Homework 3 Answers

1. Portfolio choice

With state claims price vector p the value of asset a is 16 and the value of asset b is 24.

We seek a portfolio (a synthetic asset) that has a return on (1,0)

$$q_a\begin{bmatrix}4\\3\end{bmatrix}+q_b\begin{bmatrix}12\\3\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

Note that $(q_a,q_a)=(-1,1)$ has a return of $(8,0)$ and so $(q_a,q_a)=(-\frac{1}{8},\frac{1}{8})$ has a return of $(1,0)$.
Or

$$q_A 4+q_B 12=1 \text{ and } q_a 3+q_B 3=0 \text{ .}$$

The solution is $(q_a,q_a)=(-\frac{1}{8},\frac{1}{8})$

The market value of this portfolio is $P_a q_a+P_b q_b=-16\times\frac{1}{8}+24\times\frac{1}{8}=1$. Therefore $p_1=1$.

An almost identical argument shows that $p_2=4$.

(b) The budget constraint is

$$p\cdot x=x_1+4x_2=W=96$$

$$U(x,\pi)=\tfrac{3}{8}\ln x_1+\tfrac{5}{8}\ln x_2 \text{ .}$$

Therefore the investor solves

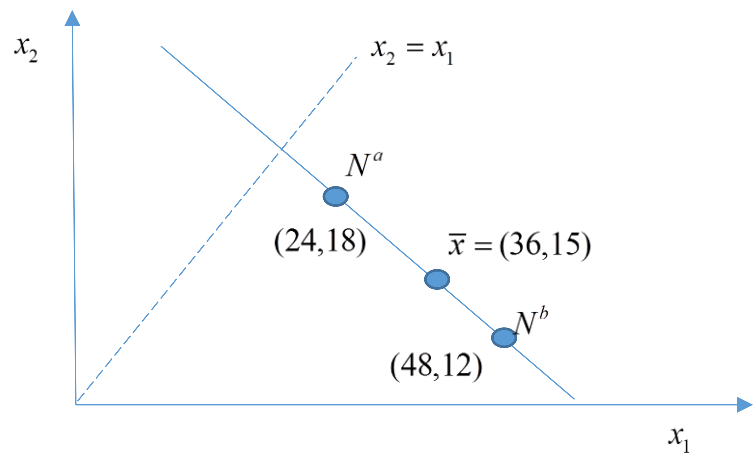
$$\underset{x}{Max}\{U(x,\pi)=\tfrac{3}{8}\ln x_1+\tfrac{5}{8}\ln x_2 \mid p\cdot x\le 96\}$$

FOC

$$\frac{\frac{3}{8}}{p_1x_1}=\frac{\frac{3}{8}}{x_1}=\frac{\frac{5}{8}}{4x_2}=\frac{1}{96} \text{ .}$$

Solving, $\bar{x}=(36,15)$.

(c) With a wealth of 96 the investor can purchase 6 units of asset a with a payoff of $6z^a=(24,18)$ or 4 units of asset b with a payoff of $4z^b=(48,12)$. These two non-diversified portfolios are depicted below.



Note that $\bar{x}$ is the mid point. More generally, from the portfolio constraint,

$$q_A = 6 - \tfrac{3}{2} q_B \ .$$

With this portfolio, the investor's state 1 return is

$$4q_a + 12q_b = 12q_b + 4(6 - \tfrac{3}{2} q_b) = 24 + 6q_b = 36 \ .$$

Therefore $q_b = 2$ and so $q_A = 3$.

The new maximization problem is

$$\underset{x}{Max}\{U(x,\pi) = \tfrac{3}{8} x_1^{1/2} + \tfrac{5}{8} x_2^{1/2} \mid p \cdot x \le 96\}$$

FOC

$$\frac{\frac{3}{8}}{x_1^{1/2}} = \frac{\frac{5}{8}}{4x_2^{1/2}} \quad \text{i.e.} \quad \frac{9}{x_1} = \frac{25}{16x_2} \quad \text{i.e.} \quad \frac{36}{x_1} = \frac{25}{4x_2} = \frac{61}{96}$$

Then

$$x_1 = 36\frac{96}{61} > 36 \ \text{ and } \ x_2 = 24\frac{25}{61} < 24$$

So the new optimum is further from the certainty line.

(d) If you check the degree of relative risk aversion is now ½, where as it was originally 1. A less risk averse investor is less averse to taking a more risky position.

2. Betting on the game

(a) $MRS(x)=\dfrac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$.

Then

$$MRS_B(w^b, w^b - b)=\frac{\pi_1 u'(w^b)}{\pi_2 u'(w^b - b)}<\frac{\pi_1}{\pi_2} \text{ and } MRS_T(w^t - t, w^t)=\frac{\pi_1 u'(w^t - t)}{\pi_2 u'(w^t)}>\frac{\pi_1}{\pi_2}$$

Therefore on the margin Tommy is willing to give up more claims to state 2 in return for more claims to state 1.

In state 1 the Bruins win and the Trojans are sad so total wealth is $\omega_1 = w^A + w^B - t$. In state 2 the Bruins are sad so total wealth is $\omega_2 = w^A + w^B - b$

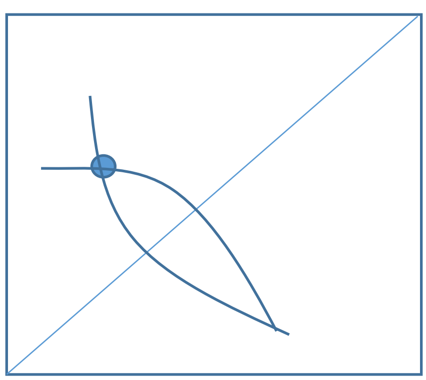
Therefore the Edgeworth Box is square if $b = t$. For a PE allocation

$$MRS_B = \frac{\pi_1}{\pi_2}\frac{x_1^a}{x_1^a} = \frac{\pi_1}{\pi_2}\frac{x_1^t}{x_1^t} = MRS_T \ .$$

Therefore

$$\frac{x_2^a}{x_1^a} = \frac{x_2^t}{x_1^t} = \text{by the Ratio Rule} = \frac{x_2^a + x_2^t}{x_1^a + x_1^b} = \frac{w^a + w^b - b}{w^a + w^b - t} = 1$$

So the PE allocations are on the diagonal.



The Pareto preferred bets are in the lens shaped region.

(b) If $b > t$

$$\frac{x_2^a}{x_1^a} = \frac{x_2^t}{x_1^t} = \frac{x_2^a + x_2^t}{x_1^a + x_1^b} = \frac{w^a + w^b - b}{w^a + w^b - t} < 1$$

The PE allocations are still on the diagaonal.

(c) From (b)

$$MRS_B = MRS_T = \frac{\pi_1}{\pi_2}\frac{\omega_2}{\omega_1} = \frac{\pi_1}{\pi_2}$$

So the price ratio os equal to the odds.

(d) From (c)

$$MRS_B = MRS_T = \frac{\pi_1}{\pi_2}\frac{\omega_2}{\omega_1} < \frac{\pi_1}{\pi_2}$$

So the price ratio is less then the odds.

As noted in (a), starting from his endowment point Tommy is willing to pay more for an increase in his claims to state 1 so is the net buyer of state 1 claims. Thus Tommy bets on the Bruins!

Arguing as above

$$\frac{p_1}{p_2} = MRS_B = \frac{\pi_1}{\pi_2}(\frac{x_2^a}{x_1^a})^{-R} = \frac{\pi_1}{\pi_2}(\frac{x_2^t}{x_1^t})^{-R} = MRS_T$$

And

$$\frac{x_2^a}{x_1^a} = \frac{x_2^t}{x_1^t} = \frac{w^a + w^b - b}{w^a + w^b - b} \ .$$

Therefore

$$\frac{p_1}{p_2} = MRS_B = MRS_T = \frac{\pi_1}{\pi_2}(\frac{\omega_2}{\omega_1})^{-R}$$

The bigger is R the more the price ratio deviates from the odds.

3. Two asset economy

(a) Consumer h chooses $\overline{x}^h$ to solve $Max\{\frac{1}{3}\ln x_1+\frac{1}{3}\ln x_2+\frac{1}{3}\ln x_3 \mid p\cdot x\le p\cdot\omega^h\}$

FOC

$$\frac{\frac{1}{3}}{p_1\overline{x}_i^h}=\frac{\frac{1}{3}}{p_2\overline{x}_2^h}=\frac{\frac{1}{3}}{p_2\overline{x}_3^h}=\frac{1}{p\cdot\overline{x}^h}=\frac{1}{p\cdot\omega^h} .$$

Therefore

$$p_1\overline{x}_1^h=\tfrac{1}{3}p\cdot\omega^h$$

Summing ove consumers, aggregate demand is

$$p_1\overline{x}_1=\sum_{h=1}^{H} p_1\overline{x}^h=\tfrac{1}{3}p\cdot\omega .$$

For equilibrium supply = demand so $\overline{x}_1=\omega_1$. Then

$$p_1\omega_1=\tfrac{1}{3}p\cdot\omega .$$

Arguing symmetrically,

$p_2\omega_2=\tfrac{1}{3}p\cdot\omega$ and $p_3\omega_3=\tfrac{1}{3}p\cdot\omega$.

Therefore

$$p_1\omega_1=p_2\omega_2=p_3\omega_3$$

And so

$$(p_1,p_2,p_3)=(1,\frac{\omega_1}{\omega_2},\frac{\omega_1}{\omega_3})=(1,\frac{1}{10},\frac{1}{20}) .$$

A quicker way to proceed is to note that the utility function is homothetic. To confirm this, note that

$$U(\theta x)=\tfrac{1}{3}(\ln\theta x_1+\ln\theta x_2+\ln\theta x_3)=\ln\theta+\tfrac{1}{3}(\ln x_1+\ln x_2+\ln x_3)=\ln\theta+U(x)$$

Therefore $U(y)\ge U(x)$ implies that $U(\theta y)\ge U(\theta x)$.

We can then solve for the WE using the representative consumer. His wealth is ω and his equilibrium consumption $\overline{x}$ must also be ω since she has noo ne to trade with.

From the FOC

$$\frac{\frac{1}{3}}{p_1\bar{x}_i^h} = \frac{\frac{1}{3}}{p_2\bar{x}_2^h} = \frac{\frac{1}{3}}{p_2\bar{x}_3^h} \ .$$

Setting $\bar{x} = \omega$

$$\frac{\frac{1}{3}}{p_1\omega_1} = \frac{\frac{1}{3}}{p_2\omega_2} = \frac{\frac{1}{3}}{p_3\omega_3} = \frac{1}{p \cdot \omega}$$

(b) Using the WE prices, it is easy to confirm that asset 1 is more valuable.

(c) Derivation follws the discussion above.

(d) FOC

$$\frac{\frac{1}{3}}{p_1(\bar{x}_i^h)^{-R}} = \frac{\frac{1}{3}}{p_2(\bar{x}_2^h)^{-R}} = \frac{\frac{1}{3}}{p_2(\bar{x}_3^h)^{-R}} \ .$$

For the representative agent $\bar{x} = \omega$.

Therefore

$$\frac{\frac{1}{3}}{p_1(3)^{-R}} = \frac{\frac{1}{3}}{p_2(30)^{-R}} = \frac{\frac{1}{3}}{p_2(60)^{-R}} \ .$$

It follows that for a WE

$$p = (1, (\tfrac{1}{10})^R, (\tfrac{1}{20})^R)$$

As R grows large the state 2 and state 3 pries approach zero so in the limit.

(e) It is therefore possible to buy only a tiny amount of state 1 claims using claims to the other states.

So in the limit there are no gains from exchange.

(f) For a homothetic function if the MRS are equal then the consumption ratios are equal and so equal to the aggregate endowment ratios.

(g) Alex chooses a portfolio of $k\omega$. Asset a and asset b together have a return of ω . Therefore holding a fraction k of this “market portfolio yields exactly this return. Similarly Bev wishes to hold a fraction 1-k of the market portfolio.

Thus Alex and Bev can achieve this outcome by trading in the asset markets.

(h) The set of feasible trades is smaller with asset markets then with the 3 state claims markets so this must be a WE.