Homework 4 Answers

1. WE in an economy with constant rturns to scale and identical homothetic preferences

(a) $$U(q_a,q_b)=\ln q_a+\ln q_b=\ln(z_1^a)^{1/3}(z_2^a)^{2/3}+\ \ln(z_1^b)^{2/3}(z_2^b)^{1/3}$$

$$=\tfrac{1}{3}\ln z_1^a+\tfrac{2}{3}\ln z_2^a+\tfrac{2}{3}\ln z_1^b+\tfrac{1}{3}\ln z_2^b$$

Choose $\bar{z}^a,\bar{z}^b$ to solve $Max\{U(q_a,q_b)\,|\,z_1^a+z_1^b\le 1,\ \ z_2^a+z_2^b\le 64\}$

i.e.

$$Max\{\tfrac{1}{3}\ln z_1^a+\tfrac{2}{3}\ln z_2^a+\tfrac{2}{3}\ln z_1^b+\tfrac{1}{3}\ln z_2^b\,|\,z_1^a+z_1^b\le 1,\ \ z_2^a+z_2^b\le 64\}$$

$$L=\tfrac{1}{3}\ln z_1^a+\tfrac{2}{3}\ln z_2^a+\tfrac{2}{3}\ln z_1^b+\tfrac{1}{3}\ln z_2^b+\mu_1(1-z_1^a-z_1^b)+\mu_2(64-z_2^a-z_2^b)$$

FOC (after some work)

$$\frac{\frac{1}{3}}{z_1^a}=\frac{\frac{2}{3}}{z_1^b}=\frac{1}{z_1^a+z_1^b}=\frac{1}{1}\ ,\qquad \frac{\frac{2}{3}}{z_2^a}=\frac{\frac{1}{3}}{z_2^b}=\frac{1}{z_2^a+z_2^b}=\frac{1}{64}\ .$$

Therefore

$$(\bar{z}_1^a,\bar{z}_2^a)=(\tfrac{1}{3},\frac{2\times 64}{3})\ ,\ \ (\bar{z}_1^b,\bar{z}_2^b)=(\tfrac{2}{3},\frac{64}{3})$$

(b) A necessary condition for profit-maximization is cost minimization

$\bar{z}^a$ solves $\underset{z}{Min}\{r\cdot z\,|\,F_a(z)\ge \bar{q}_a\}$.

After some work the FOC implies that

$$\frac{r_1}{r_2}=MTRS_a(\bar{z}^a)=\frac{\frac{\partial F_a}{\partial z_1}}{\frac{\partial F_a}{\partial z_2}}=\tfrac{1}{2}\frac{z_2^a}{z_1^a}=64$$

(c) Solve for the maximizing outputs $(\bar{q}_a,\bar{q}_b)$ and show that their ratio is 4.

$$q_a=(z_1^a)^{1/3}(z_2^a)^{2/3}=(\tfrac{1}{3})^{1/3}(\frac{2\times 64}{3})^{2/3}=(2)^{2/3}\,\frac{64^{2/3}}{3}=\frac{2^{2/3}}{3}16$$

$$q_b=(z_1^b)^{2/3}(z_2^b)^{1/3}=(\tfrac{2}{3})^{2/3}(\frac{64}{3})^{1/3}=2^{2/3}\,\frac{4}{3}$$

Therefore $q_a=4q_b$.

(d) Maximization by the representative consumer.

FOC

$$\frac{p_a}{p_b} = MRS(\overline{q}) = \frac{\dfrac{\partial U}{\partial q_a}}{\dfrac{\partial U}{\partial q_b}} = \frac{\overline{q}_b}{\overline{q}_a} = \tfrac{1}{4} \qquad (*)$$

(e) Solve for the WE input prices.

For profit maximization

$$p_1 \frac{\partial F_a}{\partial z_1}(\overline{z}^a) - r_1 = 0 \quad \text{and} \quad p_1 \frac{\partial F_a}{\partial z_2}(\overline{z}^a) - r_2 = 0 .$$

Since $p_1 = \frac{1}{4}$ these determine the input prices.

Suppose that commodity a is produced in the north of the country and those who live in the north own the inputs used in its production. Commodity b is produced in the south of the country and those who live in the south own the inputs used in the south.

(f) In a CRS economy the equilibrium profit is zero so

$p_a\overline{q}_a = r\cdot\overline{z}^a$ and $p_b\overline{q}_b = r\cdot\overline{z}^b$

Appeakling to (*) in (d) $p_a\overline{q}_a = p_b\overline{q}_b$. Therefore $r\cdot\overline{z}^a = r\cdot\overline{z}^b$.

Since

$$r\cdot\overline{z}^a + r\cdot\overline{z}^b = r\cdot(\overline{z}^a + \overline{z}^b) = r\cdot\omega$$

It follows that

$$r\cdot\overline{z}^a = r\cdot\overline{z}^b = \tfrac{1}{2} r\cdot\omega$$

(g) The representative consumer spend have of his wealth on each commodity.

Therefore in each region half the wealth is spent on each commodity. Wealth in each region is $r\cdot\overline{z}^a = r\cdot\overline{z}^b = \frac{1}{2} r\cdot\omega$. Then imports in each region are worth $\frac{1}{4} r\cdot\omega$.

Bonus

(h) On the supply side, if inputs in each region are tripled to $\omega^S = (1,128)$ and $\omega^N = (2,64)$ then because of CRS, all the necessary conditions for profit maximization continue to hold if all prices are unchanged and output in both regions triples. On the demand side, the demand of the representative consumer triples since total wealth triples. So markets continue to clear.

(i) If the two regions become independent countries the economies are the same as in part (h) so the WE of (h) remains a WE.

2. Pricing game

(a) The profit of firm 1 is

$$\pi_1(p) = (p_1 - 4)q_1(p) = (p_1 - 4)(60 - 20p_1 + 10p_2)$$

The profit of firm 2 is

$$\pi_2(p) = (p_2 - 7)q_2(p) = (p_2 - 7)(150 + 10p_1 - 20p_2) \ .$$

For any p_2 , the best response by firm 1 is p_1^b that solves

$$\underset{p_1}{Max}\{\pi_1(p) = (p_1 - 4)(60 - 20p_1 + 10p_2)\}$$

The partial derivative of the profit function can be obtained by multiplying out all the terms and d=then taking the derivative or by applying the product rule.

$$\frac{\partial \pi_1}{\partial p_1} = (60 - 20p_1 + 10p_2) - 20(p_1 - 4) = 140 - 40p_1 + 10p_2 \ .$$

For a maximum this must be zero. Then

$$p_1^b = \tfrac{1}{4}(14 + p_2) \ .$$

$$\frac{\partial \pi_2}{\partial p_2} = (150 + 10p_1 - 20p_2) - 20(p_2 - 7) = 290 + 10p_1 - 40p_2 \, .$$

For a maximum this must be zero. Then

$$p_2^b = \tfrac{1}{4}(29 + p_1) \ .$$

(b) If both prices are mutual best responses then

$p_2^b = \tfrac{1}{4}(29 + p_1^b)$ and $p_1^b = \tfrac{1}{4}(14 + p_2^b)$.

Therefore

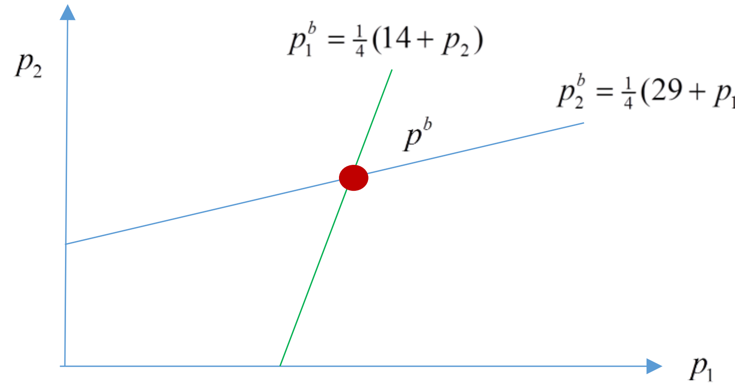
$$p_2^b = \tfrac{1}{4}(29 + \tfrac{1}{4}(14 + p_2^b)) = \frac{29}{4} + \frac{14}{16} + \frac{1}{16}p_2^b = \frac{130}{16} + \frac{1}{16}p_2^b \, .$$

Hence

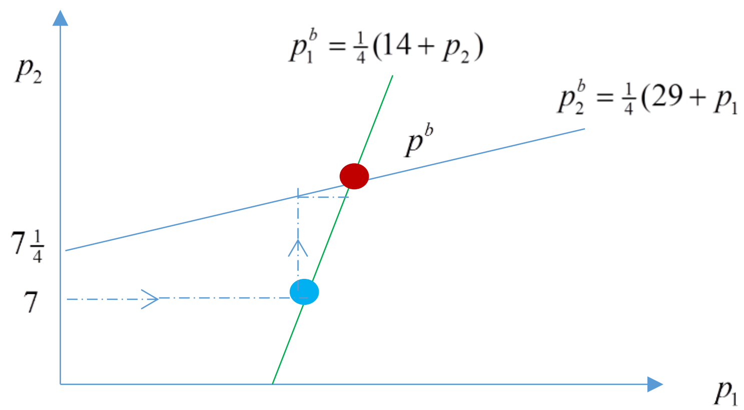
$15p_2^b = 130$ and so $p_2^b = \dfrac{26}{3}$.

$4p_1^b = 14 + p_2^b = 14 + \dfrac{26}{3} = \dfrac{68}{3}$ and so $p_1^b = \dfrac{17}{3}$

The two best response functions are depicted below.



(c) The initial price for p_2 is depicted below along with the best response by firm 1.



(d) If the figure is blown up it is clear that the Cournot adjustment process converges to the equilibrium point. At each step the responding price rises.

(e) From the slide, in the quantity setting game the equilibrium is $(q_1, q_2) = (30, 30)$. Substituting into the demand price functions, equilibrium prices are $(p_1, p_2) = (6, 9)$. These prices are both (slightly) higher.

3. Sealed high-bid auction with more than 2 buyers

(a) The bid function of buyer 2 or buyer 3 is $B(\theta_i) = k\theta_i$ so the maximum bid is $k\theta_{\max} = 10k$. This is bid with probability zero that buyer 1 wins with probaility 1 if he makes this bid. Thus he has no incentive to bid higher and paty more.

(b) If buyer 1 bids b the probaility that she wins is the probability that both of the other buyers bid lower. Buyer 2 bids lower if $B_2(\theta_2) \le b$, i.e. $k\theta_2 \le b$ i.e. $\theta_2 \le \dfrac{b}{k}$. The probability of this is $F(\dfrac{b}{k}) = \dfrac{b}{10k}$.

Similarly buyer 3 bids lower with probability $F(\frac{b}{k}) = \frac{b}{10k}$. The joint probabaility is the win probability. Therefore

$$W(b) = (\frac{b}{10k})^2 .$$

Buyer 1's expected payoff is therefore

$$u_1(\theta_1, b) = W(b)(\theta_1 - b) = \frac{b^2}{100k^2}(\theta_1 - b)$$

(c) $\frac{\partial u_1}{\partial b} = \frac{1}{100k^2}(2b\theta_1 - 3b^2)$.

If this is positive at $b = 10k$ buyer 1 bid 10k. If not, buyer 1 chooses b_1 so that

$$\frac{\partial u_1}{\partial b} = \frac{1}{100k^2}(2b\theta_1 - 3b^2) = 0$$

Then $B_1(\theta) = \frac{2}{3}\theta$.

Therefore the best response is $B_1(\theta) = Min\{\frac{2}{3}\theta, 10k\}$

(d) If $k = 2/3$ then $B_1(\theta) = \frac{2}{3}\theta$

That is, if the other two buyers use the bidding strategy $B_i(\theta_i) = \frac{2}{3}\theta_i$ then buyer 1's best reponse is to use this same strategy.
The same can be said for buyer 2 and buyer 3 so the bidding stratgies are mutual best respones/

(e) Suppose all the other buyers bid according to the strategy $B_i(\theta_i) = k\theta_i$. Arguing as above, If buyer 1 bids b the probability that buyer 2's bid is lower is $F(\frac{b}{k}) = \frac{b}{10k}$. The same can be said for buyers 2 and 3. Thus the win probability is

$$W(b) = (\frac{b}{10k})^3 .$$

Arguing as in (d) the equilibrium bidding stregy is $B_i(\theta_i) = \frac{3}{4}\theta_i$.

4. All pay auction (formerly called the sad loser auction)

(a) Suppose that buyer 2's bidding strategy is $B_2(\theta_2) = k\theta_2^2$. If buyer 1 bids b her win probability is

$$\Pr\{B_2(\theta_2) \le b\} = \Pr\{k\theta_2^{\ 2} \le b\} = \Pr\{\theta_2^{\ 2} \le \frac{b}{k}\} = \Pr\{\theta_2 \le (\frac{b}{k})^{1/2}\} = F((\frac{b}{k})^{1/2}) \ .$$

Thus the win probability is

$$W(b) = \frac{b^{1/2}}{100k^{1/2}} \ .$$

Buyer 1's expected payoff is therefore

$$u_1(\theta_1, b) = W(b)\theta_1 - b = \frac{b^{1/2}}{100k^{1/2}}\theta_1 - b$$

$$\frac{\partial u_1}{\partial b} = \frac{\theta_1 \frac{1}{2} b^{-1/2}}{100k^{1/2}} - 1) = 0 \quad \text{i.e.} \quad b^{1/2} = \frac{\theta_1}{k200} \ .$$

Thus buyer 1's best reponse is

$$B_1(\theta) = \frac{1}{k200^2}\theta_1^{\ 2} \ .$$

For a symmetric equilibrium $B_2(\theta) = k\theta^2$ and $B_1(\theta) = \frac{1}{k200^2}\theta^2$

must be the same. Therefore $k = \frac{1}{k(200)^2}$ and so $k^2 = \frac{1}{200^2}$ and so $k = \frac{1}{200}$

Repeating the argument for buyer 2, if buyer 1 bids according to $B_1(\theta) = \frac{1}{200}\theta^2$, then buyer 2's best

respsonse is $B_2(\theta) = \frac{1}{200}\theta^2$.

So these strategies are mutual best reponses.

A very similar argument shows that for some $\hat{k}$, $B_i(\theta_i) = \hat{k}\theta_i^3$ is the equilibrium strategy with three bidders.