Basics of maximization in economics

*Not required reading

77 slides

A. Necessary conditions

$x = (x_1,...,x_n)$ a vector of decision variables where each component of *x* is a real number.

($x_j \in \mathbb{R}$, $j = 1,...,n$, equivalently $x \in \mathbb{R}^n$)

$f(x)$ is a mapping from the set $\mathbb{R}^n$ onto the set $\mathbb{R}$

Assume that all the partial derivatives of $f(x)$ exist

Maximization problem

$$\underset{x}{Max}\{f(x) \mid x \in \mathbb{R}^n_+\}$$

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$\underset{x}{Max}\{f(x) \mid x \in \mathbb{R}^n_+\}$

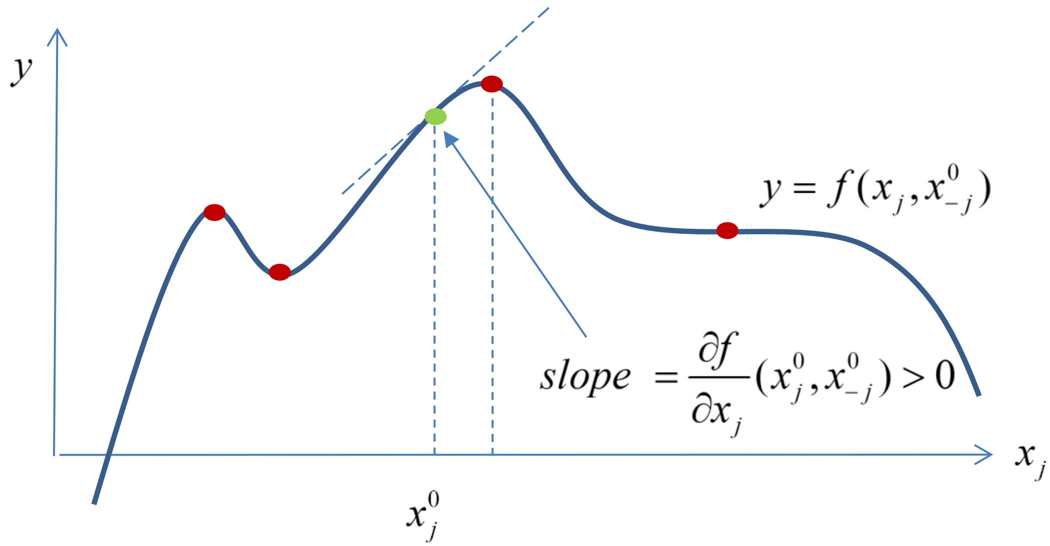
Focus on x_j . Write the vector of all other components of *x* as

$x_{-j}=(x_1,...,x_{j-1},x_{j+1},...,x_n)$

Then the function $f(x)$ can be written as follows:

$f(x)=f(x_j,x_{-j})$

Depict graph of f



Case (i) $x_j^0 > 0$

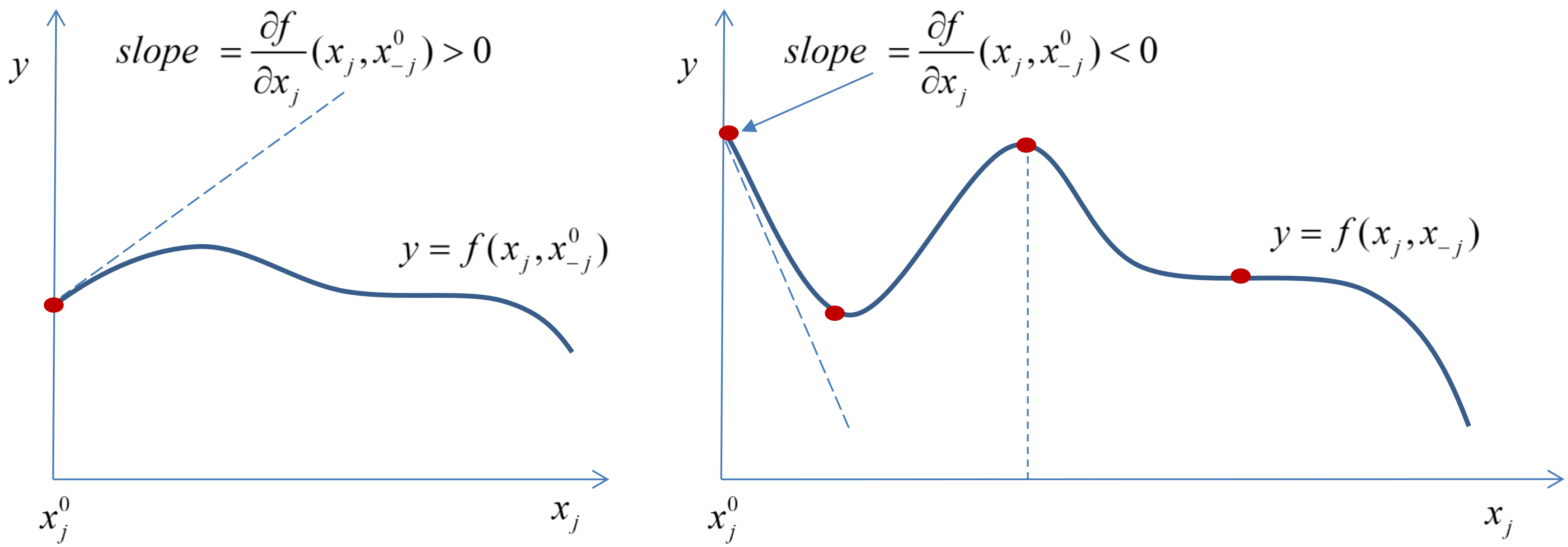
$f(x_j, x_{-j}^0)$ is a function of a single variable. Since $x_j^0 > 0$ we can consider small neighborhoods of x_j^0 in $\mathbb{R}_+$

Arguing exactly as in the one variable case the necessary condition

$$\frac{\partial f}{\partial x_j}(x_j^0, x_{-j}^0) \equiv \frac{\partial f}{\partial x_j}(x^0) = 0$$

Case (ii) $x_j = 0$

Graph of f



If the slope of the graph is strictly positive, then for $x_j > 0$ and sufficiently small,

$$f(x_j, x_{-j}^0) > f(x_j^0, x_{-j}^0).$$

Thus the necessary condition is

$$\frac{\partial f}{\partial x_j}(x_j^0, x_{-j}^0) \equiv \frac{\partial f}{\partial x_j}(x^0) \le 0$$

Therefore necessary conditions ("First order conditions") for f to take on its maximum at x^0 are as follows:

$$\frac{\partial f}{\partial x_j}(x^0) \le 0, \quad j = 1,...,n \text{ with equality if } x_j^0 > 0$$

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Equivalently,

(i) the "gradient vector" (vector of the n partial derivatives) is negative, i.e.

$$\frac{\partial f}{\partial x}(x^0) \le 0$$

(ii) the inner product of x^0 and the gradient vector is the zero vector, i.e.

$$x^0 \ge 0, \ \frac{\partial f}{\partial x}(x^0) \le 0, \quad j = 1,...,n \text{ and } x^0 \cdot \frac{\partial f}{\partial x}(x^0) = 0$$

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Since only one of the two inequality conditions above can be strict, these conditions are known as the **complementary slackness conditions**.

B. Maximization with a linear resource constraint

As a first step in the analysis of maximization with resource constraints we consider the maximization problem of a consumer who chooses among consumption vectors. The set of commodities is $\mathcal{N} = \{1,...,n\}$. Given an income I and a vector of prices $p = (p_1, p_2,..., p_n)$, the set of feasible consumption vectors is the set

$$B = \{x \geq 0 \mid p_1x_1 + ... + p_nx_n = p \cdot x \leq I\}$$

We assume that the preferences of the consumer can be represented by a continuously differentiable, strictly increasing utility function $U(x)$.

The consumer then chooses $\bar{x}$ that solves the following problem.

$$\underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq I\}$$

Note that, since $U(x)$ is strictly increasing, $p \cdot \bar{x} = I$

Example with 2 commodities: $\underset{x\geq 0}{Max}\{U(x)=x_1^{\alpha_1}x_2^{\alpha_2}\mid p_1x_1+p_2x_2\leq I\}$

Note that utility is zero if consumption of either commodity is zero. Therefore every component of the solution $\bar{x}$ is strictly positive. (We write $\bar{x} >> 0$) .

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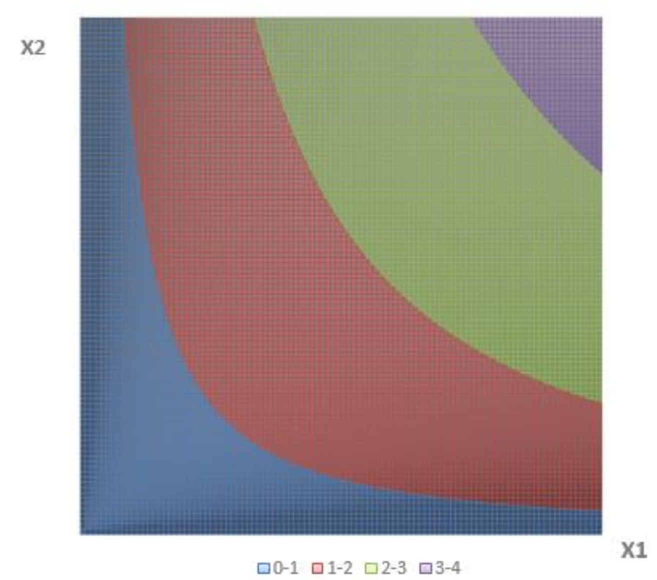
Geometry

In the 2 commodity case we can represent preferences by depicting points for which utility has the same value.

Such a set of points is called a level set. In the figure the level sets are the boundaries of the blue, red and green shaded regions.



Level sets of $U(x)=x1^{0.4}x2^{0.5}$

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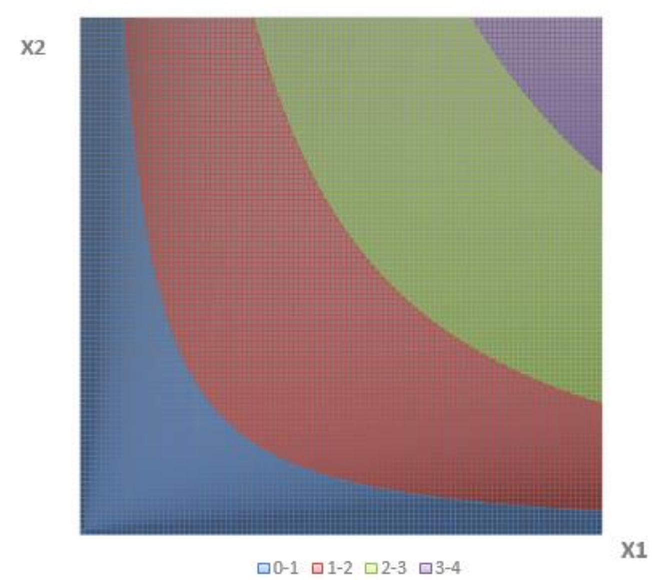


Level sets of U(x)=x1$^{0.4}$x2$^{0.5}$

In mathematical notation the 4 level sets are

$\{x\,|\,U(x)=0\},\{x\,|\,U(x)=1\},\ \{x\,|\,U(x)=2\},\{x\,|\,U(x)=3\}$.

Three of them are what economists often call indifference curves.

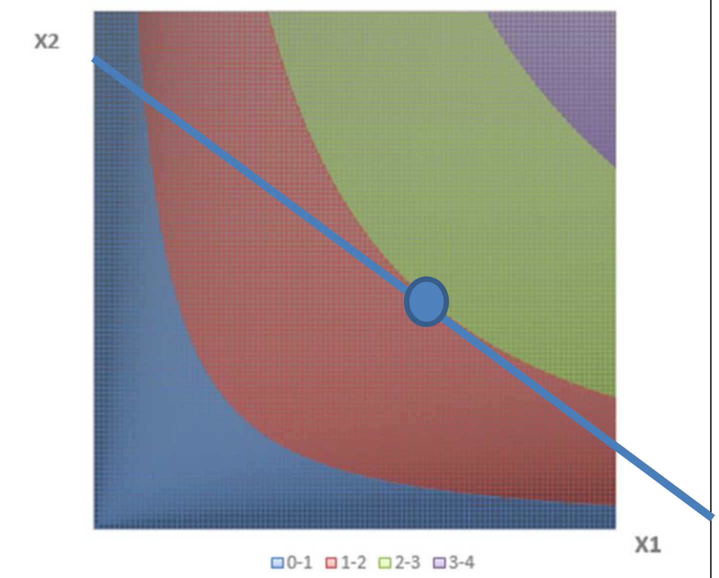
In the figure both components of the solution

$\bar{x} = (\bar{x}_1, \bar{x}_2)$ are strictly positive.

(Mathematical shorthand $\bar{x} >> 0$.)

So the slope of the budget line is

equal to the slope of the indifference curve.



Level sets of $U(x)=x1^{0.4}x2^{0.5}$

Necessary conditions for a maximum

To a first approximation, if a consumer currently, choosing $\overline{x}$ can increase consumption of commodity j by Δx_j, the change in utility is

$$\Delta U = \frac{\partial U}{\partial x_j}(\overline{x})\Delta x_j \ .$$

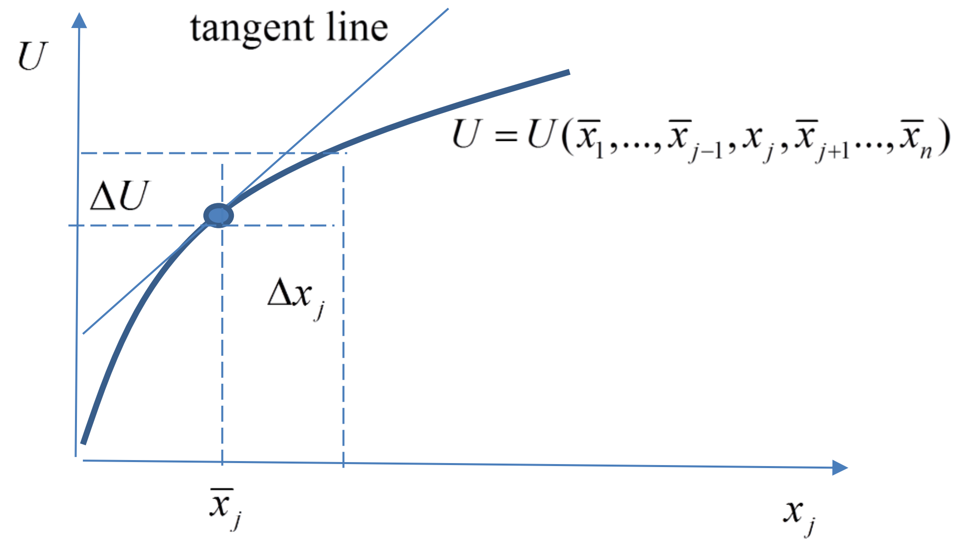


Fig. 2-3: Utility as a function of x_j

This is depicted in the figure..

The slope of the tangent line at $\overline{x}$ is

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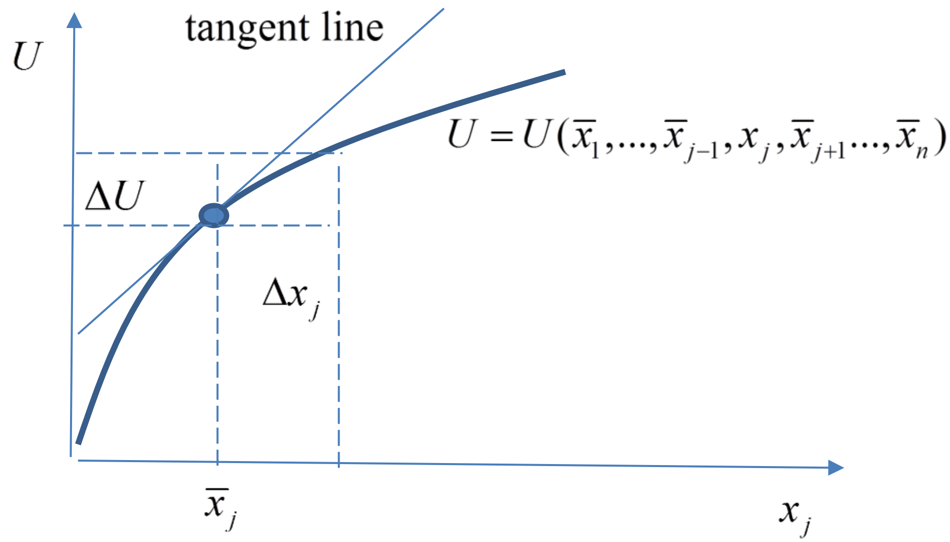

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If the consumer has an additional ΔE dollars then $\Delta E = p_j \Delta x_j$ and so $\Delta x_j = \dfrac{\Delta E}{p_j}$.

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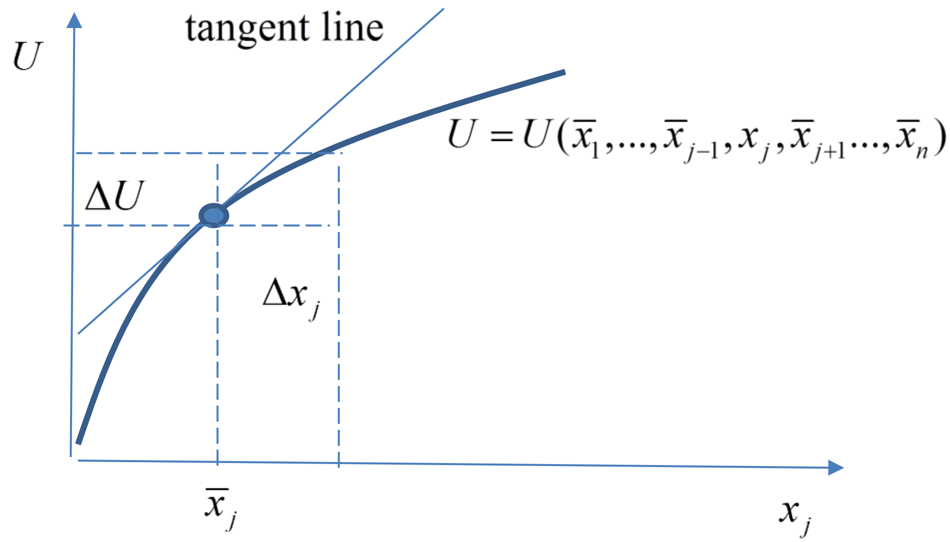

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The increase in utility is therefore $\Delta U = \dfrac{\partial U}{\partial x_j}(\bar{x})\Delta x_j = \dfrac{\partial U}{\partial x_j}(\bar{x})\dfrac{\Delta E}{p_j} = \dfrac{1}{p_j}\dfrac{\partial U}{\partial x_j}(\bar{x})\Delta E$

We have seen that

$$\Delta U = \frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})\Delta E$$

Therefore

$$\frac{\Delta U}{\Delta E} = \frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$$

In the limit as ΔE approaches zero, this becomes the rate at which utility rises as expenditure on commodity j rises.

$\frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$ is the marginal utility per dollar as expenditure on commodity j rises

Suppose that the consumer spends 1 dollar less on commodity j . His change in utility is

$-\frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$. He then spends the dollar on commodity i .

The change in utility is $\frac{1}{p_i}\frac{\partial U}{\partial x_i}(\bar{x})$. The net change in utility is therefore

$$\frac{1}{p_i}\frac{\partial U}{\partial x_i}(\bar{x})-\frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$$

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Case (i) $\bar{x}_i,\bar{x}_j>0$

If the change in utility is strictly positive the current utility can be increased by consuming more of commodity i and less of commodity j . If it is negative, utility can be increased by spending less commodity j and more on commodity i . Thus a necessary condition for $\bar{x}$ to be utility maximizing is that

$$\frac{1}{p_i}\frac{\partial U}{\partial x_i}(\bar{x})=\frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$$

Case (ii) $\bar{x}_j > \bar{x}_i = 0$

If the difference in marginal utilities is positive current $U(\bar{x})$ can be increased by spending a positive amount on commodity j. Thus a necessary condition for $\bar{x}$ to be utility maximizing is that

$$\frac{1}{p_i}\frac{\partial U}{\partial x_i}(\bar{x}) \leq \frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$$

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$$\frac{1}{p_i}\frac{\partial U}{\partial x_i}(\bar{x}) \le \frac{1}{p_j}\frac{\partial U}{\partial x_j}(\bar{x})$$

Let λ be the common marginal utility per dollar for all those commodities that are consumed in strictly positive amounts. We can therefore summarize the necessary conditions as follows:

Necessary conditions for a maximum

If $\bar{x}_j > 0$ then $\dfrac{1}{p_j}\dfrac{\partial U}{\partial x_j}(\bar{x}) = \lambda$

If $\bar{x}_j = 0$ then $\dfrac{1}{p_j}\dfrac{\partial U}{\partial x_j}(\bar{x}) \le \lambda$

Note: Since λ is the rate at which utility rises with income it is called the marginal utility of income

An alternative approach

From the argument above, if both commodity i and commodity j are consumed, then the ratio of their marginal utilities must be equal to the price ratio.

To understand this consider a change in x_i and x_j that leaves the consumer on the same level set. i.e.

$$U(\bar{x}_1+\Delta x_1,\bar{x}_2+\Delta x_2)=U(\bar{x}_1,\bar{x}_2)$$

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To understand this consider a change in x_i and x_j that leaves the consumer on the same level set. i.e.

$$U(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2) = U(\bar{x}_1, \bar{x}_2)$$

Above we showed that, to a first approximation,

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j \ .$$

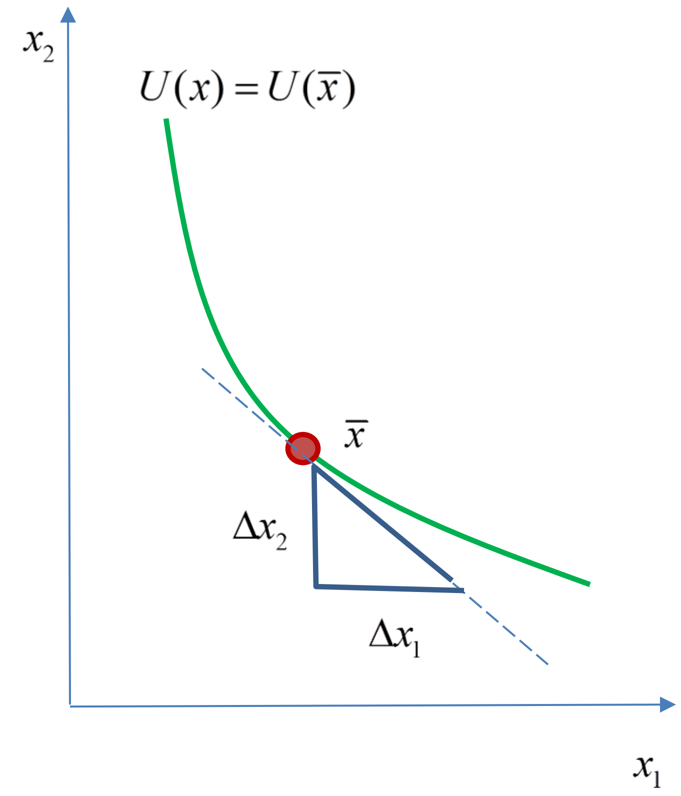
If we increase the quantity of commodity j and reduce the quantity of commodity i, then the net change in utility is

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j - \frac{\partial U}{\partial x_i}(\bar{x})\Delta x_i$$

We have argued that $\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j - \frac{\partial U}{\partial x_i}(\bar{x})\Delta x_i$

For this net change to be zero,

$$\frac{\Delta x_j}{\Delta x_i} = \frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}}$$



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For this net change to be zero,

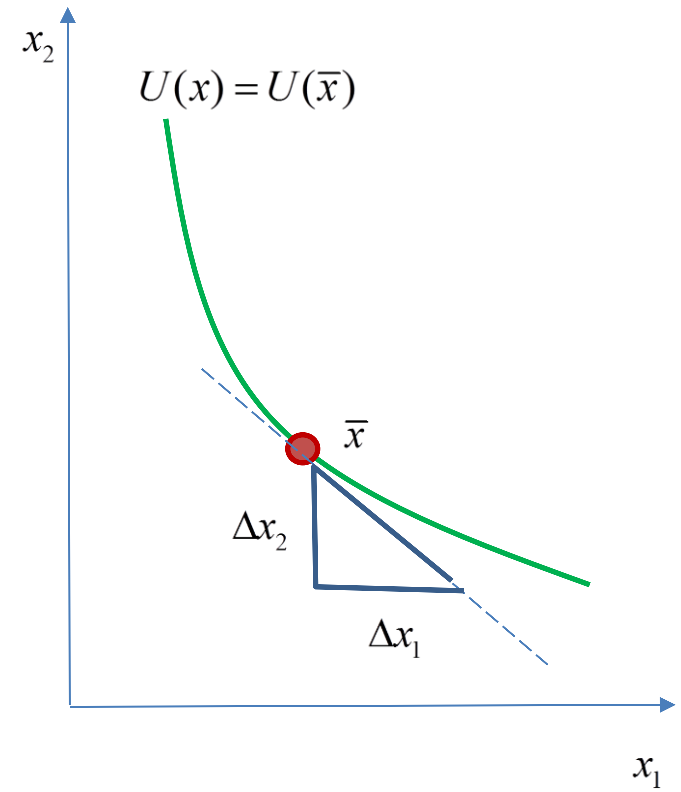
$$\frac{\Delta x_j}{\Delta x_i} = \frac{\dfrac{\partial U}{\partial x_i}}{\dfrac{\partial U}{\partial x_j}}$$

In the figure, $-\frac{\Delta x_2}{\Delta x_1}$ is the slope of the level set at $\bar{x}$.

The ratio is the rate at which x_1 must be substituted into

the consumption bundle to compensate for a reduction in x_2

Hence we call it the marginal rate of substitution of x_1 for x_2 .



Definition: Marginal rate of substitution

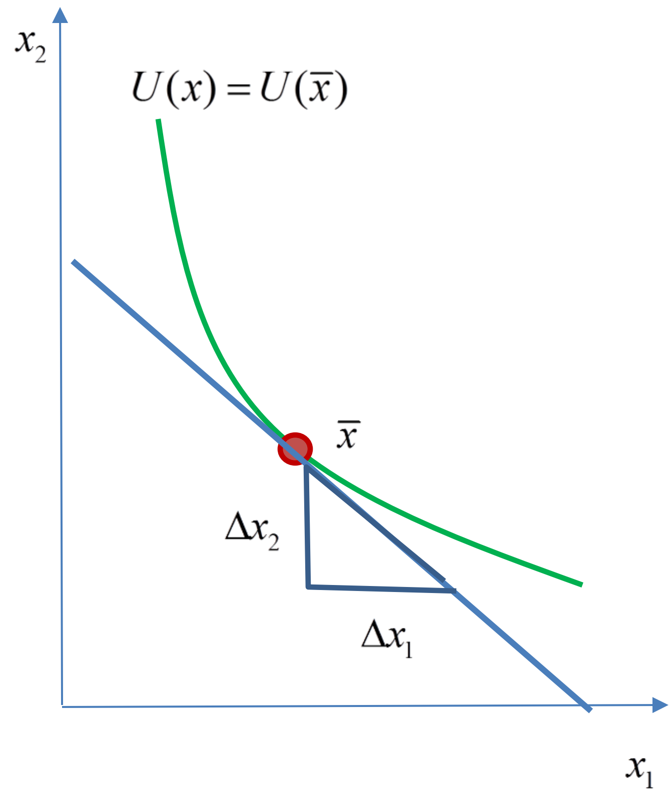
$$MRS(x_i, x_j) = \frac{\dfrac{\partial U}{\partial x_i}}{\dfrac{\partial U}{\partial x_j}}$$

For $\overline{x}$ to be the maximizer the rate at which x_1 can be substituted into the budget as x_2 is reduced must leave total expenditure on the two commodities constant, i.e.,

$$p_i \Delta x_i + p_j \Delta x_j = 0$$

Then along the budget line

$$\frac{\Delta x_j}{\Delta x_i} = -\frac{p_i}{p_j}$$



Graphically, the slope of the budget line must be equal to the slope of the indifference curve at $\overline{x}$ ie.

$$MRS(\overline{x}_i, \overline{x}_j) = \frac{\dfrac{\partial U}{\partial x_i}}{\dfrac{\partial U}{\partial x_j}} = \frac{p_i}{p_j}$$

The example: $\underset{x\geq 0}{Max}\{U(x)=x_1^{\alpha_1}x_2^{\alpha_2} \mid p_1x_1+p_2x_2\leq I\}$

Necessary conditions for a maximum

Method 1: Equalize the marginal utility per dollar

To make differentiation simple, try to find an increasing function of the utility function that is simple.

The example: $\underset{x\geq 0}{Max}\{U(x)=x_1^{\alpha_1}x_2^{\alpha_2} \mid p_1x_1+p_2x_2\leq I\}$

Necessary conditions for a maximum

Method 1: Equalize the marginal utility per dollar

To make differentiation simple, try to find an increasing function of the utility function that is simple.

Define the new utility function $u(x)=\ln U(x)$

The new maximization problem is

$$\underset{x\geq 0}{Max}\{u(x)=\ln U(x) \mid p_1x_1+p_2x_2\leq I\}$$

That is

$$\underset{x\geq 0}{Max}\{\alpha_1\ln x_1+\alpha_2\ln x_2 \mid p_1x_1+p_2x_2\leq I\}$$

Note that

$$\frac{\partial u}{\partial x_j}=\frac{\alpha_j}{x_j}\ .$$

Necessary conditions

$$\frac{1}{p_1}\frac{\partial u}{\partial x_1} = \frac{1}{p_2}\frac{\partial u}{\partial x_2} = \lambda$$

$$\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \lambda .$$

Also $p_1 x_1 + p_2 x_2 = I$.

Technical tip

Ratio Rule:

If $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $b_1 + b_2 \neq 0$ then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}$.

Therefore

$$\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\alpha_1 + \alpha_2}{p_1 x_1 + p_2 x_2} = \frac{\alpha_1 + \alpha_2}{I}$$

And so

$$x_j = \frac{\alpha_j}{\alpha_1 + \alpha_2}\frac{I}{p_j} .$$

Method 2: Equate the MRS and price ratio

$U(x) = x_1^{\alpha_1} x_2^{\alpha_2}$. Then $\dfrac{\partial U}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}$ and $\dfrac{\partial U}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}$

$$MRS(\bar{x}_1, \bar{x}_2) = \frac{\dfrac{\partial U}{\partial x_1}}{\dfrac{\partial U}{\partial x_2}} = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}} = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1} .$$

Then to be the maximizer,

$$MRS(\bar{x}_1, \bar{x}_2) = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

As we have seen, it is helpful to rewrite this as follows:

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} .$$

Then proceed as before.

Data Analytics (Taking the model to the data)

$$x_1(p,I) = \frac{\alpha_1}{\alpha_1+\alpha_2}\frac{I}{p_j}.$$

Take the logarithm

$$\ln x_j = \ln(\frac{\alpha_1}{\alpha_1+\alpha_2}) + \ln I - \ln p_j$$

The model is now linear. We can then use least squares estimation

$$\ln x_j = a_0 + a_1(\ln I - \ln p_j)$$

or

$$\ln x_j = a_0 + a_1 \ln I + a_2 \ln p_j$$

Exercise: If $U(x) = (a_1 + x_1)^{\alpha_1}(a_2 + x_2)^{\alpha_2}$, solve for the demand function $x_1(p,I)$

C. Optimization with a non-linear constraint

Problem

$$\underset{x\in\mathbb{R}^n_+}{Max}\{f(x)\,|\,g(x)\le b\}\ .$$

This is illustrated for the two variable case.

As we shall see, the necessary conditions

can be derived using a very similar argument

To that used in Section B.

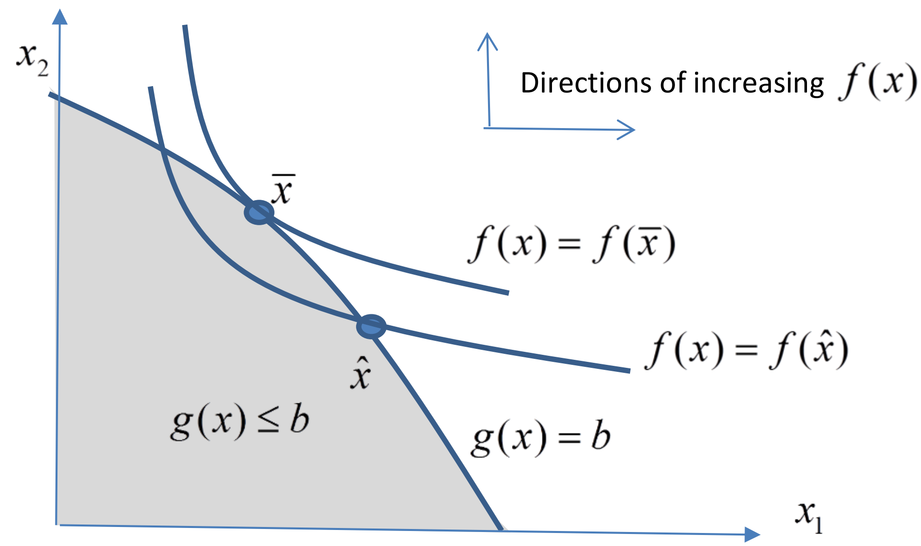


Figure C.1: Constrained maximization

In the figure the solution to the maximization problem is the vector $\bar{x}$

Interpretation:

A firm has a fixed supply of b units of some resource.

If it produces the vector of outputs x its resource use is $g(x)$ and revenue is $R=f(x)$.

Assumption 1: At any point x on the boundary of the feasible set the partial derivatives of $g(x)$ are all non-zero.*

Assumption 2: The solution to the unconstrained maximization problem $\underset{x}{Max}\{f(x)\}$ violates the resource constraint. Therefore if $\bar{x}$ solves the constrained maximization problem the constraint must be binding, i.e. $g(\bar{x}) = b$.

*Or, as a mathematician would say, the components of the gradient vector $\frac{\partial g}{\partial x}(x)$ are all nonzero.

Suppose that the firm chooses $\overline{x}$ where the constraint is binding. The figure below depicts the graph of $g(\overline{x}_1,....,\overline{x}_{j-1},x_j,\overline{x}_{j+1},...,\overline{x}_n)$ and the tangent line at $\overline{x}_j$. The slope of the tangent line at $\overline{x}$ is $\dfrac{\partial g}{\partial x_j}(\overline{x})$

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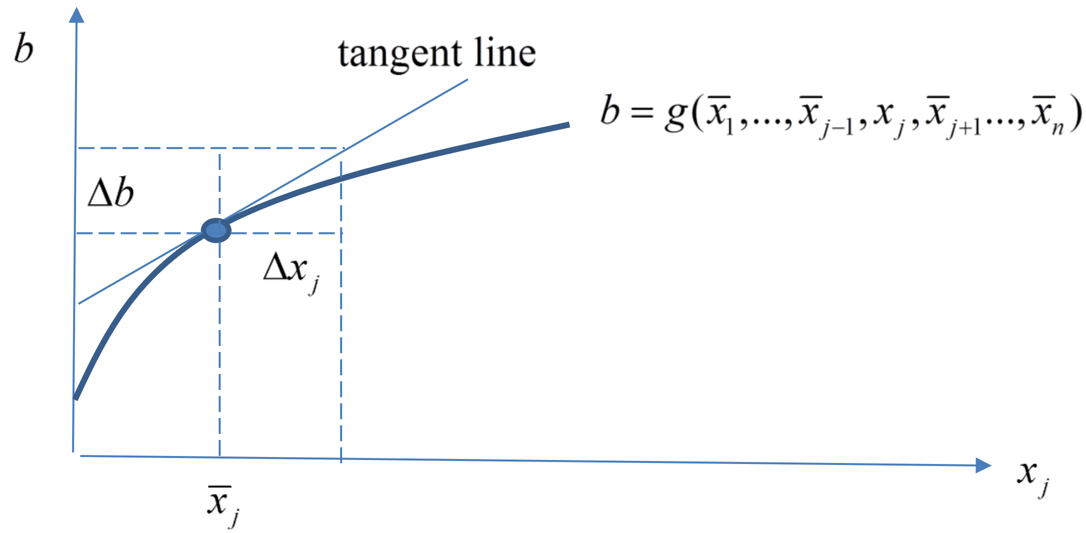

Figure C.2: Resource requirement

To a first approximation, if the firm wishes to increase output of commodity j by Δx_j it requires an extra Δb units of the resource where

$$\Delta b = \frac{\partial g}{\partial x_j}(\overline{x})\Delta x_j \ .$$

We have argued that the extra resource requirement is

$$\Delta b = \frac{\partial g}{\partial x_j}(\overline{x})\Delta x_j$$

Rearranging this expression, if follows that if the firm has Δb extra units of the resource and uses it to increase commodity j, then the increase in x_j is

$$\Delta x_j = \frac{\Delta b}{\frac{\partial g}{\partial x_j}(\overline{x})}$$

By the same argument, to a first approximation the increase in revenue is

$$\Delta R = \frac{\partial f}{\partial x_j}(\overline{x})\Delta x_j$$

This is depicted in the figure below. The slope of the tangent line at $\overline{x}$ is $\frac{\partial f}{\partial x_j}(\overline{x})$.

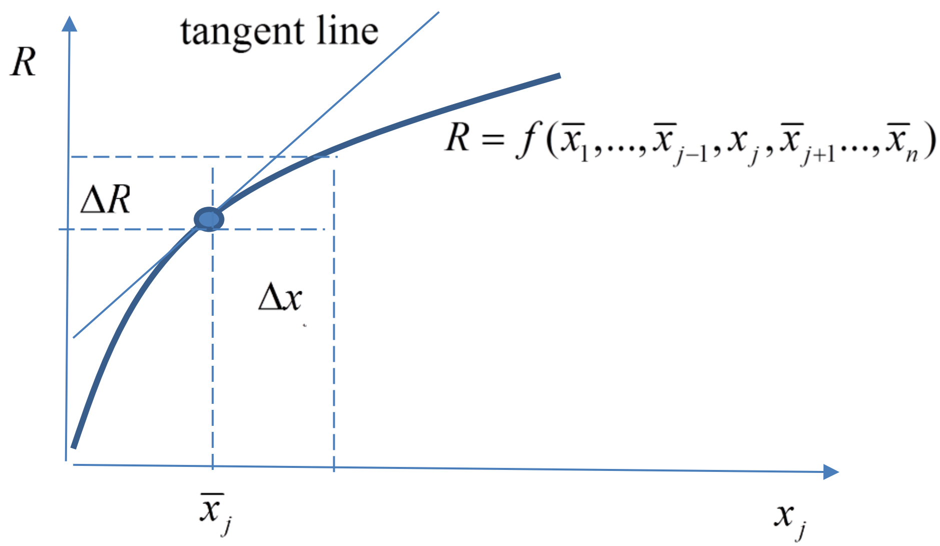

Fig. C-3: Revenue as a function of x_j

We have argued that

$$\Delta R = \frac{\partial f}{\partial x_j}(\bar{x})\Delta x_j \text{ and } \Delta x_j = \frac{\Delta b}{\frac{\partial g}{\partial x_j}(\bar{x})}$$

Combining these results, if the resource increment Δb is used to increase x_j , then the increase in revenue is

$$\Delta R = \frac{\partial f}{\partial x_j}(\bar{x})\Delta x_j = \frac{\frac{\partial f}{\partial x_j}(\bar{x})}{\frac{\partial g}{\partial x_j}(\bar{x})}\Delta b \ .$$

*

We have argued that

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$$\Delta R = \frac{\partial f}{\partial x_j}(\bar{x})\Delta x_j = \frac{\frac{\partial f}{\partial x_j}(\bar{x})}{\frac{\partial g}{\partial x_j}(\bar{x})}\Delta b \ .$$

We divide by Δb to get the marginal revenue product of the resource

$$MRP_j \equiv \frac{\Delta R}{\Delta b} = \frac{\frac{\partial f}{\partial x_j}(\bar{x})}{\frac{\partial g}{\partial x_j}(\bar{x})}$$

Case (i) $\bar{x}_i, \bar{x}_j > 0$

Suppose that the marginal revenue product is strictly lower for product j than for product i

$$MRP_j(\bar{x}) - MRP_i(\bar{x}) < 0.$$

We can decrease the allocation of the resource to commodity j by Δb and so lower revenue by

$$\Delta R = MRP_j(\bar{x})\Delta b \ .$$

*

Case (i) $\bar{x}_i, \bar{x}_j > 0$

Suppose that the marginal revenue product is strictly lower for product j than for product i

$$MRP_i(\bar{x}) - MRP_j(\bar{x}) > 0.$$

We can decrease the allocation of the resource to commodity j by Δb and so lower revenue by

$$\Delta R = MRP_j(\bar{x})\Delta b \; .$$

We can then use the Δb to increase x_i. The net gain is then

$$\Delta R = MRP_i(\bar{x})\Delta b - MRP_j(\bar{x})\Delta b = [MRP_i(\bar{x}) - MRP_j(\bar{x})]\Delta b > 0 \; .$$

It follows that $\bar{x}$ is not profit maximizing. Thus for $\bar{x}$ to solve the maximization problem we have the following necessary condition.

If $\bar{x}_i, \bar{x}_j > 0$, then $MRP_j(\bar{x}) = MRP_i(\bar{x})$

Let λ be the equalized marginal revenue product. The necessary conditions can be written as follows:

If $\bar{x}_i, \bar{x}_j > 0$ then $MRP_j(\bar{x}) = \dfrac{\dfrac{\partial f}{\partial x_j}(\bar{x})}{\dfrac{\partial g}{\partial x_j}(\bar{x})} = \lambda$.

Case (ii) $\bar{x}_i = 0$, $\bar{x}_j > 0$.

We can no longer decrease x_i and increase x_j but we can do the reverse, increasing x_i and decreasing x_j . Arguing as above, the net gain is

$$MRP_i(\bar{x})\Delta b - MRP_j(\bar{x})\Delta b = [MRP_i(\bar{x}) - MRP_j(\bar{x})]\Delta b \ .$$

If $\bar{x}$ maximizes revenue then this change cannot increase revenue. Therefore

$$MRP_i(\bar{x}) - MRP_j(\bar{x}) \le 0 \ .$$

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That is

$$MRP_i(\bar{x}) \le MRP_j(\bar{x}) = \lambda$$

Therefore

If $\bar{x}_i = 0$ and $\bar{x}_j > 0$, then $MRP_i(\bar{x}) = \dfrac{\frac{\partial f}{\partial x_i}(\bar{x})}{\frac{\partial g}{\partial x_i}(\bar{x})} \le \lambda$

Example: $\underset{x \geq 0}{Max}\{U(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 \mid p_1 x_1 + p_2 x_2 \leq I\}$

$$MRP_1(\bar{x}) = \frac{\frac{\alpha_1}{x_1}}{p_1} = \frac{\alpha_1}{p_1 x_1}\ ,\ MRP_2(\bar{x}) = \frac{\frac{\alpha_2}{x_2}}{p_2} = \frac{\alpha_2}{p_2 x_2}$$

Case (i) $x_1, x_2 > 0$

$$MRP_1(\bar{x}) = \frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = MRP_2(\bar{x})\ .$$

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Example: $\underset{x\geq 0}{Max}\{U(x)=\alpha_1 \ln x_1+\alpha_2 \ln x_2 \mid p_1x_1+p_2x_2\leq I\}$

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Ratio Rule: If $\frac{a_1}{b_1}=\frac{a_2}{b_2}$ and $b_1+b_2\neq 0$ then $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_1+a_2}{b_1+b_2}$.

Therefore

$$\frac{\alpha_1}{p_1x_1}=\frac{\alpha_2}{p_2x_2}=\frac{\alpha_1+\alpha_2}{p_1x_1+p_2x_2}=\frac{\alpha_1+\alpha_2}{I}$$

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Case (ii) $\bar{x}_i=0,\ \bar{x}_j>0$.

Exercise: Show that the necessary conditions cannot be satisfied in this case

D. Constrained Optimization with multiple constraints –an intuitive approach

$$\underset{x\geq 0}{Max}\{f(x)\,|\,b_i - g_i(x)\geq 0,\ \ i=1,...,m\}.$$

Economic Interpretation of maximization problem

profit maximizing multi-product firm with fixed inputs.

x = vector of outputs $\quad x\geq 0$

$f(x)$ revenue

$b=(b_1,...,b_m)$ = vector of inputs (fixed in short run)

$g(x)=(g_1(x),...,g_m(x))$ inputs needed to produce output vector x

D. Constrained Optimization – an intuitive approach

$\underset{x\geq 0}{Max}\{f(x)\,|\,b_i - g_i(x) \geq 0,\ i=1,...,m\}$. In vector notation $\underset{x\geq 0}{Max}\{f(x)\,|\,b-g(x)\geq 0\}$

Economic Interpretation of maximization problem

profit maximizing multi-product firm with fixed inputs.

x = vector of outputs $\quad x \geq 0$

$f(x)$ revenue

b = vector of inputs available (fixed in short run)

$g(x)$ = vector of inputs needed to produce output vector x

constraints: $\quad g(x) \leq b$.

Example: m linear constraints.

Each unit of x_j requires a_{ij} units of resource b_i .

One constraint:

Suppose that $\bar{x}$ solves the optimization problem.

If the firm increases x_j, the direct effect on profit is the marginal revenue $\dfrac{\partial f}{\partial x_j}$.

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The net gain to increasing x_j is therefore

$$\frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x})$$

If the optimum for commodity *j*, $\bar{x}_j$, is strictly positive, this marginal net gain must be zero.

That is

$$\bar{x}_j > 0 \Rightarrow \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) = 0$$

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Summarizing

$\frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \le 0$, with equality if $\bar{x}_j > 0$.

Since $\bar{x}$ must be feasible $b - g(\bar{x}) \geq 0$.

Moreover, we have defined λ to be the opportunity cost of additional resource use.

Then if not all the resource is used, λ must be zero.

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Summarizing,

$b - g(\bar{x}) \geq 0$, with equality if $\lambda > 0$.

Multiple constraints:

Introduce a shadow price for each constraint.

The marginal net gain to increasing x_j is then

$$\frac{\partial f}{\partial x_j}(\overline{x}) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(\overline{x}) = \frac{\partial f}{\partial x_j}(\overline{x}) - \lambda \cdot \frac{\partial g}{\partial x_j}(\overline{x}).$$

The Intuitive argument then proceeds as in the one constraint case.

There is a convenient way to remember these conditions. First write the *i*-th constraint in the form $h_i(x) - b_i - g_i(x) \geq 0$, $i = 1,...,m$. In vector notation $h(x) \geq 0$. Thus in our example we write the constraints as $h(x) = b - g(x) \geq 0$.

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Then introduce a vector of "Lagrange multipliers" or shadow prices λ and define the Lagrangian

$$\mathcal{L}(x,\lambda) = f(x) + \lambda \cdot h(x)$$

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$$\mathcal{L}(x,\lambda)=f(x)+\lambda\cdot h(x)$$

The first order conditions are then all restrictions on the partial derivatives of $\mathcal{L}(x,\lambda)$.

(i) $\dfrac{\partial\mathfrak{L}}{\partial x_j}=\dfrac{\partial f}{\partial x_i}+\lambda\cdot\dfrac{\partial h}{\partial x_j}\leq 0,$ with equality if $\bar{x}_j$ >0, $j=1,...,n$.

(ii) $\dfrac{\partial\mathfrak{L}}{\partial\lambda_i}=h_i(\bar{x})\geq 0,$ with equality if λ_i >0, $i=1,...,m$.

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(i) $\dfrac{\partial \mathfrak{L}}{\partial x_j} = \dfrac{\partial f}{\partial x_i} + \lambda \cdot \dfrac{\partial h}{\partial x_j} \leq 0$, with equality if $\bar{x}_j > 0$, $j = 1,...,n$.

(ii) $\dfrac{\partial \mathfrak{L}}{\partial \lambda_i} = h_i(\bar{x}) \geq 0$, with equality if $\lambda_i > 0$, $i = 1,...,m$.

Equivalently,

(i) $\dfrac{\partial \mathfrak{L}}{\partial x}(\bar{x},\lambda) \leq 0$ and $\bar{x} \cdot \dfrac{\partial \mathfrak{L}}{\partial x_j}(\bar{x},\lambda) = 0$.

(ii) $\frac{\partial \mathfrak{L}}{\partial \lambda}(\bar{x},\lambda) \geq 0$ and $\lambda \cdot \frac{\partial \mathfrak{L}}{\partial \lambda}(\bar{x},\lambda) = 0$

Exercise: Solve the following problem. $Max\{U(x) = \ln x_1 + \ln(x_2 + 2x_3) \mid p_1x_1 + p_2x_2 + p_3x_3 \leq 60\}$

(i) if $p = (1,2,6)$ (ii) $p = (1,2,2)$ (iii) $p = (1,2,4)$

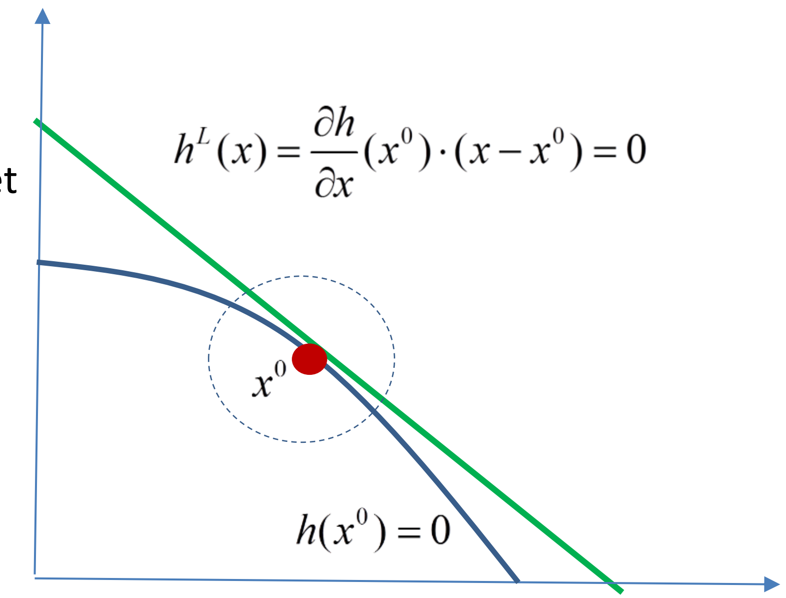
E. The Constraint Qualifications*

Suppose that the constraint $h_i(x) \geq 0$ is binding at x^0

Then the constraint is satisfied if $x \geq 0$ is in the superlevel set

$$\{x \in \mathbb{R}^n \mid h(x) \geq h(x^0) = 0\}$$

where x^0 is a boundary point.



We replace the constraint by its linear approximation.

***Technical section. Not required reading**

Linear approximation of a function $h(x)$ in a neighborhood of x^0

$$h_i^L(x) = h_i(x^0) + \frac{\partial h_i}{\partial x}(x^0) \cdot (x - x^0) = h_i(x^0) + \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j}(x^0)(x_j - x_j^0)$$

**

Linear approximation of a function $h(x)$

$$h_i^L(x) = h_i(x^0) + \frac{\partial h_i}{\partial x}(x^0)\cdot(x - x^0) = h_i(x^0) + \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j}(x^0)(x_j - x_j^0)$$

Value and partial derivatives of $h_i(x)$ at x^0 :

$$h_i(x^0),\ \frac{\partial h_i}{\partial x_j}(x^0),\ \ j = 1,...,n$$

*

Linear approximation of a function at x^0 (where $h_i(x^0)=0$

$$h_i(x)$$

$$h_i^L(x)=h_i(x^0)+\frac{\partial h_i}{\partial x}(x^0)\cdot(x-x^0)=\frac{\partial h_i}{\partial x}(x^0)\cdot(x-x^0)=\sum_{j=1}^{n}\frac{\partial h_i}{\partial x_j}(x^0)\cdot(x_j-x_j^0)$$

Value and partial derivatives of $h_i(x)$ at x^0 :

$$h_i(x^0),\ \frac{\partial h_i}{\partial x_j}(x^0),\ \ j=1,...,n$$

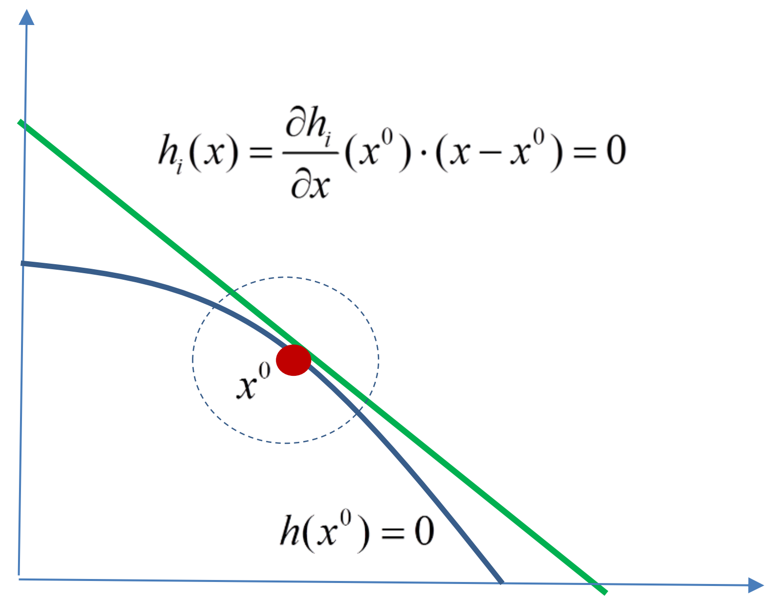
Value and partial derivatives of $h_i^L(x)$ at x^0 :

$$h_i(x^0),\ \frac{\partial h_i}{\partial x_j}(x^0),\ \ j=1,...,n$$

We replace a binding constraint

$$h_i(x) \geq h_i(x^0) = 0$$

by its linear approximation

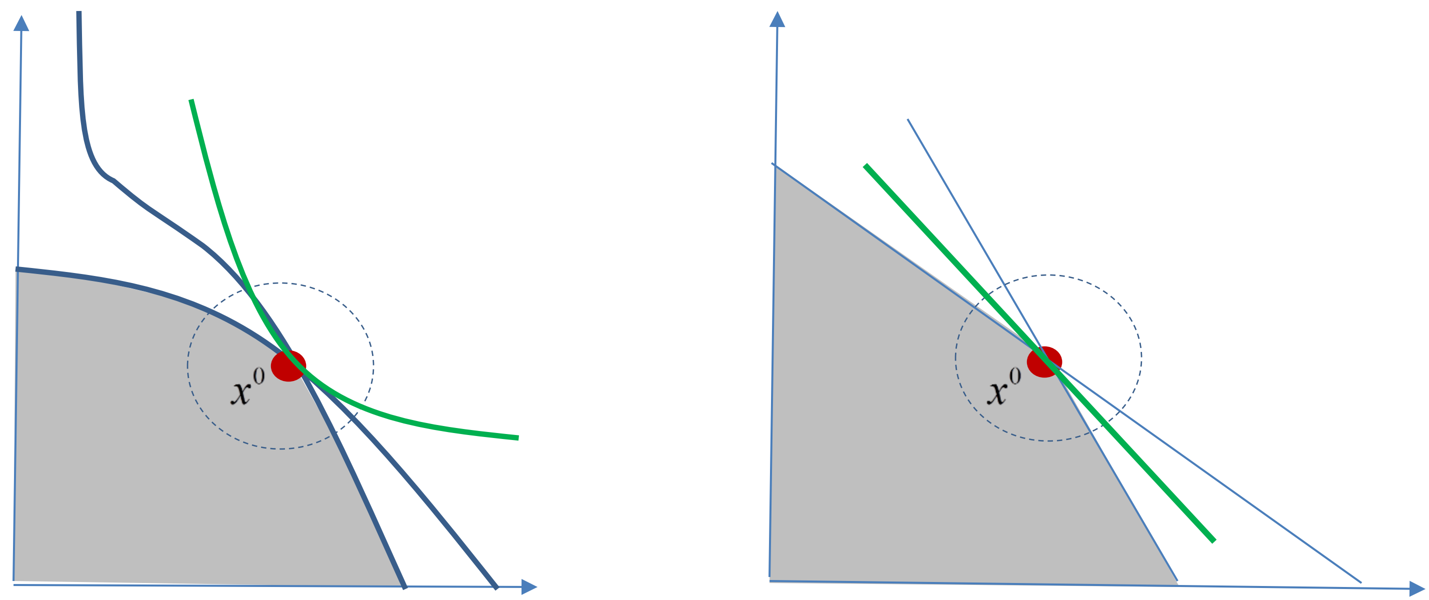


Two binding constraints

Suppose that two constraints are both binding at x^0 . This is depicted below.

Note that locally the linearized feasible set approximates the original feasible set.

(slopes of the original and the linearized feasible sets are the same at x^0)



Intuitively, replacing binding functions and the maximand by their linearized approximations should yield the necessary conditions.

This is almost true but the argument fails in two cases.

In each case the linearization drastically changes the constraints.

Case 1: Disappearing constraint

Suppose that the gradient vector is zero: $\frac{\partial h}{\partial x}(x^0) = 0$

The linearized constraint is

$$\sum_{j=1}^{n} \frac{\partial h}{\partial x_j}(x^0)(x_j - x_j^0) \geq 0$$

Thus if the gradient vector is zero the constraint disappears!

Example: Consider the following two optimizations problems

(i) $\underset{x\in\mathbb{R}^2_+}{Max}\{u(x)=x_1x_2\,|\,10-x_1-x_2\geq 0\}$ (ii) $\underset{x\in\mathbb{R}^2_+}{Max}\{u(x)=x_1x_2\,|\,(10-x_1-x_2)^3\geq 0\}$

You should convince yourself that the feasible sets are the same and so the solutions are the same. Given the symmetry of the problem consider $x^0=(5,5)$.

Write down the derivatives of the Lagrangian in each case. You will find the x^0 satisfies the intuitively derived necessary conditions in problem (i) but not in problem (ii)

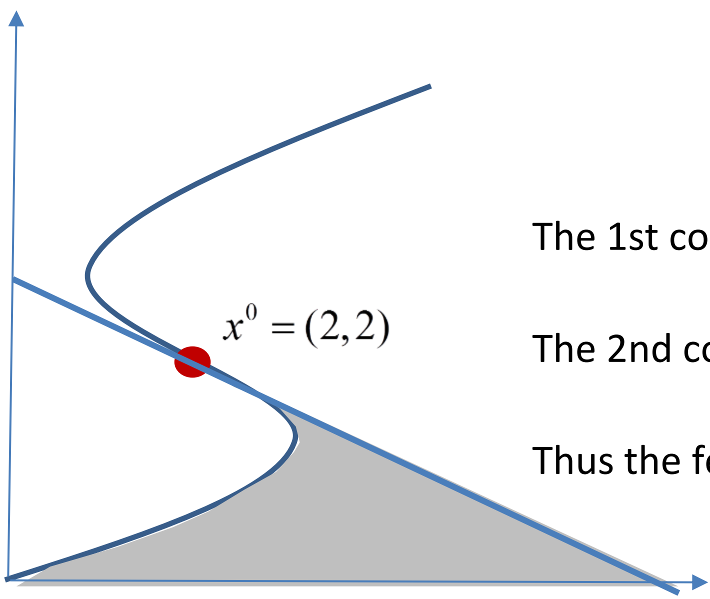
Case 2: Disappearing vertex

Suppose that two constraints are both binding at x^0 .

If, as depicted, the two constraints have the same slope at x^0 then there can be a problem.

To illustrate consider the following example with solution $x^0 = (2,2)$

$$Max\{ f(x) = x_2 \mid h_1(x) = x_1 - 9x_2 + 6x_2^{\ 2} - x_2^{\ 3} \geq 0,\ h_2(x) = 8 - x_1 - 3x_2 \geq 0\}$$



The 1st constraint holds for all x_1 to the right of the boundary $h_1(x) = 0$

The 2nd constraint holds for all x_1 to the left of the boundary $h_2(x) = 0$

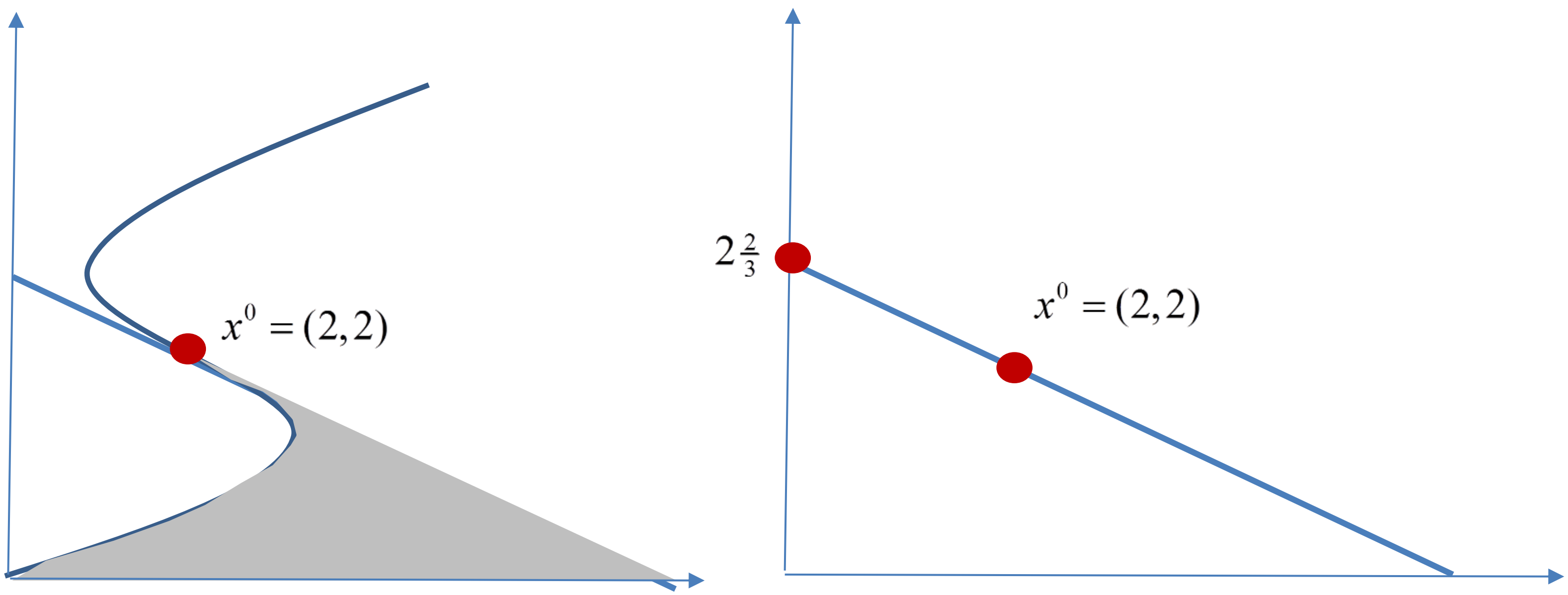
Thus the feasible set is the shaded area.

$$Max\{ f(x) = x_2 \mid h_1(x) = x_1 - 9x_2 + 6x_2^{\ 2} - x_2^{\ 3} \geq 0,\ \ h_2(x) = 8 - x_1 - 3x_2 \geq 0\}$$

We linearize the first constraint

$$\frac{\partial h_1}{\partial x}(x) = (1, -9 + 12x_2 - 3x_2^{\ 2}) \quad \frac{\partial h_1}{\partial x}(x^0) = (1,3) \quad \text{Then} \quad \frac{\partial h_1}{\partial x}(x^0)\cdot(x - x^0) = 1(x_1 - 2) + 3(x_2 - 2) \geq 0$$

i.e. $x_1 + 3x_2 \geq 8$



The linearized feasible set is the line $x_1 + 3x_2 = 8$. The vertex disappears.

Constraint Qualifications

Formally, we must check the following “constraint qualifications” If they are satisfied the intuitively derived conditions are indeed necessary conditions.

1. Suppose that constraint *i* is binding at x^0 but the gradient vector at x^0, $\frac{\partial h_i}{\partial x}(x^0) = 0$. Then there is no associated linearized constraint.

Thus to apply this approach we require that $\frac{\partial h_i}{\partial x}(x^0) \neq 0$ for each binding constraint

2. Suppose that the i-th constraint is binding if and only if $i \in I$. Check that the feasible set of binding linearized constraints has a non-empty interior. That is, there exists $\hat{x}$ such that

$\frac{\partial h_i}{\partial x}(x^0) \cdot (\hat{x} - x^0) > 0$ for all $i \in I$.

Constraint Qualifications

Define *X* to be the set of feasible vectors, that is $X = \{x \mid x \geq 0,\ h_i(x) \geq 0,\ i = 1,...,m\}$.

The constraint qualifications holds at $x^0 \in X$ if

(i) for each constraint that is binding at x^0 the associated gradient vector $\frac{\partial h_i}{\partial x}(x^0) \neq 0$.

(ii) $\bar{X}$, the set of non-negative vectors satisfying the linearized binding constraints has a non-empty interior.

As long as the constraint qualifications hold, the intuitively derived conditions are indeed necessary conditions. This is summarized below.

Proposition: Necessary Conditions for a Constrained Maximum

Suppose x^0 solves $\underset{x}{Max}\{f(x)\,|\,x \in X\}$. If the constraint qualifications hold at x^0 then there exists a vector of shadow prices $\lambda \geq 0$ such that

$$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial x_j}(x^0,\lambda) \leq 0, \quad j=1,...,n \text{ with equality if } \; x_j^0 > 0 \\ \text{and} \\ \dfrac{\partial \mathcal{L}}{\partial \lambda}(x^0,\lambda) \geq 0, \quad i=1,...,m \text{ with equality if } \; \lambda_i > 0 \end{array}\right\} \text{ Kuhn-Tucker conditions}$$

Since Kuhn and Tucker* were the first to provide a complete set of constraint qualifications, the first order conditions (FOC) are often called the Kuhn-Tucker Conditions.

*These conditions are also called the Karush-Kuhn-Tucker conditions