Econ 401A: Economic Theory Mid-term

Answers

1. Labor supply

(a) Let z be labor supply. Then $x_1 = 24 - z$

The key step is setting up the budget constraint.

$$p_2 x_2 = wz = w(24 - x_1)$$

Thus the budget constraint can be rewritten as follows:

$$wx_1 + p_2 x_2 = w24$$

value of consumption = value of endowments

Remark: It is not immediately intuitive to think of the wage as the price of time and the day as your endowment of time. But on the margin, if you decide to enjoy another hour of leisure, then your income goes down by w.

Transform the utility function into the following concave function

$$u = \ln U = \ln x_1 + \ln(a + x_2)$$

FOC

$$\frac{mu_1}{w} = \frac{mu_2}{p_2} \quad \text{i.e.} \quad \frac{1}{wx_1} = \frac{1}{p_2(a + x_2)} = \frac{2}{w24 + p_2 a}$$

Therefore

$$wx_1 = \tfrac{1}{2}(w24 + p_2 a)$$

Hence

$$x_1 = \frac{1}{2}[24 + \frac{p_2}{w}a)] = 12 + \frac{1}{2}\frac{p_2}{w}a \quad \text{and so} \quad z_1^* = 24 - x_1 = 12 - \frac{1}{2}\frac{p_2}{w}a$$

(b) As the price of leisure (the wage rises, the substitution effect is to consumer more of the relatively cheaper commodity, x_2, and less leisure. Hence labor supply rises. The worker is better off as the wage rises. So income must be increased to move the consumer to her new optimal choice. In this homothetic model consumption rises proportionally with income so the income effect is to increases both x_1 and x_2 .

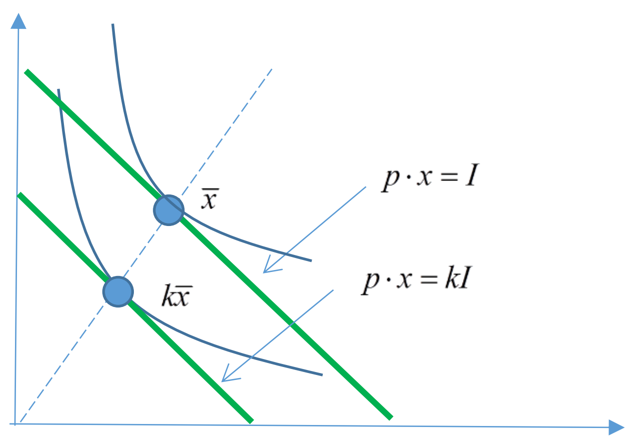
So the two effects are offsetting on leisure and hence on labor supply.

(c) $z_1^* = 12 - \frac{1}{2}\frac{p_2}{w}a$. This must be positive.

2. Walrasian Equilibrium (WE) in an exchange economy

(a) If a utility function is homothetic, then the marginal rate of substitution is constant along a ray. (See below.) Let $\overline{x}$ be the consumption choice of a consumer with income I . Consider two consumers with incomes I^A and I^B. Let I be the total. Then, for some k, $I^A = kI$ and $I^B = (1-k)I$. Therefore A's choice is $\overline{x}^A = k\overline{x}$ and B's choice is $\overline{x}^B = (1-k)\overline{x}$.

A's choice is depicted below.



It follows that $\overline{x}^A + \overline{x}^B = k\overline{x} + (1-k)\overline{x} = \overline{x}$

Thus rather than solve for the two consumption vectors and add them, it is equivalent to solve for the demand of one consumer with income $I = I^A + I^B$.

Remark: I did not emphasize this intuitive explanation strongly enough in my lecture on homothetic preferences. Next time I will do better.

For the representative consumer there can be no trade in a WE since there is no one to trade with. Thus $\overline{x} = \omega$.

(b) Consumer h, $h = 1,...,H$ has utility function $U(x_1^h, x_2^h) = \alpha_1 (x_1^h)^{1/2} + \alpha_2 (x_2^h)^{1/2}$.

$$MRS(x) = \frac{\alpha_1}{\alpha_2}\left(\frac{x_2^h}{x_1^h}\right)^{1/2}$$ and so is constant along a ray.

(b) For the representative consumer there can be no trade in a WE since there is no one to trade with.

$$\bar{x} = \omega = (100, 400) \ .$$

FOC

$$\frac{p_1}{p_2} = MRS(\bar{x}) = MRS(\omega) = \frac{\alpha_1}{\alpha_2}(\frac{400}{100})^{1/2} = \frac{2\alpha_1}{\alpha_2}$$

(c) $U(x^h, \pi) = \pi_1 (x_1^h)^{1/2} + \pi_2 (x_2^h)^{1/2}$

For an efficient allocation

$$MRS_A(x^A) = \frac{\pi_1}{\pi_2}(\frac{x_2^A}{x_1^A})^{1/2} = MRS_B(x^B) = \frac{\pi_1}{\pi_2}(\frac{x_2^B}{x_1^B})^{1/2}$$

Therefore

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{x_2^A + x_2^B}{x_1^A + x_1^B} = \frac{\omega_2}{\omega_1} = \frac{1}{4} \, .$$

Thus the PE allocations lie on the diagonal of the Edgeworth Box.

(d)

EITHER:

A WE is Pareto Efficient. Therefore risks are shared proportionally.

OR

$$MRS_A(\bar{x}^A) = \frac{p_1}{p_2} = MRS_B(\bar{x}^B)$$

Therefore a WE allocation is a PE allocation.

3. Multi-product monopoly

(a) If f and g are concave then for any x^0 , x^1 and convex combination x^λ ,

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$

and

$$g(x^\lambda) \geq (1-\lambda) g(x^0) + \lambda g(x^1) \, .$$

Summing these inequalities,

$$f(x^\lambda) + g(x^\lambda) \geq (1-\lambda)(f(x^0) + g(x^0)) + \lambda(f(x^1) + g(x^1))$$

Define the sum $h = f + g$.

Then

$$h(x^{\lambda}) \geq (1-\lambda)h(x^0) + \lambda h(x^1)$$

Remark: If you considered only functions of one commodity you had 0.5 points deducted.

(b) $\pi(q) = p(q_1)q_1 + p_2(q_2)q_2 - C(q) = 60q_1 - q_1^{\ 2} + 90q_2 - q_2^{\ 2} - (q_1+q_2)^2 - 2q_2^{\ 2}$ (*)

Using the second derivative condition is fine for all terms except $-(q_1+q_2)^2$. $q_1 + q_2$ is a concave function since it is the sum of two linear (and hence concave) functions. An increasing concave function of a concave function is concave.

Remark: If you missed this you had 1 point deducted.

(c) Since $\pi(q)$ is concave, the FOC are both necessary and sufficient

$$\frac{\partial \pi}{\partial q_1} = 60 - 2q_1 - 2(q_1+q_2) = 60 - 4q_1 - 2q_2 = 0$$

$$\frac{\partial \pi}{\partial q_1} = 90 - 4q_2 - 2(q_1+q_2) - 4q_2 = 90 - 2q_1 - 10q_2 = 0$$

The two FOC can be written as follows:

$$4q_1 + 2q_2 = 60 \qquad (1)$$

$$2q_1 + 10q_2 = 90 \qquad (2)$$

Double the second equation

$$4q_1 + 20q_2 = 180 \qquad 2\times\ (2)$$

$$4q_1 + 2q_2 = 60 \qquad (1)$$

Subtract the bottom equation from the top.

$18q_2 = 120$ so $q_2 = \frac{120}{18} = \frac{20}{3}$.

Then $2q_1 = 90 - 10q_2 = 90 - \frac{200}{3} = \frac{270-200}{3}$ so $q_1 = \frac{35}{3}$

(c) Substitute into (*) to obtain the maximized profit $\pi(q^*)$.

Remark: I feel bad. The profit maximizing outputs were intended to be integers to make this easy.

(d) As the constant K increases, the new profit is $\pi(q^*)-K$. So produce as long as $\pi(q^*)-K\geq 0$

4. Walrasian Equilibrium in a three commodity model with production

(a) All of input 2 is used in production so

$$z_2^*=16$$

Then

$$q=8z_1^{1/2}(16)^{1/2}=32z_1^{1/2}$$

Also

$$x_1=\omega_1-z_1=16-z_1$$

Then

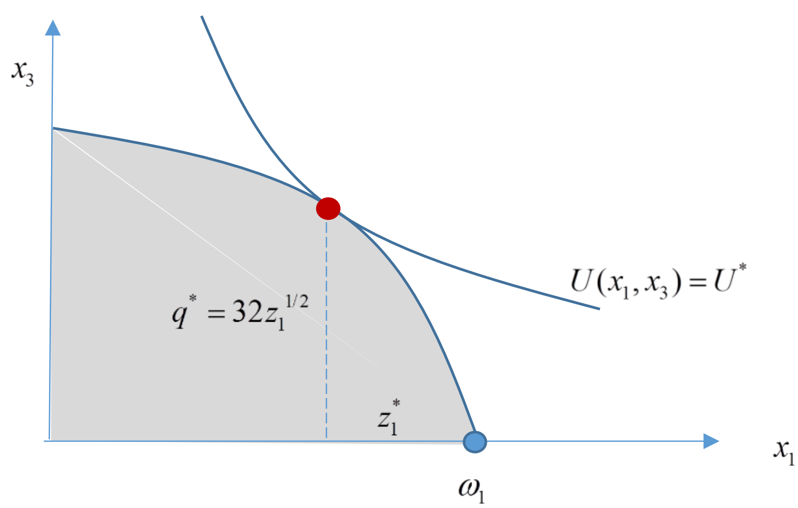
$$U=\ln(16-z_1)+2\ln 32z_1^{1/2}=\ln(16-z_1)+2\ln 32+\ln z_1$$

The derivative is

$$U'(z_1)=\frac{1}{16-z}-\frac{1}{z}=0 \text{ for a maximum.}$$

Solving, $z_1^*=8$ and so $q^*=32z_1^{1/2}=64\sqrt{2}$. We have already argued that $z_2^*=16$

(b)



(c) $\pi(z,p) = p_3 q - p_1 z_1 - p_2 z_2 = p_3 8 z_1^{1/2} z_2^{1/2} - p_1 z_1 - p_2 z_2$

FOC

$$\frac{\partial \pi}{\partial z_1}(z,p) = p_3 4 z_1^{-1/2} z_2^{1/2} - p_1 = p_3 4 (\frac{z_2}{z_1})^{1/2} - p_1 = 0$$

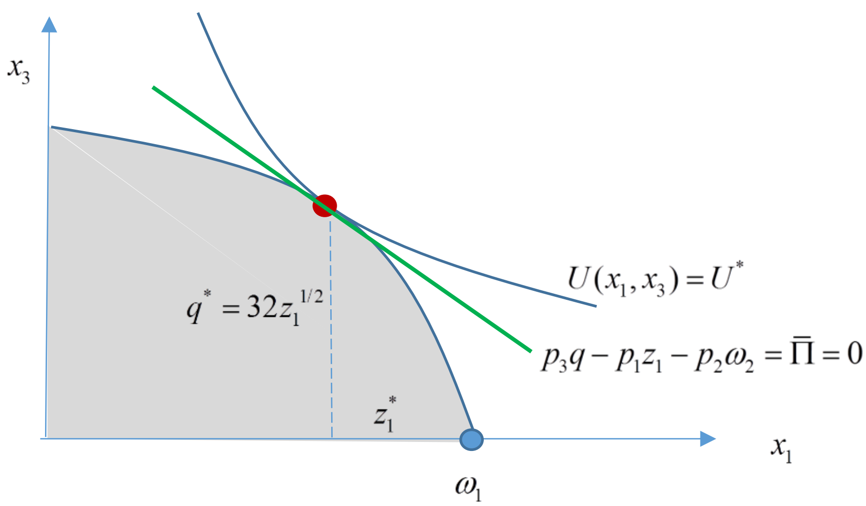
$$\frac{\partial \pi}{\partial z_2}(z,p) = p_3 4 z_1^{1/2} z_2^{-1/2} - p_2 = p_3 4 (\frac{z_1}{z_2})^{1/2} - p_2 = 0.$$

To support the optimum, these conditions must hold at $z^* = (8,16)$

$$\frac{\partial \pi}{\partial z_1}(z,p) = 4\sqrt{2} p_3 - p_1 = 0$$

$$\frac{\partial \pi}{\partial z_2}(z,p) = p_3 4 (\frac{1}{2})^{1/2} - p_2 = p_3 \frac{4}{\sqrt{2}} - p_2 = 2\sqrt{2} p_3 - p_2 = 0.$$

Therefore if $p_2 = 1$ it follows that $p_1 = 2$ and $p_3 = \frac{1}{2\sqrt{2}}$.



(d) Profit is $\pi = p_3 q^* - p_1 z_1^* - p_2 z_2^* = (\frac{1}{2\sqrt{2}})(64\sqrt{2}) - 2 \times 8 - 1 \times 16 = 0$.

Remark: The production function exhibits constant returns to scale. Thus if z yields a profit $2z$ yields twice as much. So WE profit must be zero.

(e) If you drew the diagram it was enough to point out that the maximum profit level set (the heavy green line) is also the boundary of the budget set for the representative consumer. From the figure, (x_1^*, x_3^*) solves the consumer's maximization problem.

Here is a more formal treatment.

For any feasible production plan profit cannot exceed maximized profit so

$$\pi = p_3 q - p_1 z_1 - p_2 z_2 \leq \bar{\Pi} = 0$$

Note that $z_1 = \omega_1 - x_1,\ z_2^* = \omega_2,\ q = x_3$. Therefore

$$p_3 x_3 - p_1(\omega_1 - x_1) - p_2\omega_2 \leq \bar{\Pi} .$$

Rearranging this expression,

$$p_1 x_1 + p_3 x_3 \leq p_2 \omega_2 + \bar{\Pi} .$$

Thus the maximum profit level set is the budget constraint for the representative consumer.

From the figure, the red marker is a WE since demand is equal to supply in all markets.