33 pages

See also Calculus of Econ Modules 7 and 8.

Introduction

Four questions

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What do economists do?

Discuss in 3 person groups

2. Maximization

Suppose $\bar{x}=(\bar{x}_1,...,\bar{x}_n)$ solves $\underset{x}{Max}\{f(x_1,...,x_n)\}$.

Define $\bar{x}_{-1}=(\bar{x}_2,...,\bar{x}_n)$

Then $\bar{x}_1$ solves $\underset{x_1}{Max}\{f(x_1,\bar{x}_{-1})\}$.

This is a one variable problem.

The necessary condition for a maximum is the

zero slope condition $\dfrac{\partial f}{\partial x_1}(\bar{x}_1,\bar{x}_{-1})=0$

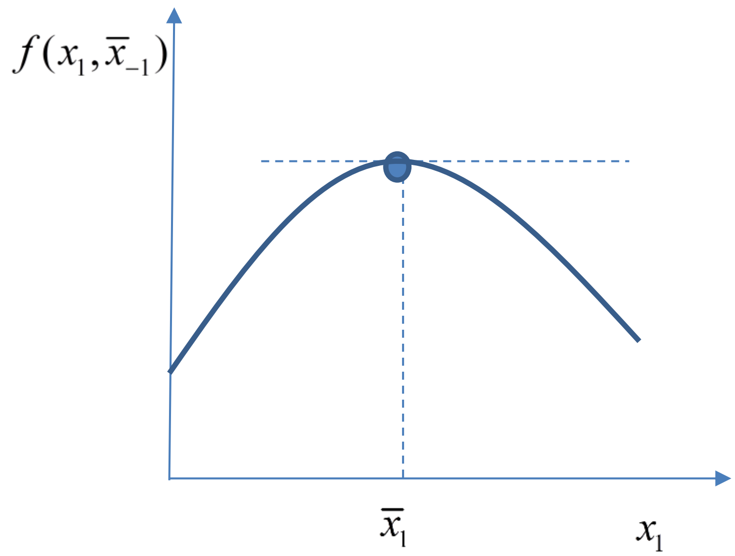


Fig.1 -Necessary condition for a maximum

Making an identical argument for each of the other variables we have the following result.

First order necessary conditions for a maximum

For $\bar{x}$ to be a maximizer the following conditions must hold

$$\frac{\partial f}{\partial x_j}(\bar{x})=0,\ \ j=1,...,n \qquad (2\text{-}1)$$

Consider Fig 2.

The first order necessary condition holds at $\bar{x}_1$.

If the slope is strictly increasing at $\bar{x}_1$, then the slope

is strictly negative to the left and strictly positive

to the right. Therefore $f(x_1, \bar{x}_1)$ is not maximized at $\bar{x}_1$.

Thus a further necessary condition for a maximum is that

the slope must be decreasing at $\bar{x}_1$.

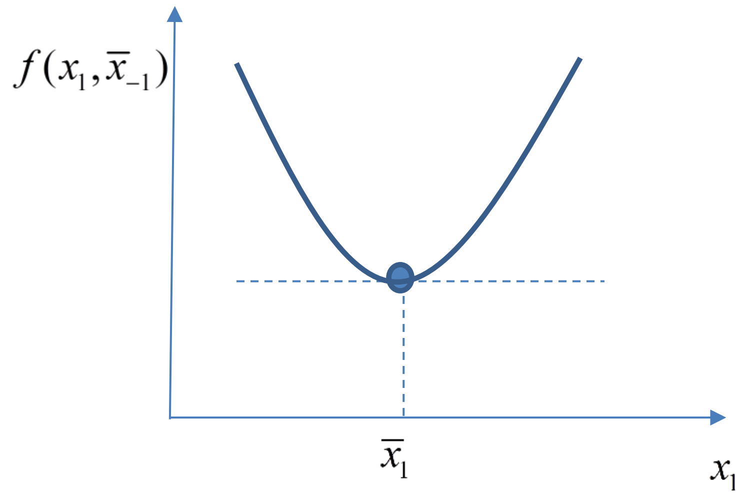


Fig. 2: Local minimum

We therefore have a second set of necessary conditions for a maximum. Since they are restrictions on second derivatives they are called the second order necessary conditions.

Second order necessary conditions for a maximum

If $\bar{x}$ is a maximizer of $f(x)$, then

$$\frac{\partial^2 f}{\partial x_j^{\,2}} = \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_j}(\bar{x}) \le 0, \quad j = 1,...,n \qquad \text{(2-2)}$$

3. Non-negativity constraints

Many economic variables cannot be negative. Suppose this is true for all variables

Let $\bar{x}=(\bar{x}_1,...,\bar{x}_n)$ solve $\underset{x\ge 0}{Max}\{f(x)\}$.

We will consider the first variable.

It is helpful to write the optimal value

of all the other variables as $\bar{x}_{-1}$. Then

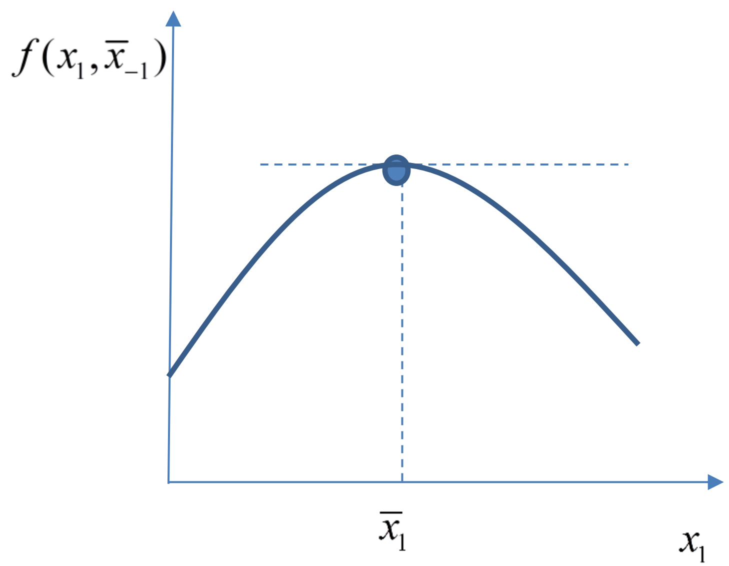
$\bar{x}_1$ solves $\underset{x_1\ge 0}{Max}\{f(x_1,\bar{x}_{-1})\}$.

Case (i) $\bar{x}_1>0$

This is depicted opposite.

For $\bar{x}_1$ to be the maximizer,

the graph of $f(x_1,\bar{x}_{-1})$ must be zero at $\bar{x}_1$.



Case (i): Necessary condition for a maximum

Case (ii) $\bar{x}_1 = 0$

This is depicted opposite.

For $\bar{x}_1$ to be the maximizer,

the graph of $f(x_1, \bar{x}_{-1})$ cannot be strictly

positive at $\bar{x}_1$.

Taking the two cases together,

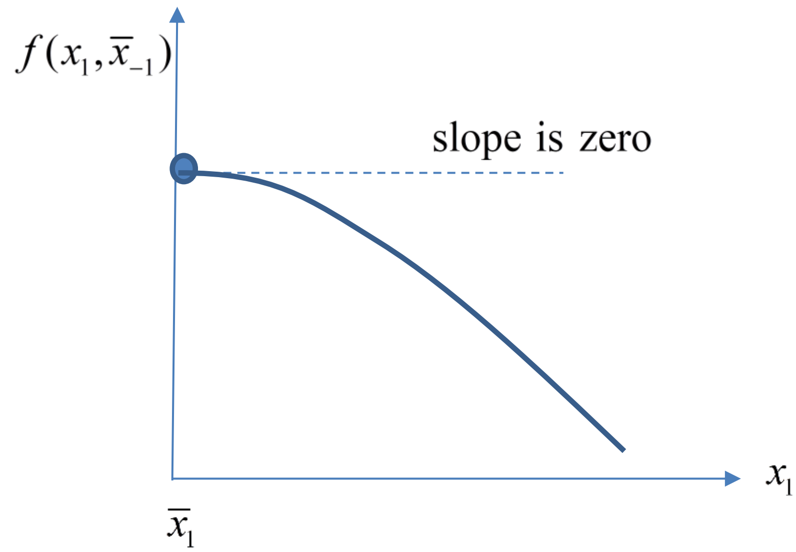
$$\frac{\partial f}{\partial x_1}(\bar{x}) \le 0 \text{ , with equality if } \bar{x}_1 > 0$$

An identical argument holds for all of the

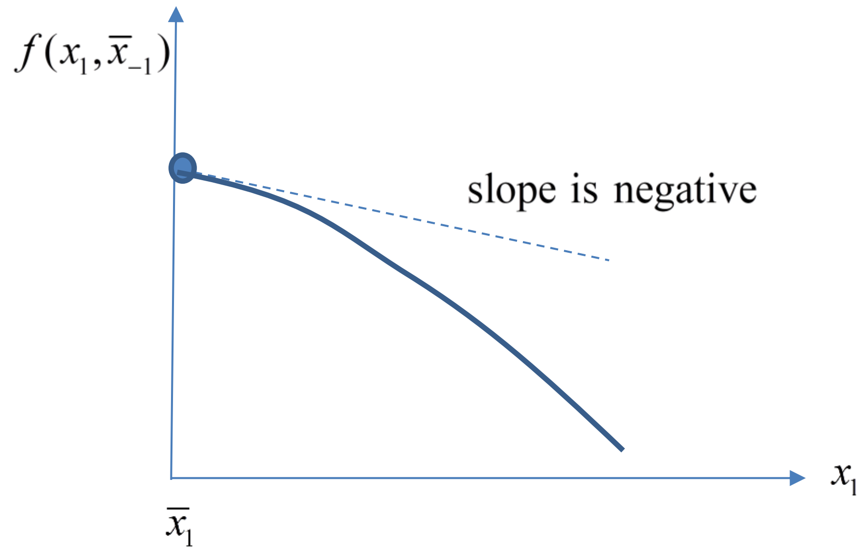
variables.

Necessary conditions

$$\frac{\partial f}{\partial x_j}(\bar{x}) \le 0 \text{ , with equality if } \bar{x}_j > 0 \text{ , } j = 1,...,n$$



Case (ii): Necessary condition for a maximum



Case (ii): Necessary condition for a maximum

4. Examples: Profit-maximizing firm

Example 1:

Cost function

$$C(q) = 5 + 12q + 3q^2$$

Demand price function

$$p(q) = 20 - q$$

Group exercise: Solve for the profit maximizing output and price.

Example 2: Two products

MODEL 1

Cost function

$$C(q) = 10q_1 + 15q_2 + 2q_1^2 + 3q_1q_2 + 2q_2^2$$

Demand price functions

$p_1 = 85 - \frac{1}{4}q_1$ and $p_2 = 90 - \frac{1}{4}q_2$

Group 1 exercise: How might you solve for the profit maximizing outputs?

MODEL 2

Cost function

$$C(q) = 10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2$$

Demand price functions

$p_1 = 65 - \frac{1}{4}q_1$ and $p_2 = 70 - \frac{1}{4}q_2$

Group 2 exercise: How might you solve for the profit maximizing outputs?

MODEL 1:

Revenue

$$R_1 = p_1q_1 = (85 - \tfrac{1}{4}q_1)q_1 = 85q_1 - \tfrac{1}{4}q_1^{\,2} \ , \quad R_2 = p_2q_2 = (90 - \tfrac{1}{4}q_2)q_2 = 90q_2 - \tfrac{1}{4}q_2^{\,2}$$

Profit

$$\pi = R_1 + R_2 - C$$

$$= 85q_1 - \tfrac{1}{4}q_1^{\,2} + 90q_2 - \tfrac{1}{4}q_2^{\,2} - (10q_1 + 15q_2 + 2q_1^{\,2} + 3q_1q_2 + 2q_2^{\,2})$$

$$= 75q_1 + 75q_2 - \tfrac{9}{4}q_1^{\,2} - \tfrac{9}{4}q_2^{\,2} - 3q_1q_2 \,.$$

Note that profit is positive for small q.

Class exercise: Explain why profit is negative for large q.

It follows that a maximum exists.

$$\pi = 75q_1 + 75q_2 - \tfrac{9}{4}q_1^{\,2} - \tfrac{9}{4}q_2^{\,2} - 3q_1q_2$$

Think on the margin

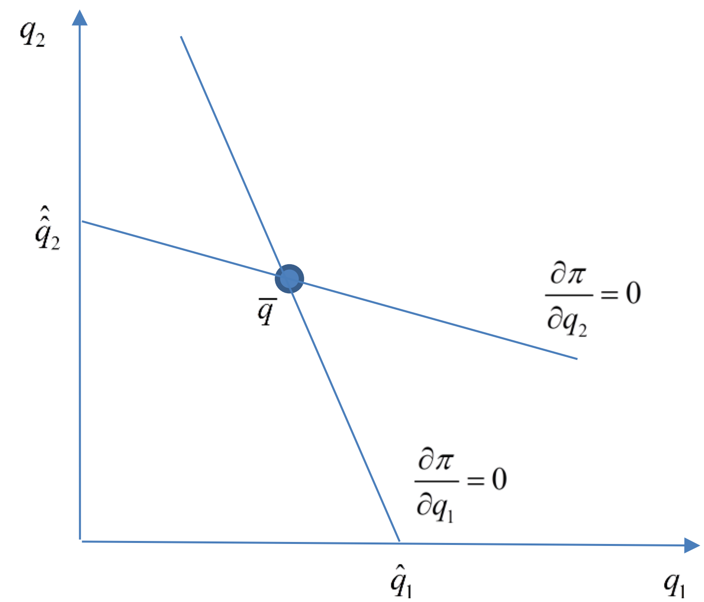
Marginal profit from increasing q_1

$$\frac{\partial \pi}{\partial q_1} = 75 - \tfrac{9}{2}q_1 - 3q_2 \ .$$

Marginal profit from increasing q_2

$$\frac{\partial \pi}{\partial q_2} = 75 - 3q_1 - \tfrac{9}{2}q_2 \ .$$

Therefore a critical point $\bar{q} \gg 0$ must satisfy



Model 1: Necessary conditions

the following two necessary conditions

$$\frac{\partial \pi}{\partial q_1} = 75 - \tfrac{9}{2}q_1 - 3q_2 = 0, \quad \frac{\partial \pi}{\partial q_2} = 75 - 3q_1 - \tfrac{9}{2}q_2 = 0$$

The graphs of these two lines are depicted. If you solve for $\bar{q}$ satisfying both equations you will find that the unique solution is $\bar{q} = (\bar{q}_1, \bar{q}_2) = (10,10)$.

MODEL 2

Cost function

$$C(q) = 10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2$$

Demand price functions

$p_1 = 65 - \frac{1}{4}q_1$ and $p_2 = 70 - \frac{1}{4}q_2$

Revenue

$$R_1 = p_1q_1 = (65 - \tfrac{1}{4}q_1)q_1 = 65q_1 - \tfrac{1}{4}q_1^2 \;\;,\;\; R_2 = p_2q_2 = (70 - \tfrac{1}{4}q_2)q_2 = 70q_2 - \tfrac{1}{4}q_2^2$$

Profit

$$\pi = R_1 + R_2 - C$$

$$= 65q_1 - \tfrac{1}{4}q_1^2 + 70q_2 - \tfrac{1}{4}q_2^2 - (10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2)$$

$$= 55q_1 + 55q_2 - \tfrac{5}{4}q_1^2 - \tfrac{5}{4}q_2^2 - 3q_1q_2$$

Profit is strictly positive for small q and negative for large q so there is a solution.

$$\pi = 55q_1 + 55q_2 - \tfrac{5}{4}q_1^2 - \tfrac{5}{4}q_2^2 - 3q_1q_2$$

Think on the margin

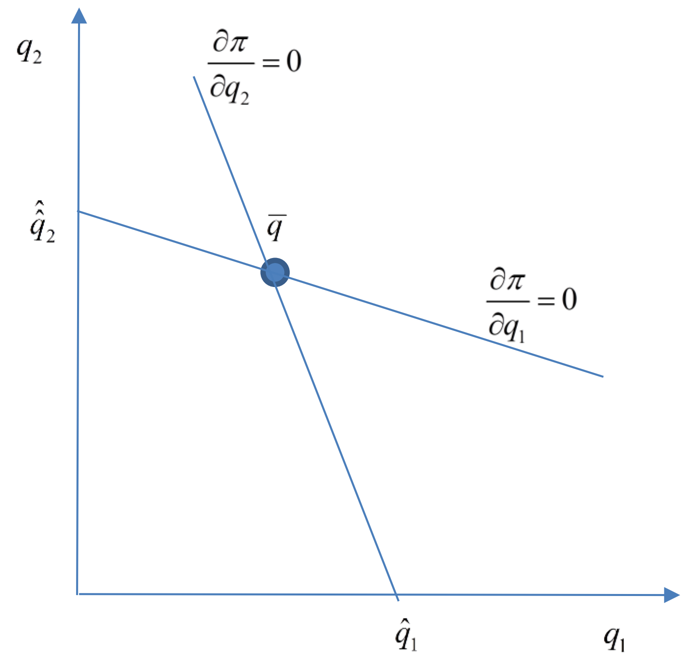
Marginal profit of increasing q_1

$$\frac{\partial \pi}{\partial q_1} = 55 - \tfrac{5}{2}q_1 - 3q_2 \ .$$

Marginal profit of increasing q_2

$$\frac{\partial \pi}{\partial q_2} = 55 - 3q_1 - \tfrac{5}{2}q_2 \ .$$

Therefore a critical point $\bar{q} >> 0$ satisfies



Model 2: Necessary conditions

the following two necessary conditions

$$\frac{\partial \pi}{\partial q_1} = 55 - \tfrac{5}{2}q_1 - 3q_2 = 0 \text{ and } \frac{\partial \pi}{\partial q_2} = 55 - 3q_1 - \tfrac{5}{2}q_2 = 0$$

If you solve for $\bar{q}$ satisfying both equations you will find that the unique critical point $\bar{q} >> 0$ is

$\bar{q} = (\bar{q}_1, \bar{q}_2) = (10,10)$.

However, we must also check to see if there are any other output vectors satisfying the necessary conditions.

Look along the boundaries.

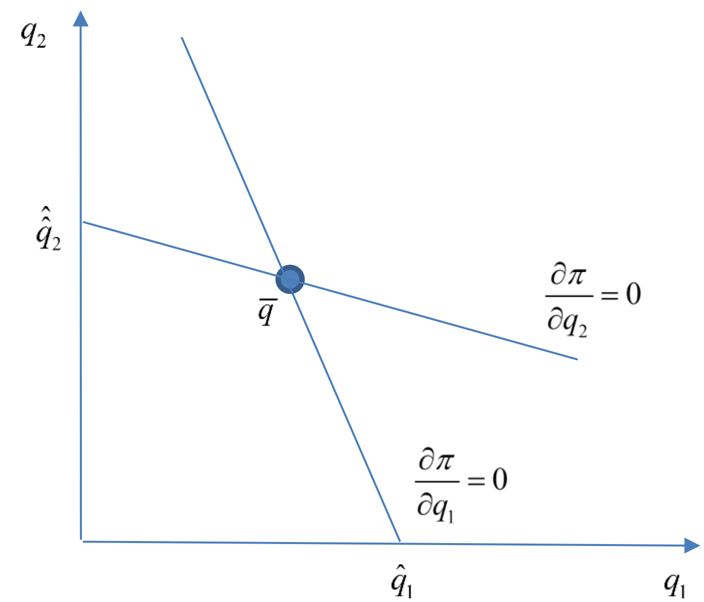
Necessary conditions for a maximum at $\hat{q} = (\hat{q}_1, 0)$

Model 1:

$\hat{q} = (\hat{q}_1, 0)$ must satisfy the following two conditions

$$\frac{\partial \pi}{\partial q_1} = 75 - \tfrac{9}{2}\hat{q}_1 = 0, \quad \frac{\partial \pi}{\partial q_2} = 75 - 3\hat{q}_1 \leq 0$$

$\frac{\partial \pi}{\partial q_1} = 0$ implies that $\hat{q}_1 = \frac{50}{3}$. But then $\frac{\partial \pi}{\partial q_2} > 0$.



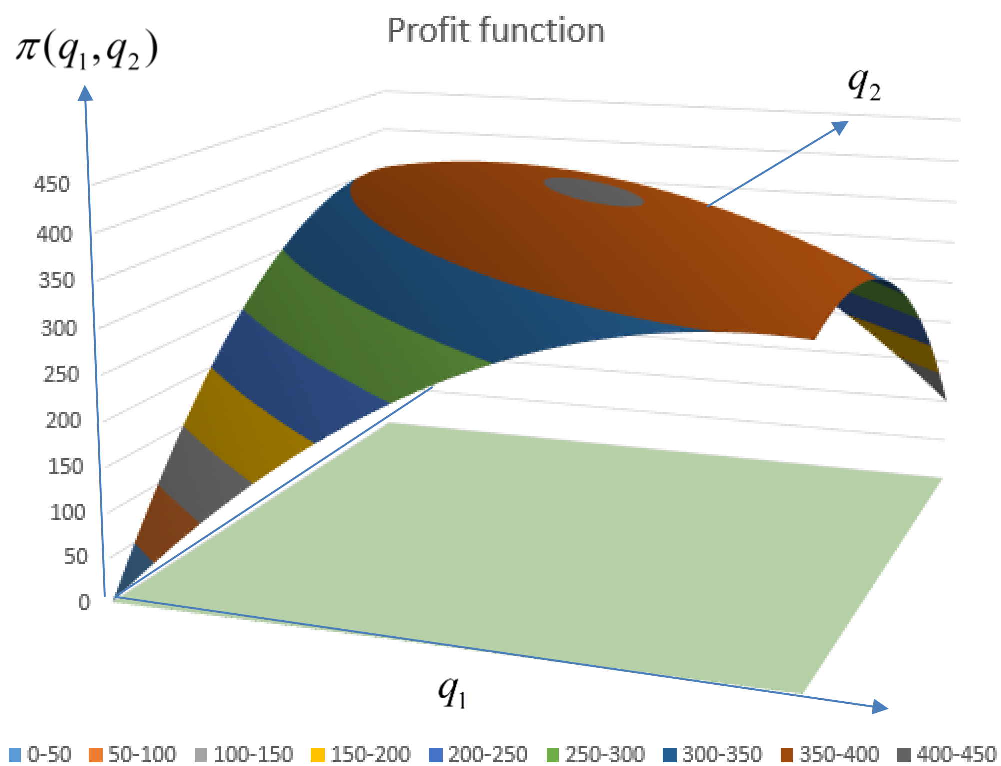
Model 1: Necessary conditions

A symmetrical argument shows that the necessary conditions cannot both hold at $\hat{\hat{q}} = (0, \hat{\hat{q}}_2)$ either.

Thus the necessary conditions only hold at $\bar{q} = (10, 10)$.

Hence this must be the maximizer.

The profit function is depicted below (using a spread-sheet)



Model 2:

At $\hat{q}=(\hat{q}_1,0)$ the necessary conditions for a maximum are

$$\frac{\partial\pi}{\partial q_1}=55-\tfrac{5}{2}q_1-3q_2=55-\tfrac{5}{2}\hat{q}_1=0$$

and

$$\frac{\partial\pi}{\partial q_2}=55-3q_1-\tfrac{5}{2}q_2=55-3\hat{q}_1\leq 0$$

Note that $\hat{q}=(22,0)$ satisfies both conditions.

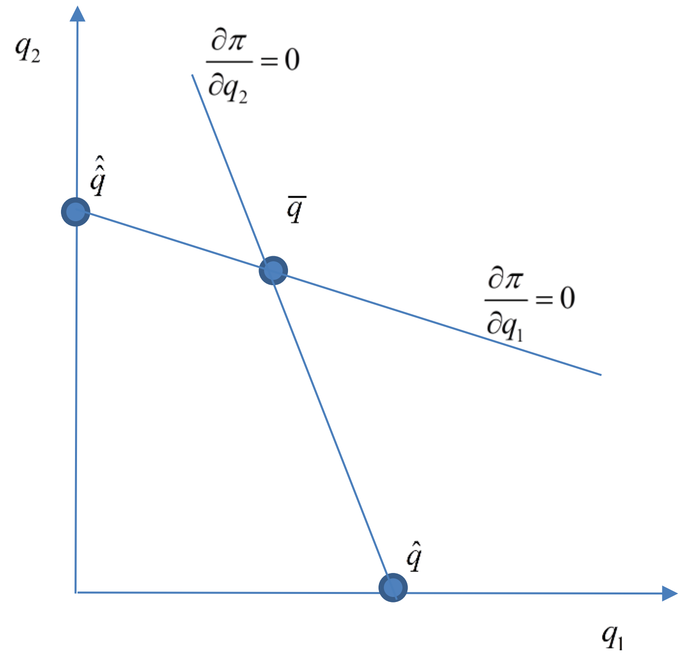
A symmetrical argument establishes that $\hat{\hat{q}}=(0,22)$

satisfies the necessary conditions as well.

If you check you will find that

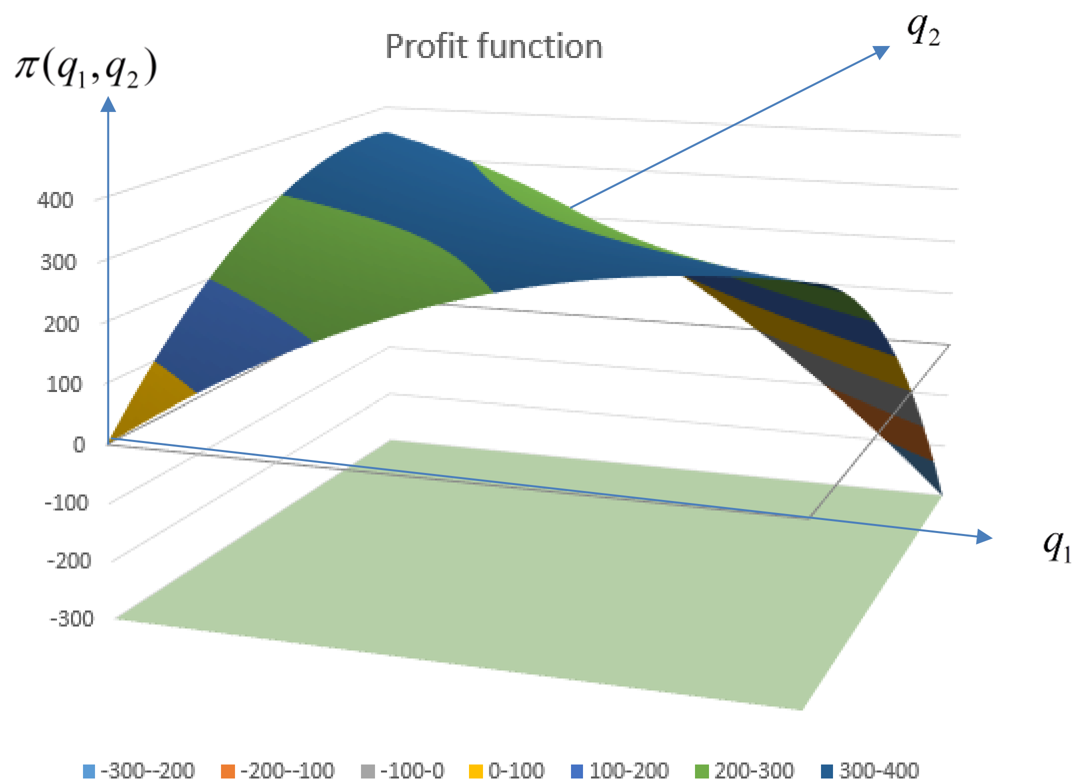
$$\pi(\bar{q})<\pi(\hat{q})=\pi(\hat{\hat{q}})$$

Therefore profit is maximized by producing only one commodity.



Model 2: Necessary conditions

The profit function has the shape of a saddle. The output vector $\bar{q}$ where the slope in the direction of each axis is zero is called a saddle-point.



5. Resource constrained maximization – a graphical approach

2 variables so $x=(x_1,x_2)$

Problem: $\underset{x\geq 0}{Max}\{f(x)\,|\,b-g(x)\geq 0\}$.

Assumption: Both $f(x)$ and $g(x)$

are strictly increasing functions.

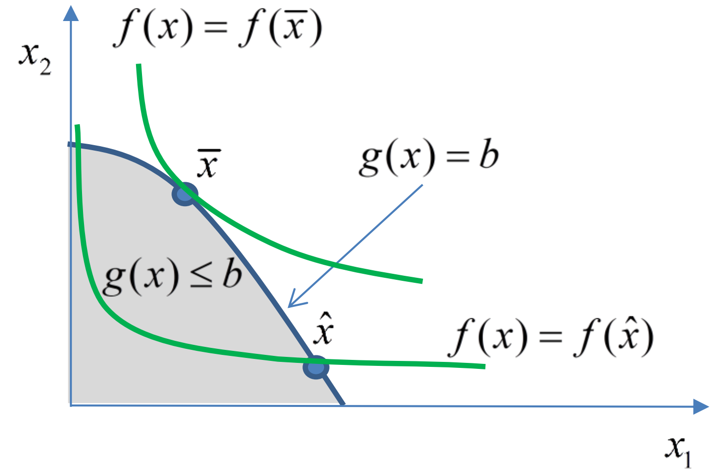
For $\bar{x}>>0$ to be the solution, the slopes of

the level sets

$$f(x)=f(\bar{x}) \text{ and } g(x)=g(\bar{x})=b$$

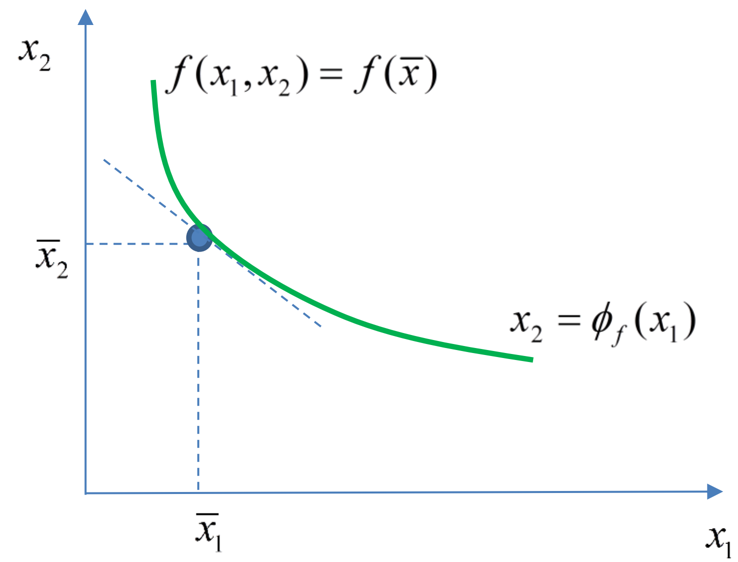
must be equal at $\bar{x}$.

But what is the slope of a level set?



Level sets

The slope of a level set



Slope of a level set

The level set $f(x_1,x_2)=f(\bar{x})$ implicitly defines

a mapping $x_2=\phi_f(x_1)$. That is,

$$f(x_1,\phi_f(x_1))=f(\bar{x}) \ .$$

Differentiate by x_1 .

$$\frac{\partial f}{\partial x_1}(x)+\frac{\partial f}{\partial x_2}(x)\phi_f{}'(x_1)=0$$

Thus the slope of the level set at $\bar{x}$ is $\phi_f{}'(\bar{x}_1)=-\dfrac{\dfrac{\partial f}{\partial x_1}(\bar{x})}{\dfrac{\partial f}{\partial x_2}(\bar{x})}$.

Then the Necessary condition for a maximum is

$$-\frac{\dfrac{\partial f}{\partial x_1}(\bar{x})}{\dfrac{\partial f}{\partial x_2}(\bar{x})}=\phi_f{}'(\bar{x}_1)=\phi_g{}'(\bar{x}_1)=-\frac{\dfrac{\partial g}{\partial x_1}(\bar{x})}{\dfrac{\partial g}{\partial x_2}(\bar{x})}$$

Class discussion: Can the graphical approach be used when there are more than 2 variables?

6. The shadow price (Lagrange multiplier) method and an economic interpretation

Problem: $\underset{x\geq 0}{Max}\{f(x) \mid b-g(x)\geq 0\}$

NOTE: Always write a resource constraint in the form $h(x)\geq 0$

Introduce a shadow price $\lambda \geq 0$. The "Lagrangian" of the problem is

$$\mathfrak{L} = f(x)+\lambda(b-g(x))$$

Necessary conditions for $\bar{x}$ to solve the resource constrained problem

For some $\lambda \geq 0$

$$\frac{\partial \mathfrak{L}}{\partial x_j}(\bar{x},\lambda) = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0, \text{ with equality if } \bar{x}_j > 0$$

Moreover, if the constraint is not binding (i.e. $b-g(\bar{x})>0$) then $\lambda = 0$.

Problem: $\underset{x \geq 0}{Max}\{f(x) \mid b - g(x) \geq 0\}$

Economic interpretation of the problem.

If the firm chooses x it requires $g(x)$ units of a resource that is fixed in supply (i.e. floor space of plant, highly skilled workers)

We interpret $f(x)$ as the revenue of the firm. The fixed supply of this input is b.

The relaxed problem

Suppose first that the firm could purchase or sells units of the resource at the market price λ .

If the firm purchases $g(x)-b$ additional units of the resource its profit is then

$$\mathfrak{L}=f(x)-\lambda(g(x)-b)=f(x)+\lambda(b-g(x))$$

If the firm sells $b-g(x)$ units of the resource its profit is then

$$\mathfrak{L}=f(x)+\lambda(b-g(x))$$

The firm then solves the following problem.

$$\underset{x\geq 0}{Max}\{\mathfrak{L}=f(x)+\lambda(b-g(x))\}$$

Necessary conditions for $\bar{x}$ to solve the relaxed problem

$\frac{\partial \mathfrak{L}}{\partial x_j}(\bar{x})=\frac{\partial f}{\partial x_j}(\bar{x})-\lambda\frac{\partial g}{\partial x_j}(\bar{x})\leq 0$, with equality if $\bar{x}_j>0$, $j=1,...,n$

It is a beautiful (and deep) theorem that, under very weak assumptions, there exists a "shadow price" $\lambda \geq 0$ such that these are also the necessary conditions when there is no market in which to purchase or sell the resource.

The shadow price is the marginal value of an additional unit of the resource.

If the constraint is not binding at the maximum ($b - g(\bar{x}) > 0$) then the value of an additional unit is zero, i.e. $\lambda = 0$

The 2 variable case with $\bar{x} \gg 0$

$$\frac{\partial \mathfrak{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0 \, .$$

$$\frac{\partial \mathfrak{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0.$$

Equivalently, $\dfrac{\partial f}{\partial x_1} = \lambda \dfrac{\partial g}{\partial x_1}$ and $\dfrac{\partial f}{\partial x_2} = \lambda \dfrac{\partial g}{\partial x_2}$

Therefore

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial x_2}} = \frac{\lambda \dfrac{\partial g}{\partial x_1}}{\lambda \dfrac{\partial g}{\partial x_2}} = \frac{\dfrac{\partial g}{\partial x_1}}{\dfrac{\partial g}{\partial x_2}}$$

This is the Necessary Condition obtained using the graphical approach.

Remark:

With multiple constraints $g_i(x) \le b_i,\ i = 1,...,m$ the necessary conditions are the same. The Lagrangian becomes

$$\mathfrak{L} = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) \ .$$

For any constraint that is not binding, the associated shadow price must be zero.

Solving for the maximum

Example 1: Consumer choice with 2 commodities

A consumer's preferences are represented by the utility function $f(x)=\sum_{j=1}^{2}\alpha_j \ln x_j$

where $\alpha >> 0$ and $\sum_{i=1}^{2}\alpha_i = 1$. The price vector is p. The consumer has an income I.

$$\underset{x\geq 0}{Max}\{f(x) \mid I - p\cdot x \geq 0\}$$

We write down the Lagrangian

$$\mathfrak{L} = \sum_{j=1}^{2}\alpha_j \ln x_j + \lambda(I - \sum_{j=1}^{2} p_j x_j)$$

Necessary conditions

$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\alpha_j}{x_j} - \lambda p_j \leq 0, \text{ with equality if } \bar{x}_j > 0, \quad j = 1,2 . \tag{6-1}$$

Necessary conditions

$$\frac{\partial \mathfrak{L}}{\partial x_j} = \frac{\alpha_j}{x_j} - \lambda p_j \le 0, \text{ with equality if } \bar{x}_j > 0, \quad j = 1,2 .$$

Note that $\lim_{x_j \to 0} \frac{\alpha_j}{x_j} = \infty$. Therefore the inequality cannot hold at $x_j = 0$. Then $\bar{x}_j > 0$ and so the necessary conditions are equalities. Hence

$$p_j x_j = \frac{\alpha_j}{\lambda} \text{ , } j = 1,2 \qquad (6\text{-}2)$$

Summing over the commodities,

$$I = \sum_{j=1}^{2} p_j x_j = \sum_{j=1}^{2} \frac{\alpha_j}{\lambda} = \frac{1}{\lambda}, \text{ since } \sum_{j=1}^{2} \alpha_j = 1 \qquad (6\text{-}3)$$

Appealing to (6-2) and (6-3) it follows that

$$p_j \bar{x}_j = \alpha_j I \text{ and so } \bar{x}_j = \frac{\alpha_j}{p_j} I, \ j = 1,2$$

Example 2: Consumer choice with 3 commodities

A consumer's preferences are represented by a utility function $f(x)=\sum_{j=1}^{3}\alpha_j \ln x_j$

where $\alpha >> 0$ and $\sum_{i=1}^{3}\alpha_i = 1$. The price vector is p. The consumer has an income I.

Group Exercise:

Show that the solution is

$$\bar{x}_j = \frac{\alpha_j}{p_j} I,\ j = 1,2,3$$

Example 3: Output maximization with a budget constraint

A firm has a production function $q = g(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ where $\alpha >> 0$ and $\sum_{i=1}^{2} \alpha_i = 1$.

The input price vector is p. The manager is given a budget of $\bar{b}$ and tasked with maximizing output.

$$\underset{x \geq 0}{Max}\{g(x) \mid \bar{b} - p \cdot x \geq 0\}$$

Group Exercise:

Explain why the solution is $\bar{x}_j = \frac{\alpha_j}{p_j}\bar{b}$, $j = 1,2$

Then maximized output is

$$\bar{q} = g(\bar{x}) = (\frac{\alpha_1 \bar{b}}{p_1})^{\alpha_1} (\frac{\alpha_2 \bar{b}}{p_2})^{\alpha_2} = \bar{b}(\frac{\alpha_1}{p_1})^{\alpha_1} (\frac{\alpha_2}{p_2})^{\alpha_2}$$