Homework 3 Due Tuesday, Nov 15

1. Portfolio choice

An investor with wealth of 96 has a choice of two risky assets. Asset a has a return of $z^a = (4,3)$. That is, in state 1 its value is 4 and in state 2 its value is 3. Asset b has return of $z^b = (12,3)$. The probability of state 1 is 3/8. The price of a claim to wealth in state s is p_s. Therefore the market value of asset a with return z^a is $P_a = p \cdot z^a$ and the market value of asset b is $P_b = p \cdot z^b$.

Suppose that the market value of asset a is $P^a = 16$ and the market value of asset b is $P_b = 24$. The investor's utility function is $u(x) = \ln x$.

(a) Explain why the price of a claim to state 1 must be $p_1 = 1$ and solve for the equilibrium price of a claim to state 2.

(b) Using the state claims prices, solve for the investor's optimal vector of state claims $\bar{x}$.

(c) The portfolio constraint is

$$P_a q_a + P_b q_b = 16 q_a + 24 q_b \leq 96 .$$

What portfolio results in a state claims vector of $\bar{x}$?

(d) Suppose that $u(x) = x^{1/2}$. What is the new optimum $\bar{\bar{x}}$?

(e) Show that the absolute value of $\bar{\bar{x}}_1 - \bar{\bar{x}}_2$ is bigger than the absolute value of $\bar{x}_1 - \bar{x}_2$ so the consumer is choosing a more risky outcome. Why is this?

2. Betting on "The Game".

In state 1 the Bruins win. In state 2 the Trojans win. Bev (a Bruin) and Tommy (a Trojan) think that the odds of a Bruin victory are $\dfrac{\pi_1}{\pi_2}$. Bev has a wealth of w^B and Tommy has a wealth of w^T. If Trojans lose they feel sad. The sadness is equivalent to a wealth loss of t. Therefore, if Tommy does not bet he has an expected utility of

$$U(w^T - t, w^T, \pi) = \pi_1 u_T(w^T - t) + \pi_2 u_T(w^T).$$

If the Bruins win the sadness for Bev is equivalent to a wealth loss of b. Therefore, if she does not bet, her expected utility is

$$U_B(w^B, w^B - b, \pi) = \pi_1 u_B(w^B) + \pi_2 u_B(w^B - b).$$

(a) Explain why there are bets that are strictly Pareto preferred.

(b) Depict the Pareto preferred bets in an Edgeworth Box diagram if $b=t$

(c) Also depict the Pareto Efficient allocations for $b>t$ if $u_B(x)=u_T(x)=\ln x$

(d) If $b=t$ what is the equilibrium price ratio? Who be*s on a Bruin's victory?

(e) Suppose that the utility function $u_B(x)=u_T(x)=(1-1/R)x^{1-1/R}, R>0, R\neq 1$. Show that the PE allocations are on the line joining the two corners of the box. Suppose that $b>t$. How does the equilibrium price ratio (the market odds) vary as R increases?

3. Two Asset economy

Asset a has a return $z^a=(1,27,56)$. Asset b has a return $z^b=(2,3,4)$. Thus the total **return** in the three states is $\omega=(3,30,60)$. There are H consumers, each with some initial share of each asset. Each consumer has the same utility function $u(x)=\ln x$ and the three states are equally likely.

(a) Solve for the equilibrium state claims prices if $p_1=1$

(b) Hence show that asset a has a higher market value than asset b.

Suppose henceforth that $u(x)=(1-1/R)x^{1-1/R}, R>0, R\neq 1$.

(c) Explain why each individual has the same homothetic preferences using the following general definition:

$U(x,\pi)\geq U(y,\pi)$ implies that $U(\theta x,\pi)\geq U(\theta y,\pi)$ for all $\theta>0$.

(d) Solve for the equilibrium state claims prices if $p_1=1$. Hence show that if R is sufficiently high, then asset b has a higher market value. What is the ratio of market values in the limit as $R\to\infty$.

(e) How much do consumers gain from trade in the limit?

(f) Consider the two consumer case. In the state claims equilibrium, if the value of Alex's assets is a fraction k of the total value, expaln why his final holding's of state claims will be $\bar{x}=k\omega$.

(g) Suppose that the three state claims markets are closed and instead the two consumers trade only in the two asset markets. If the asset prices are the same as in the state claim's market equilibrium, is the outcome from the state claims equilibrium still feasible?

(h) Is it the equilibrium outcome?

Remarks on question 1

The direct way of analyzing this question is to maximize the expected utility of a portfolio $u(q_a, q_b)$ subect to the portfolio constraint, i.e. the cost of the portfolio cannot exceed person wealth

$$P \cdot q = P_a q_a + P_b q_b \leq W \ .$$

But what is the expected utility of a portfolio? Each asset has a payoff depending upon some exogenous event like a who wins the election. If you hold a lot of asset b you are happy if you end up in state 1 (Trump wins) If you hold a lot of asset a you are not greatly affected by the outcome of the election. You then choose a portfolio based on your beliefs, the asset prices and risk aversion.

Writing the asset returns as column vectors, the portfolio return is

$$\begin{bmatrix} x_1(q) \\ x_2(q) \end{bmatrix} = q_a \begin{bmatrix} z_1^a \\ z_2^a \end{bmatrix} + q_b \begin{bmatrix} z_1^a \\ z_2^a \end{bmatrix} = \begin{bmatrix} q_a z_1^a + q_b z_1^b \\ q_a z_2^a + q_b z_2^b \end{bmatrix}$$

The top row is the return in state 1 and the second row is the return in state 2. The consumer then ends with $x_1(q)$ in state 1 and $x_2(q)$ in state 2. His expected utility is therefore a function of the consumer's portfolio

$$u(q) \equiv U(x(q);\pi) = \pi_1 u(x_1(q)) + \pi_2 u(x_2(q))$$

If you know the uitility fucntion $u(x)$ you can easily solve this problem numerically. How to do so is discussed below. But it is not easy to solve the problem analytically.

There is another way.

Setp 1: Create two synthetic assets (portfolios of the two assets) that payoff 1 unit in one state and 0 in the other and compute that market value of each of these synthetic assets, p_1 and p_2 . You are going to purchases quantities of these two sythetic assets. (You might think of each as a special mutual fund.) If you buy x_1 units of the first asset and x_2 of the second asset your budget constraint is

$$p_1 x_1 + p_2 x_2 \leq W$$

and your final return is x_1 in state 1 and x_2 in state 2. Expected utility is therefore

$$U(x,\pi) = \pi_1 u(x_1) + \pi_2 u(x_2)$$

so you solve a standard utility maximization problem.

But how do you know the price of the synthetic assets?

The two assets have a positive return in both eventualities (states). But a portfolio of $(q_a, q_b) = (-1,1)$ has a return of 8 units in state 1 and 0 in state 2 and its value is $P_b - P_a =$ 24 – 16 =8

Thus the portfolio of $(q_a, q_b) = (-\frac{1}{8}, \frac{1}{8})$ has a return of (1,0) That is, it pays off 1 unit in state 1 and nothing in state 2. If you own x1 units of this portfolio you get x1 units in state 1.

In (a) you need to also create a second portfolio that pays off 1 unit in state 2 and costs p2.

Numerical solution using Solver in Excel

Step 1: Enter all the data and the varibles in a spread-sheet (see below).

Data are colored yellow. The changing cells are green.

Step 2: Use Solver to compute the portfolio return in each state, the utility in each state, cost of the portfolio. Finally, enter some probbaility of state 1 and compure the resultuing expected utility.

In the figure the formula for the state 1 return can be seen in the formula bar at the top. All the cells with a formula have the "Calculation" style on the TAB.

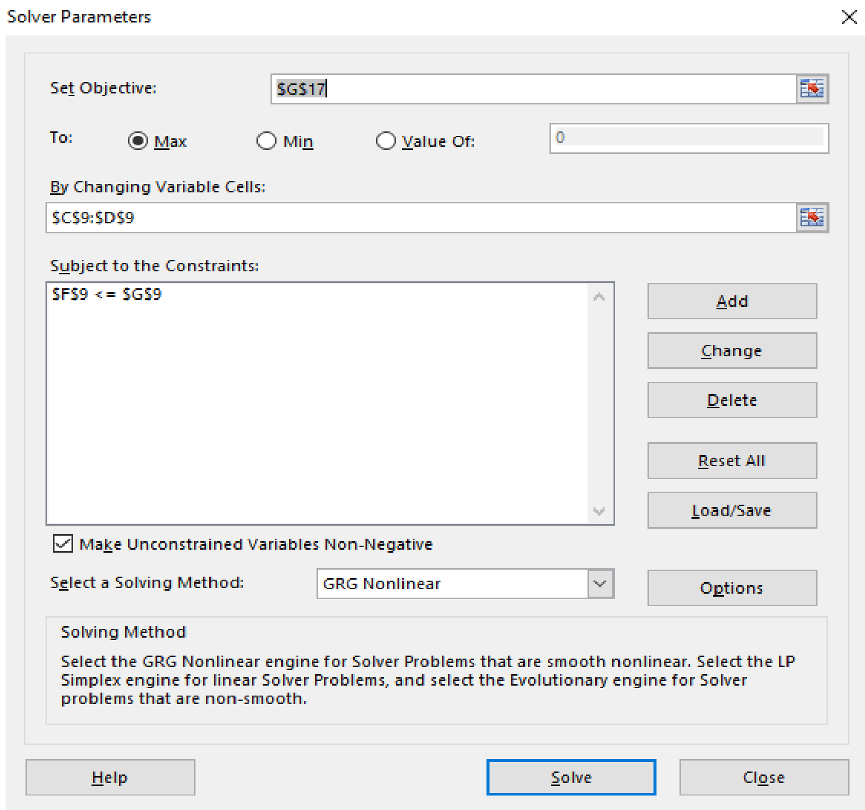
F13 =SUMPRODUCT(C$9:D$9,C13:D13)

	A	B	C	D	E	F	G	H
1	**Case (i)**		**u(x)=lnx**					
2								
3		wealth	96					
4								
5		assets	a	b				
6		asset prices	Pa	Pb				
7			56	40		portfolio	Wealth	
8		asset quantities	qa	qb		cost	w	
9			1	1		96	96.00	
10						portfolio		
11						return	utility	probabilities
12		asset returns	za	zb		x	u(x)	π
13		state 1	40	8		48	3.87	0.5
14		state 2	8	16		24	3.18	0.5
15								
16						expected utility		
17						U(x,π)	3.52	
18								

Step 3: From the DATA tab select Solver.

Complete as shown. You are maximizing expected utility (Cell G17) subject to the budget constraint

F9 <= G9 using the two changing variaible cells range[C9:D9]


Solver Parameters
Set Objective:
$G$17
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$C$9:$D$9
Subject to the Constraints:
$F$9 <= $G$9
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
GRG Nonlinear
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
Close