utility per dollar. It follows that

$$p_i \frac{\partial x_i^c}{\partial p_i} + \frac{\partial y^c}{\partial p_i} = 0.$$



Elasticity of substitution

In the last slide we showed that

$$p_i \frac{\partial x_i^c}{\partial p_i} + \frac{\partial y^c}{\partial p_i} = 0$$

We convert this equation into elasticities,

$$x_i\left(\frac{p_i}{x_i}\frac{\partial x_i^c}{\partial p_i}\right) + \frac{y}{p_i}\left(\frac{p_i}{y}\frac{\partial y^c}{\partial p_i}\right) = x_i \mathcal{E}(x_i^c, p_i) + \frac{y}{p_i}\mathcal{E}(y^c, p_i) = 0.$$

**

In the last slide we showed that

$$p_i \frac{\partial x_i^c}{\partial p_i} + \frac{\partial y^c}{\partial p_i} = 0$$

We convert this equation into elasticities,

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Multiply both terms by $\dfrac{p_i}{I}$.

$$\frac{p_i x_i}{I}\mathcal{E}(x_i^c, p_i) + \frac{y}{I}\mathcal{E}(y^c, p_i) = 0$$

*

In the last slide we showed that

$$p_i \frac{\partial x_i^c}{\partial p_i} + \frac{\partial y^c}{\partial p_i} = 0$$

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Multiply both terms by $\frac{p_i}{I}$.

$$\frac{p_i x_i}{I}\mathcal{E}(x_i^c, p_i) + \frac{y}{I}\mathcal{E}(y^c, p_i) = 0$$

Define the expenditure share $k_i = \frac{p_i x_i}{p \cdot x}$. Then $1 - k_i = 1 - \frac{p_i x_i}{I} = \frac{y}{I}$

Therefore

$$k_i\mathcal{E}(x_i^c, p_i) + (1 - k_i)\mathcal{E}(y^c, p_i) = 0$$

In the last slide we showed that

$$k_i\mathcal{E}(x_i^c, p_i) + (1-k_i)\mathcal{E}(y^c, p_i) = 0 \ .$$

Add $(1-k_i)\mathcal{E}(x_i^c, p_i)$ to the first term and subtract it from the second.

Then

$$\mathcal{E}(x_i^c, p_i) + (1-k_i)[\mathcal{E}(y, p_i) - \mathcal{E}(x_i^c, p_i)] = 0 .$$

In the last slide we showed that

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Add $(1-k_i)\mathcal{E}(x_i^c, p_i)$ to the first term and subtract it from the second.

Then

$$\mathcal{E}(x_i^c, p_i) + (1-k_i)[\mathcal{E}(y, p_i) - \mathcal{E}(x_i^c, p_i)] = 0 .$$

The term in brackets is the elasticity of substitution, $\sigma_i = \mathcal{E}(\frac{y^c}{x_i^c}, p_i) = \mathcal{E}(y^c, p_i) - \mathcal{E}(x_i^c, p_i)$.

Therefore

$$\mathcal{E}(x_i^c, p_i) + (1-k_i)\sigma_i = 0$$

Proposition: Price elasticity of compensated demand

The own price elasticity of compensated demand is

$$\mathcal{E}(x_i^c, p_i) = -(1-k_i)\sigma_i .$$

Decomposition of the own price elasticity of demand

Let $x_i(p_i, I)$ he the consumer's uncompensated demand for commodity i. In section E we defined $M(p_i, \bar{u}))$ to be the income the consumer would need to maintain a constant utility.

Then the consumer's compensated demand for commodity i is

$$x_i^c = x_i(p_i, M(p_i, \bar{u})) \ .$$

Differentiating by p_i

$$\frac{\partial x_i^c}{\partial p_i} = \frac{\partial x_i}{\partial p_i} + \frac{\partial x_i}{\partial I}\frac{\partial M}{\partial p_i} \ .$$

*

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Differentiating by p_i

$$\frac{\partial x_i^c}{\partial p_i} = \frac{\partial x_i}{\partial p_i} + \frac{\partial x_i}{\partial I}\frac{\partial M}{\partial p_i}\ .$$

Appealing to the Envelope Theorem, $\dfrac{\partial M}{\partial p_i} = x_i$.

We therefore have the following result

Slutsky equation

$$\frac{\partial x_i^c}{\partial p_i} = \frac{\partial x_i}{\partial p_i} + x_i\frac{\partial x_i}{\partial I}\ .$$

Rewrite the Slutsky equation as follows:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^c}{\partial p_i} - x_i \frac{\partial x_i}{\partial I}$$

Converting into elasticities,

$$\frac{p_i}{x_i}\frac{\partial x_i}{\partial p_i} = \frac{p_i}{x_i}\frac{\partial x_i^c}{\partial p_i} - p_i x_i \frac{1}{p_i}\frac{\partial x_i}{\partial I} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\left(\frac{I}{p_1}\frac{\partial x_1}{\partial I}\right).$$

Therefore

$$\mathcal{E}(x_i, p_i) = \mathcal{E}(x_i^c, p_i) - k_i \mathcal{E}(x_i, I)$$

Appealing to our earlier result, we have the following proposition.

Proposition: Decomposition of the own price elasticity of demand

$$\mathcal{E}(x_i, p_i) = -((1-k_i)\sigma_i + k_i \mathcal{E}(x_i, I))$$

The own price elasticity of demand is a convex combination of the elasticity of substitution and the income elasticity of demand.

Remark: If the fraction spent on commodity 1 is small, then the own price elasticity is approximately equal to $-\sigma_i$.

3. Derived utility function and convex superlevel sets

Proposition: Convexity of the derived utility function

Define $r=(p_2,...,p_n)$ and $z=(x_2,...,x_n)$. If $U(x)$ is strictly increasing and the superlevel sets of $U(x)$ are convex then the superlevel sets of

$$u(x_1,y)\equiv \underset{z}{Max}\{U(x_1,z)\,|\,r\cdot z\le y\}$$

are also convex.

Proof:

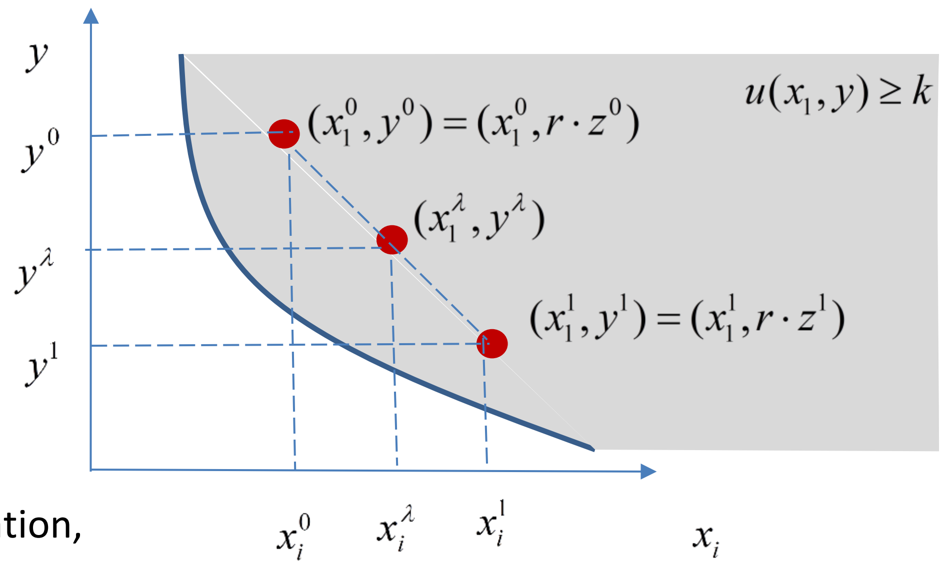
Suppose that (x_i^0,y^0) and (x_i^1,y^1)

are, as depicted, in the superlevel set

$S=\{(x_i,y)\,|\,u(x_i,y)\ge\bar{u}\}$.

i.e.

$$u(x_i^0,y^0)\ge\bar{u} \text{ and } u(x_i^1,y^1)\ge\bar{u} \quad (*)$$



We need to show that for any convex combination,

(x_i^λ,y^λ), $u(x_i^\lambda,y^\lambda)>\bar{u}$.

From (*)

(i) for some z^0 costing y^0, $U(x_1^0, z^0) \geq \bar{u}$ and (ii) for some z^1 costing y^1, $U(x_1^1, z^1) \geq \bar{u}$

Define the convex combinations $x_1^\lambda = (1-\lambda)x_1^0 + \lambda x_1^1$ and $z^\lambda = (1-\lambda)z^0 + \lambda z^1$

By hypothesis the superlevel sets of U are convex so $U(x_1^\lambda, z^\lambda) \geq \bar{u}$.

It remain to show that z^λ is feasible with income $y^\lambda = (1-\lambda)y^0 + \lambda y^1$.

z^0 costs y^0 so $(1-\lambda)z^0$ costs $(1-\lambda)y^0$

z^1 costs y^1 so λz^1 costs λy^1

Then $z^\lambda = (1-\lambda)z^0 + \lambda z^1$ costs $y^\lambda = (1-\lambda)y^0 + \lambda y^1$.

So z^λ is feasible.

QED