More uncertainty

A. Net demands

Consider a two commodity model.

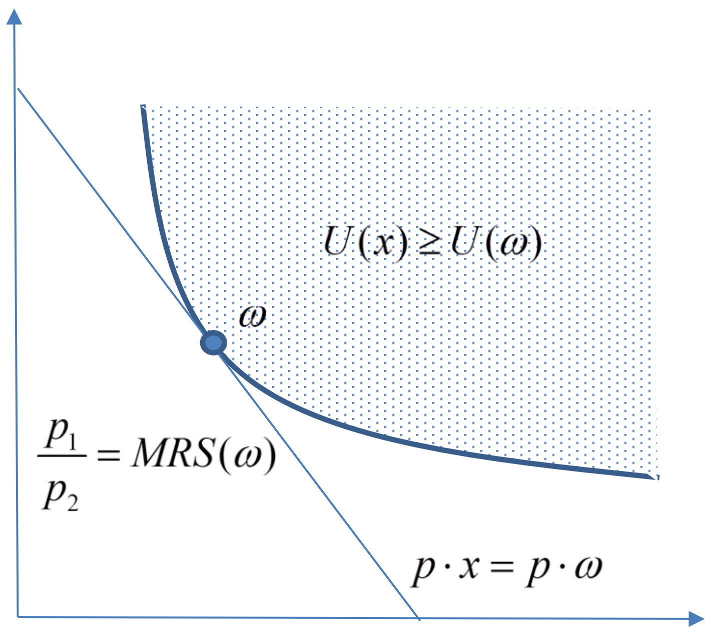
A consumer has an endowment $\omega = (\omega_1, \omega_2)$.

Let $x(p)$ be her market demand.

If $\dfrac{p_1}{p_2} = MRS(\omega)$ then $x(p) - \omega = 0$.

The consumer is better off not buying

more of either commodity.



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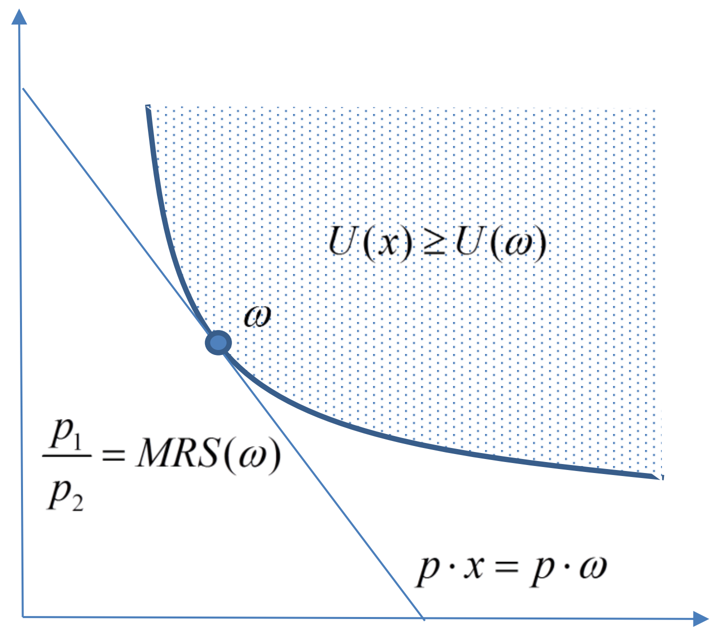

However, if the price ratio is not equal to $MRS(\omega)$, then the consumer will want to exchange some of one commodity for the other. These exchanges are called net trades

$$n_1(p) \equiv x_1(p) - \omega_1 \text{ and } n_2(p) \equiv x_2(p) - \omega_2$$

If $\frac{p_1}{p_2} < MRS(\omega)$

then the "net demands" are

$n_1(p) \equiv x_1(p) - \omega_1 > 0$ and $n_2(p) \equiv x_2(p) - \omega_2 < 0$

The net demand for commodity 1 is depicted in the lower diagram.

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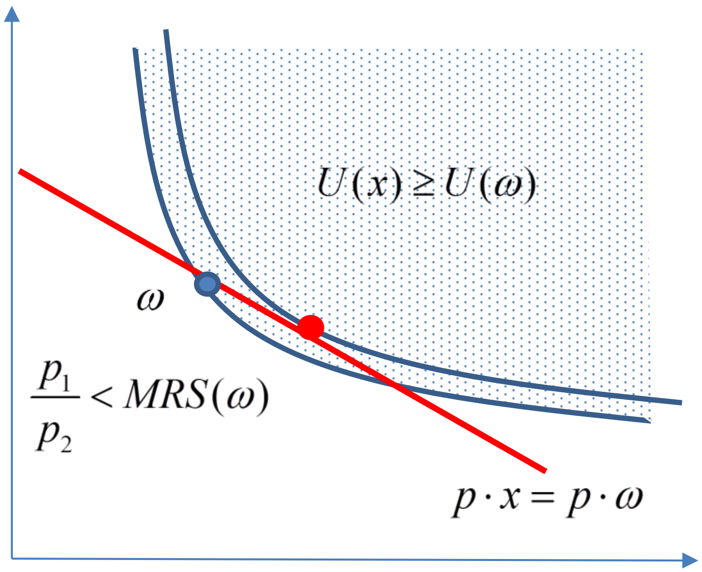

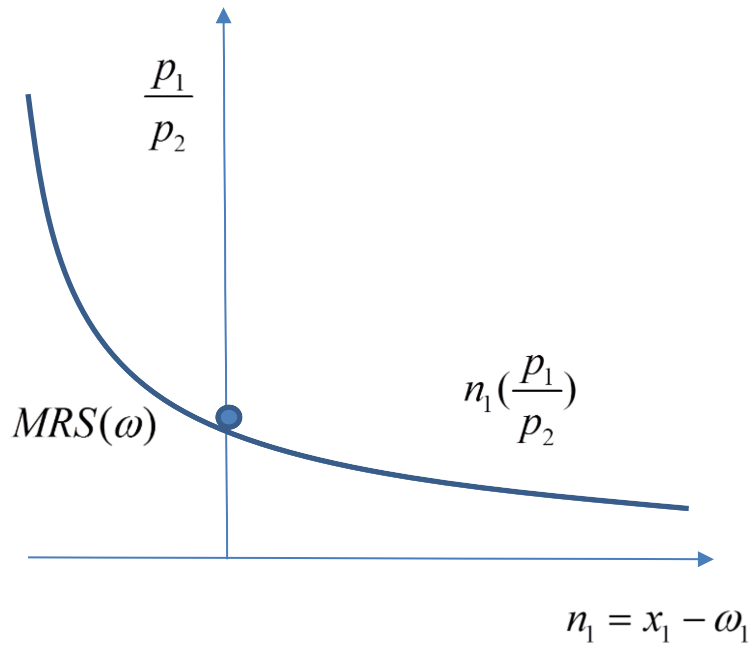

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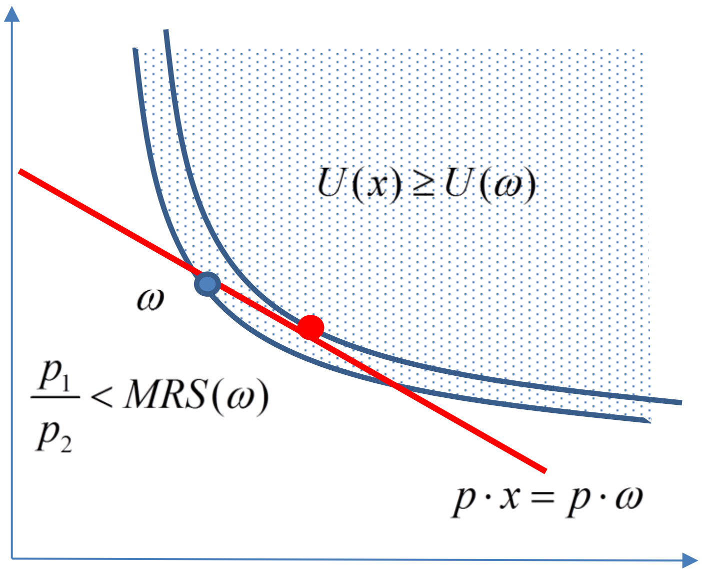
The net demand for commodity 1 is depicted in the lower diagram.

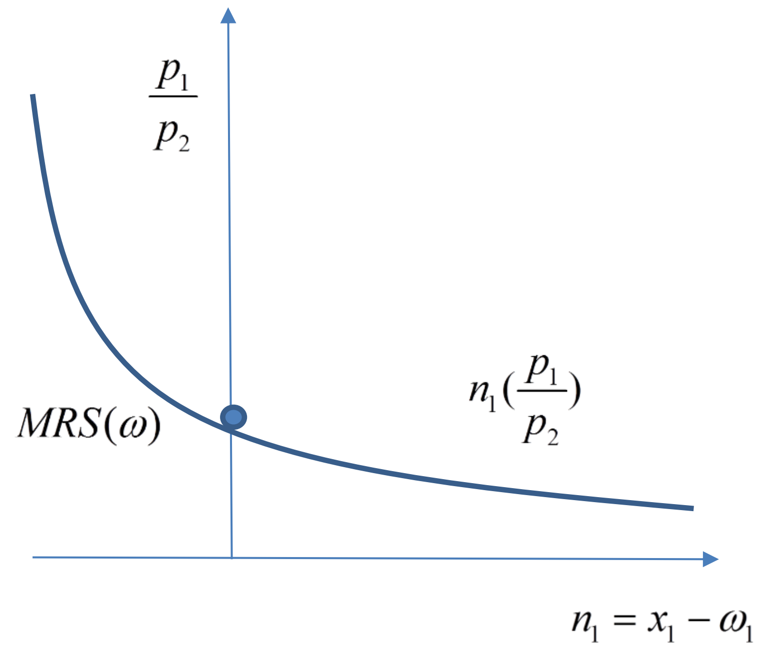
Note that

$p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0$.

Therefore

$$n_2 = x_2 - \omega_2 = -\frac{p_1}{p_2} n_1$$





Example:

$x(p)$ solves $\underset{x}{Max}\{U(x) = \alpha \ln x_1 + (1-\alpha)\ln x_2 \mid p \cdot \leq p \cdot \omega\}$

As you may confirm,

$$p_1 x_1 = \alpha(p_1\omega_1 + p_2\omega_2)$$

*

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As you may confirm,

$$p_1 x_1 = \alpha(p_1\omega_1 + p_2\omega_2)$$

Therefore

$$x_1 = \alpha(\omega_1 + \frac{p_2}{p_1}\omega_2)$$

and so

$$n_1 = x_1 - \omega_1 = -(1-\alpha)\omega_1 + \alpha\frac{p_2}{p_1}\omega_2$$

B. Betting on "The Game"

Alex is a rabid Bruins fan. He thinks that the probability of state 1 (Bruin victory) is high. Bev, who went to USC thinks that the probability of state 2 (Bruin defeat) is high. Alex's wealth is w^A and Bev's wealth is w^B.Their utility functions are as follows:

$$U_A(x;\pi^A) = \pi_1^A u_A(x_1^A) + \pi_2^A u_A(x_2^A) \text{ and } U_B(x^B;\pi^B) = \pi_1^B u_A(x_1^B) + \pi_2^B u_A(x_2^B)$$

$$MRS_A(x_1^A, x_2^A) = \frac{\pi_1^A}{\pi_2^A}\frac{u_A'(x_1^A)}{u_A'(x_2^A)} \quad \text{and} \quad MRS_B(x_1^B, x_2^B) = \frac{\pi_1^B}{\pi_2^B}\frac{u_B'(x_1^B)}{u_A'(x_2^B)}$$

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$$MRS_A(w^A, w^A) = \frac{\pi_1^A}{\pi_2^A} > \frac{\pi_1^B}{\pi_2^B} = MRS_B(w^B, w^B)$$

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$$MRS_A(w^A, w^A) = \frac{\pi_1^A}{\pi_2^A} > \frac{\pi_1^B}{\pi_2^B} = MRS_B(w^B, w^B)$$

If

$$\frac{p_1}{p_2} < \frac{\pi_1^A}{\pi_2^A}, \text{ then } n_1^A(\frac{p_1}{p_2}) > 0 \text{ and } n_2^A(\frac{p_1}{p_2}) < 0$$

$$\frac{p_1}{p_2} > \frac{\pi_1^B}{\pi_2^B}, \text{ then } n_1^B(\frac{p_1}{p_2}) < 0 \text{ and } n_2^A(\frac{p_1}{p_2}) > 0$$

A monopoly "bookmaker"

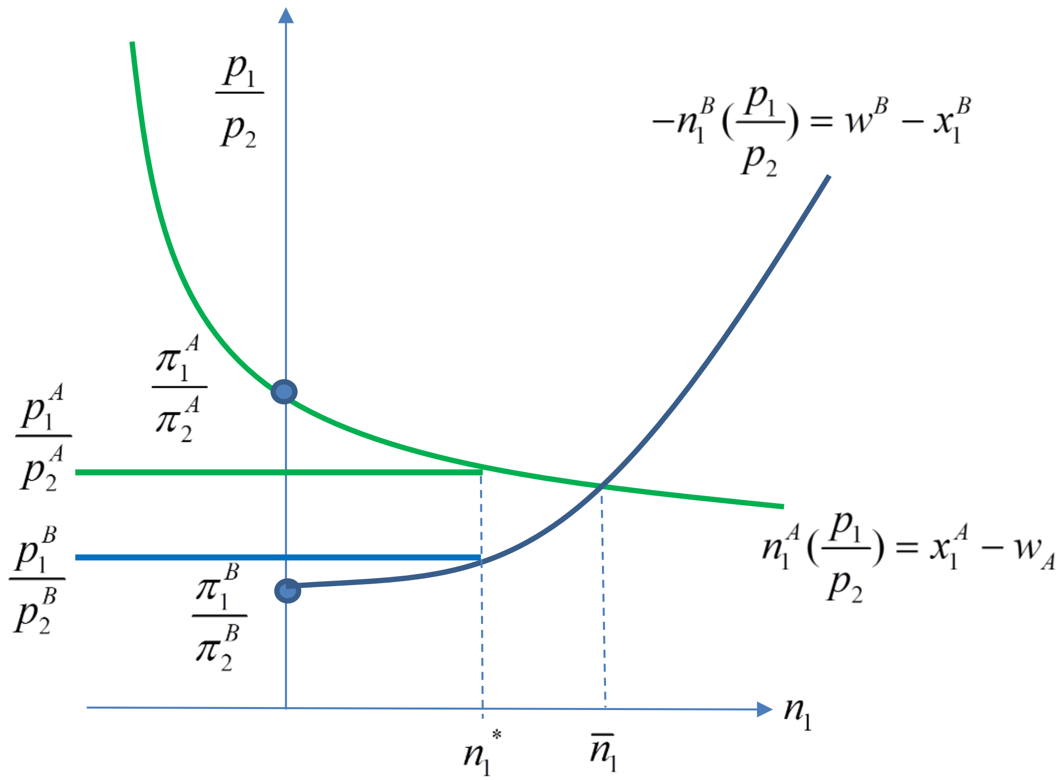
The bookmaker sets a price ratio

(the market odds for betting on a

Bruin victory) and another price ratio

(the market odds for betting on a

Trojan victory.



Competition among bookmakers lowers the difference in market odds. Ignoring bookmaker costs, the equilibrium volume of betting is $\bar{n}_1$

C. Globally more risk averse consumer

Consumer A 's aversion to a small risk is measured by his degree of risk aversion $-\dfrac{u_A''(x)}{u_A(x)}$.

Suppose that B is everywhere more risk averse than A

$$-\frac{u_B''(x)}{u_B(x)} > -\frac{u_A''(x)}{u_A(x)}$$

Intuitively, if Bev is less willing to take on any small risk,

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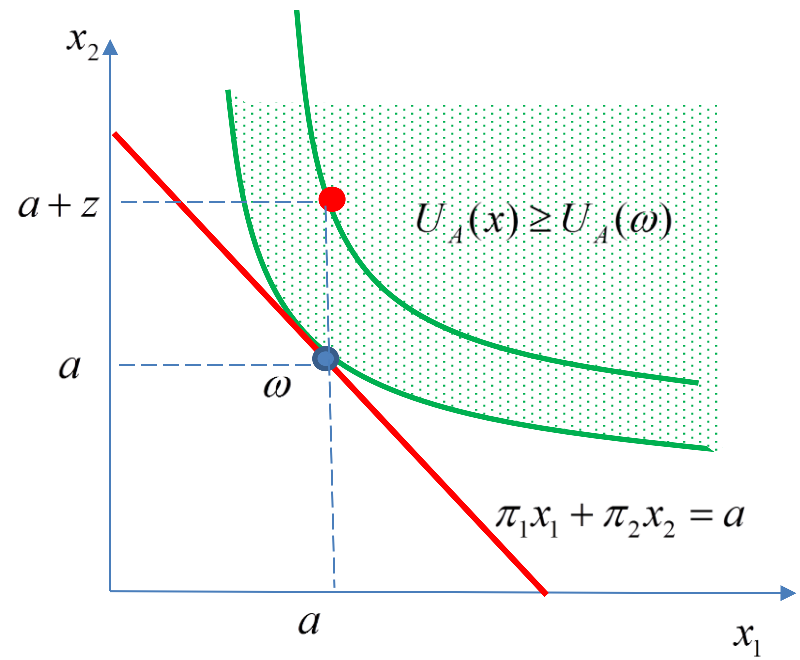
We now show that this is correct.

In the figure Alex has a riskless endowment, $\omega = (a,a)$.

His set of acceptable gambles is the superlevel set

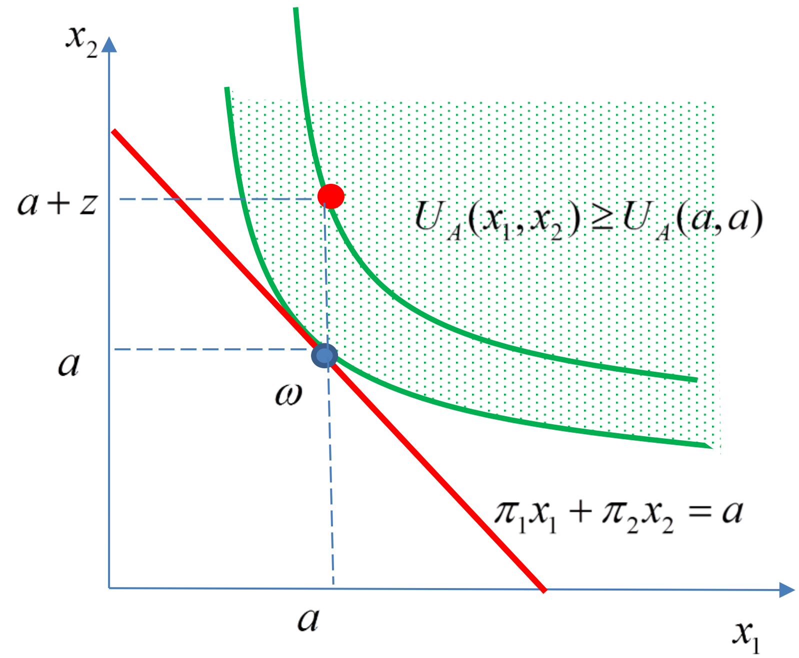
$$U_A(x_1, x_2) \geq U_A(a,a)$$



Consider the points $\omega = (a,a)$ and $x = (a, a+z)$ in the figure

$$MRS_A(a,a) = \frac{\pi_1}{\pi_2},$$

$$MRS_A(x_1,x_2) = \frac{\pi_1}{\pi_2} \frac{u_A{}'(a)}{u_A{}'(a+z)}$$



So see how rapidly the MRS increases with z

we differentiate by z.

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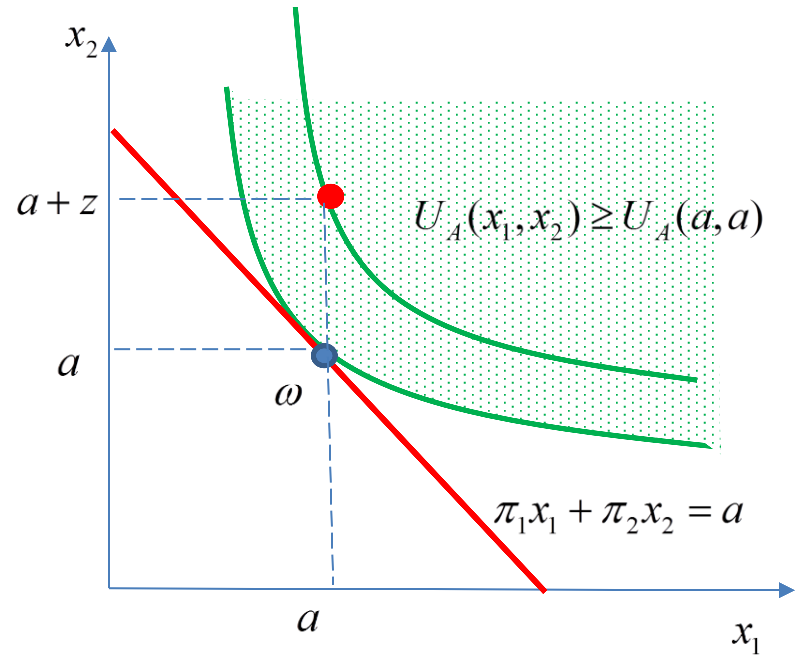

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Actually it is easier to take the logarithm

and then differentiate

$$\ln MRS_A(a,a+z) = \ln(\frac{\pi_1}{\pi_2}) + \ln u_A{}'(a) - \ln u_A{}'(a+z)$$

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Consider the points $\omega=(a,a)$ and $x=(a,a+z)$ in the figure

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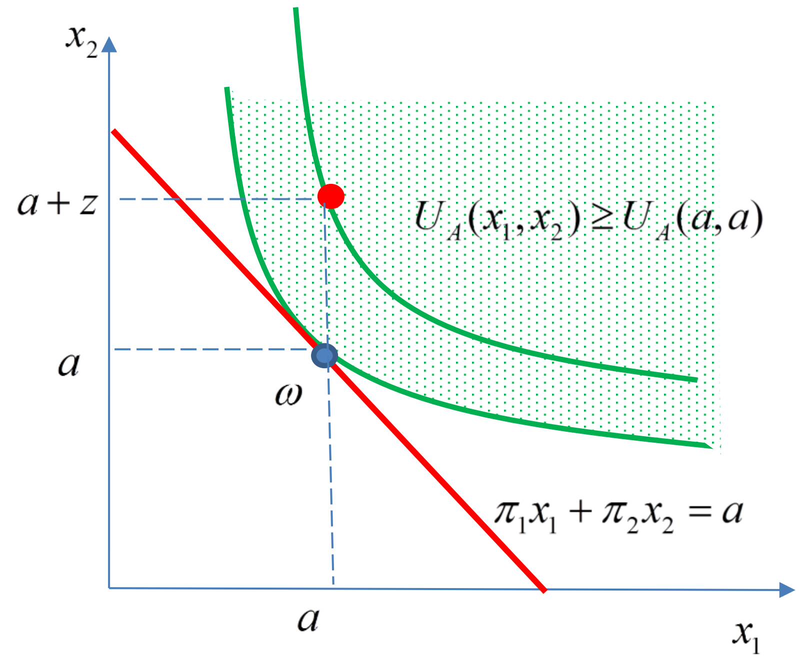

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$$\ln MRS_A(a,a+z)=\ln(\frac{\pi_1}{\pi_2})+\ln u_A{}'(a)-\ln u_A{}'(a+z)$$

$$\frac{d}{dx}\ln MRS_A(a,a+z)=-\frac{d}{dx}\ln u_A{}'(a+z)=-\frac{u_A{}''(a+z)}{u_A{}'(a+z)}.$$

Thus the greater is Alex's absolute aversion to a small risk, the more rapidly the logarithm of the MRS rises and hence the more rapidly the MRS rises.

It follows that if Bev has a higher absolute aversion

to a small risk, then her MRS rises more rapidly.

But $MRS_A(a,a) = \frac{\pi_1}{\pi_2} = MRS_B(a,a)$.

Therefore

$MRS_B(a, a+z) > MRS_A(a, +z)$, for any $z > 0$

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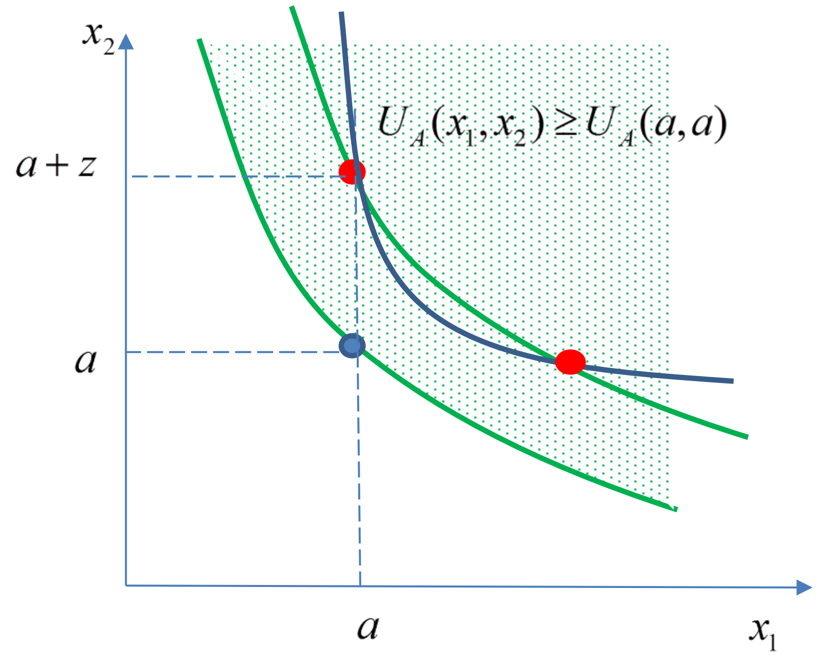

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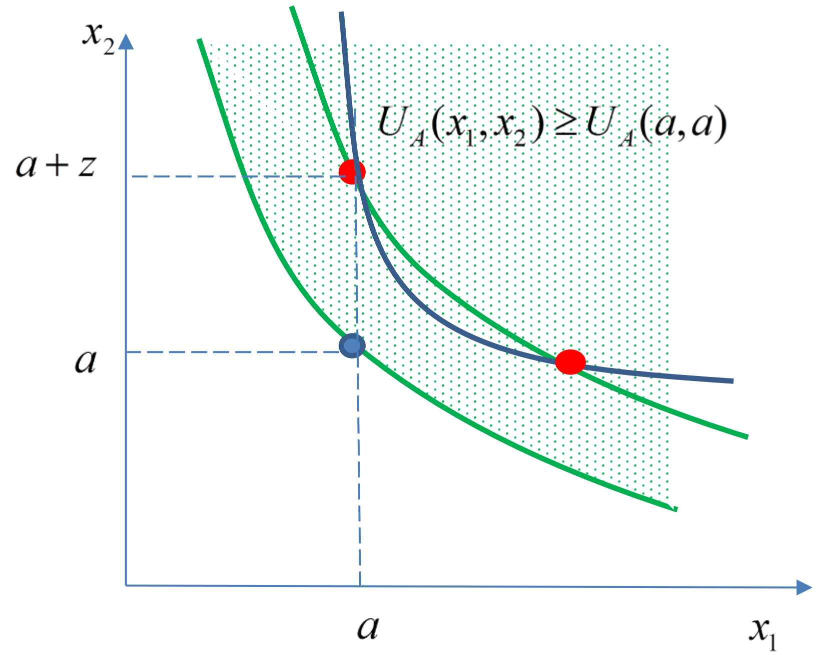
By an identical argument,

$MRS_B(a+z,a)<MRS_A(a+z,a)$, for any $z>0$

Thus any two level sets that intersect must do so as depicted.

*



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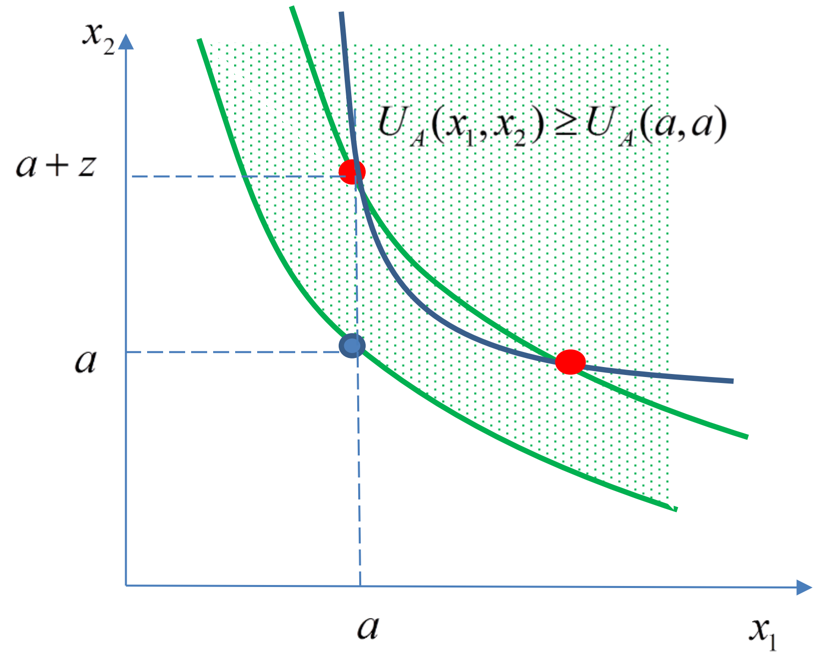
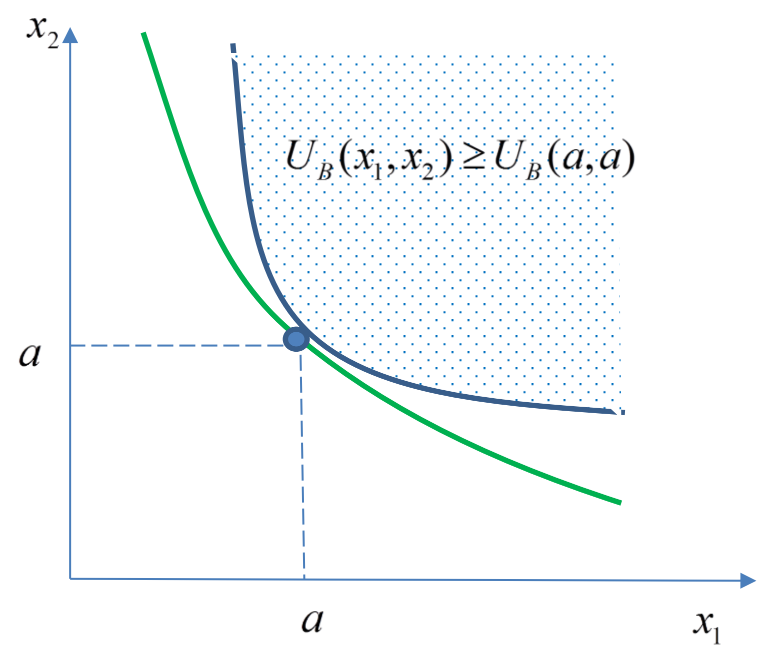




It follows that the blue acceptance set in the Lower figure must be smaller

Class Exercise

Draw an Edgeworth Box assuming that (i) Alex and Bev are both risk averse (ii) state 1 is the good state ($\omega_1 > \omega_2$) and (iii) they agree on the probabilities of the two states.

(a) Depict the PE allocations. Why must they lie between the two no risk lines (i.e. 45° lines).

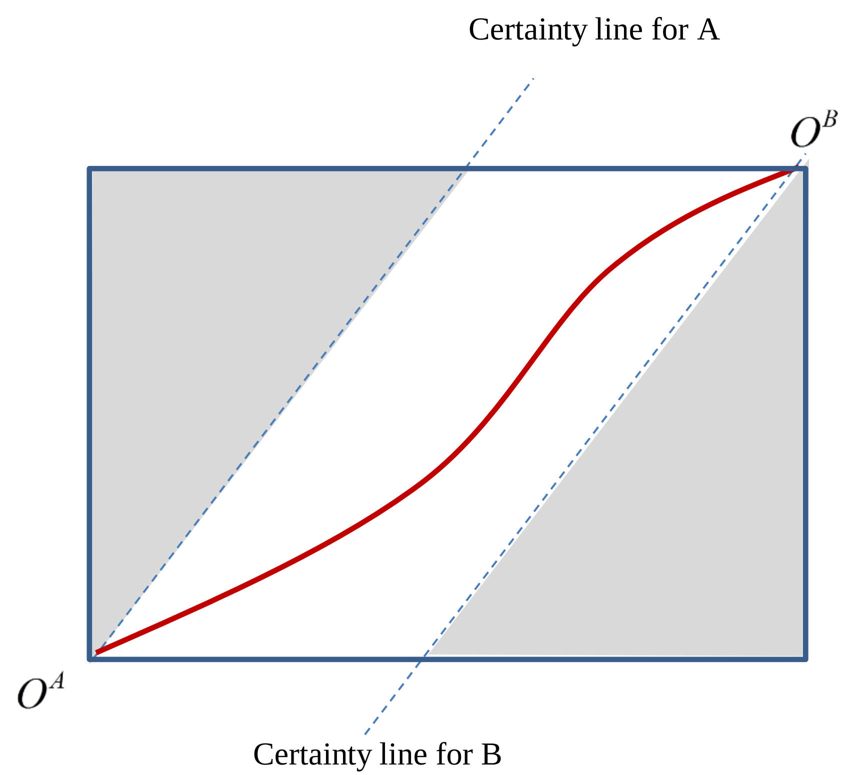
(b) Suppose that Alex's aversion to all small risks increases. What can be said about the shift in the PE allocations?

(c) If Alex is a net demander of state 1 claims what is the effect on the WE price ratio?

Answer:

The 45° line for each consumer is that consumer's "certainty line since $x_2 = x_1$. Along such a line a consumers marginal rate of substitution is equal to the ratio of the probabilities or "odds". In the lower shaded region $\hat{x}_2^A < \hat{x}_1^A$ and (using the inverted axes) $\hat{x}_2^B \geq \hat{x}_1^B$ so

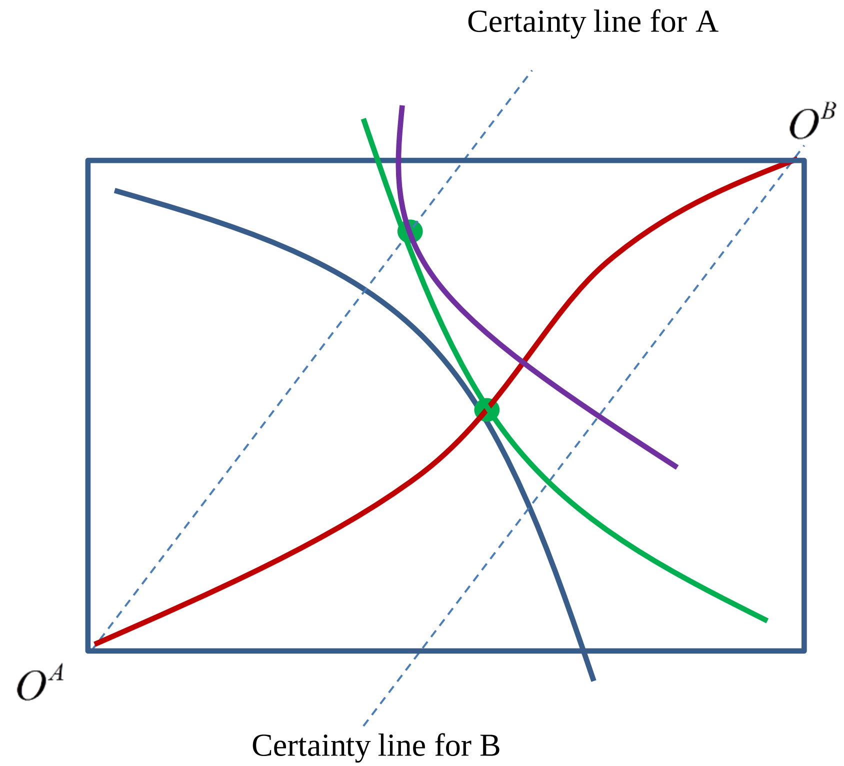
$$MRS(\hat{x}^A) < \frac{\pi_1}{\pi_2} \leq MRS(\hat{x}^B).$$



It follows that no point in the lower shaded region except the origin is a PE allocation.

The same argument can be used to show that no allocation in the upper shaded region except the origin for B is Pareto Efficient.

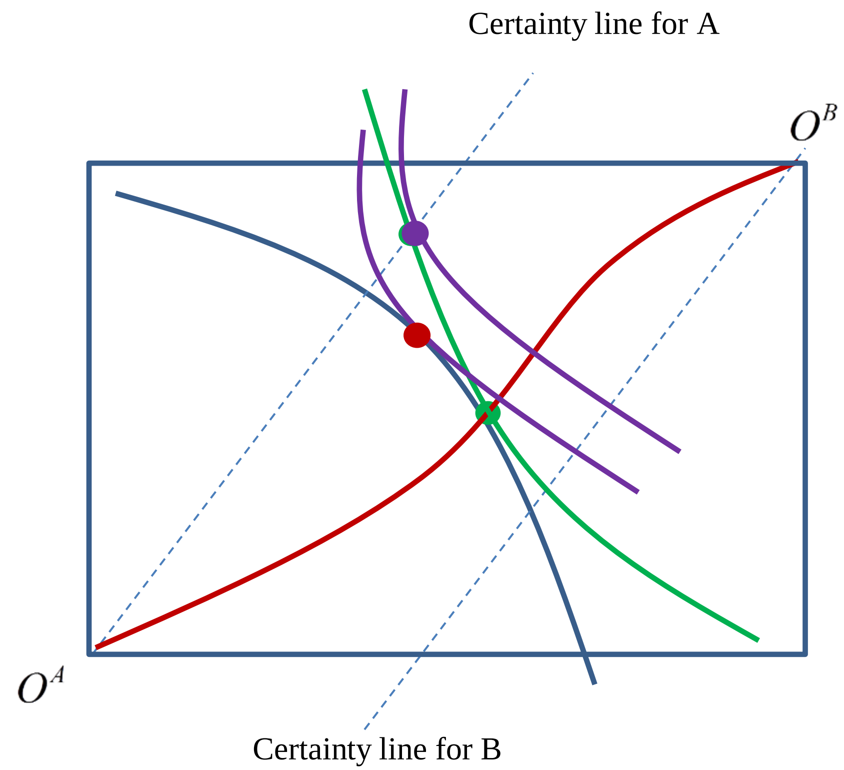
If Alex becomes more risk averse his MRS declines more rapidly below his certainty line. The green level set is therefore his initial level set and the purple one is the level set when his risk aversion increases.



Therefore the new PE allocation on Bev's Level set lies to the North-West. In the figure the red marker is to the left of the green marker.

Thus Alex's allocation is closer to his certainty line and Bev's is further from her certainty line.

The higher is Alex's aversion to risk, the less of the risk he shares.



E. Asset markets

Consider Alex and Bev on their South Pacific island. Alex's state contingent endowment is $\omega^A = (1500, 300)$ while Bev's state contingent endowment is $\omega^B = (500, 700)$. Thus $\omega = (2000, 1000)$ so there is aggregate risk. Each has the same logarithmic expected utility function $u(x) = \ln x$. The probability of the two states is $\pi = (\frac{2}{3}, \frac{1}{3})$

In the coconut plantation interpretation, in state 1 Alex has a catastrophic loss of 1200 palms. In state 2 Bev has a loss of 200.

Class Exercise

(a) Solve for the state claim equilibrium price ratio $\dfrac{p_1}{p_2}$ and WE allocation $\{\overline{x}^A(p), \overline{x}^B(p)$.

(b) Draw the budget line for Alex and depict ω^A and $\overline{x}^A$.

(c) Normalize and let $p_1 = 1$. Hence determine the market value of each plantation.

$P^A = p \cdot \omega^A$ and $P^B = p \cdot \omega^B$

The solution

Beliefs are identical and $u(x) = \ln x$.

Therefore both consumers have the

same homothetic utility function.

$U(x,\pi) = \pi_1 \ln x_1 + \pi_1 \ln x_2$

So the consumption choices $\overline{x}^A$ and $\overline{x}^B$

are on the same ray.

Budget constraints

$p \cdot x^A \leq p \cdot \omega^A = P_A$, $p \cdot x^B \leq p \cdot \omega^B = P_B$

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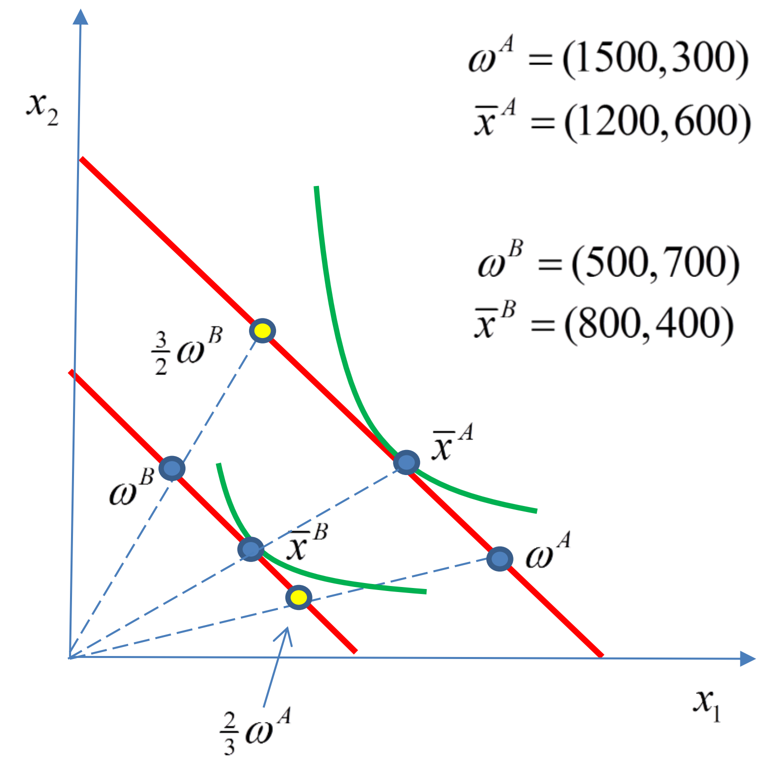

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$U(x,\pi) = \pi_1 \ln x_1 + \pi_1 \ln x_2$

So the consumption choices are on the

same ray.

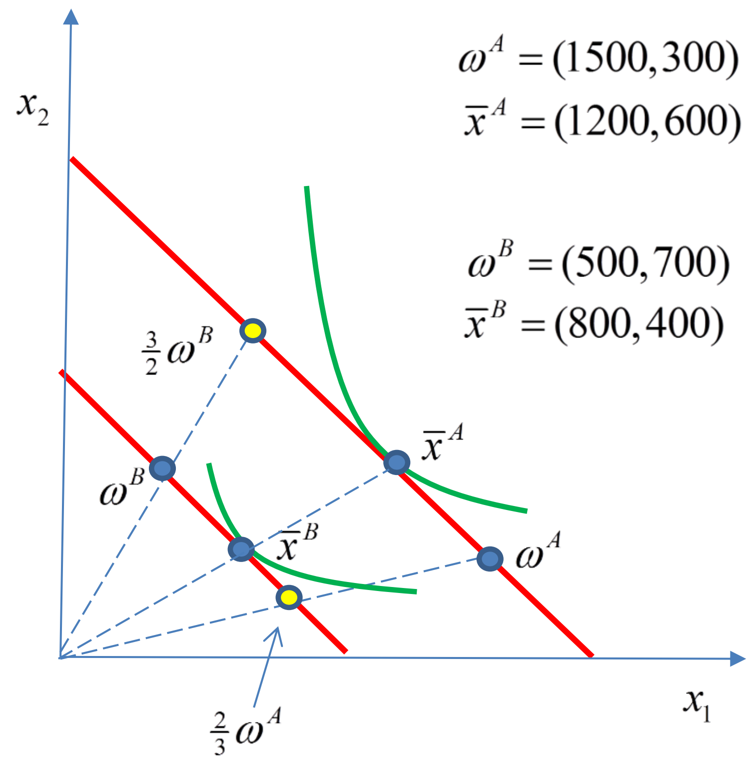
Budget constraints

$p \cdot x^A \leq p \cdot \omega^A = P_A$, $p \cdot x^B \leq p \cdot \omega^B = P_B$

In the figure $\dfrac{p_1}{p_2} = 1$

The asset prices are

$(P_A, P_B) = (1800,1200)$



Remark: Note that the yellow centered markers can be interpreted as undiversified portfolios.

Trading in assets $P=(P_A,P_B)=(1800,1200)$

Let $q=(q_1,q_2)$ be Bev's asset holdings.

Her initial holding is $\hat{q}=(1,0)$.

Then her portfolio constraint is

$$P\cdot q=P_Aq_A+P_Bq_B\le P\cdot\hat{q}=P_B$$

For our example $P=(P_A,P_B)=(1800,1200)$

so the portfolio constraint is

$$1800q_A+1200q_B=1200\ .$$

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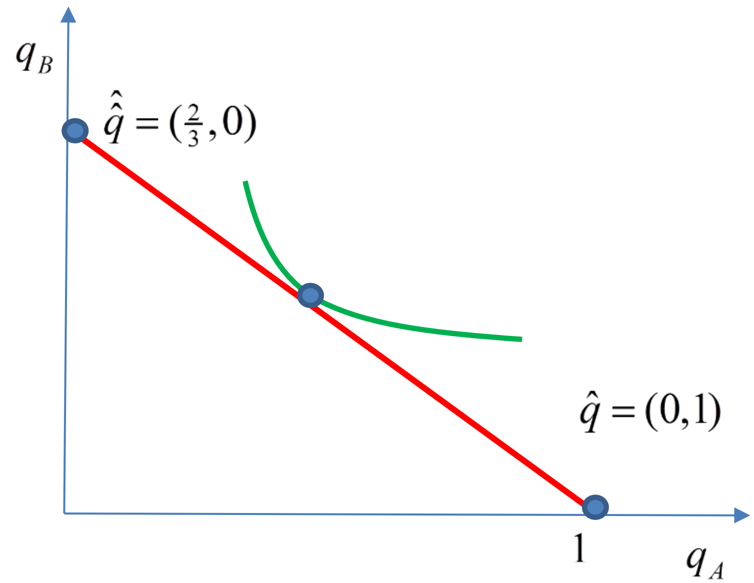

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For our example $P=(P_A,P_B)=(1800,1200)$

so the portfolio constraint is

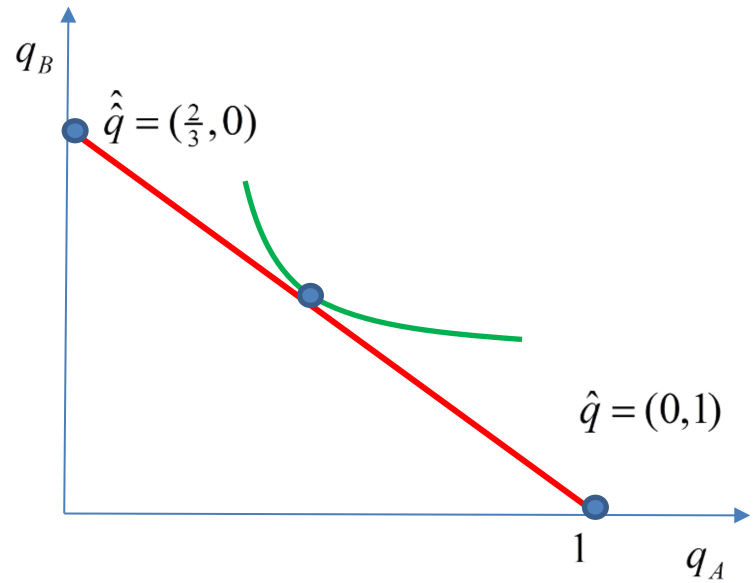
$$1800q_A+1200q_B=1200 \ .$$

Her final wealth in state s is

$$x_s=q_A\omega_s^A+q_B\omega_s^B$$

So expected utility is

$$U_B=\pi_1 u(x_1)+\pi_2 u(x_2)=\sum_{s=1}^{2}\pi_s u(q_A\omega_s^A+q_B\omega_s^B)$$



Undiversified portfolio returns for Bev

$P = (P_A, P_B) = (1800, 1200)$

$\hat{q} = (0,1)$, $\hat{x} = \omega^A$, Market value P_B

$\hat{\hat{q}} = (\frac{2}{3}, 0)$, $\hat{\hat{x}} = \frac{2}{3}\omega^B$, Market value $\frac{2}{3}P_A = P_B$

$P \cdot \hat{q} = P_B$

$P \cdot \hat{\hat{q}} = P_B$

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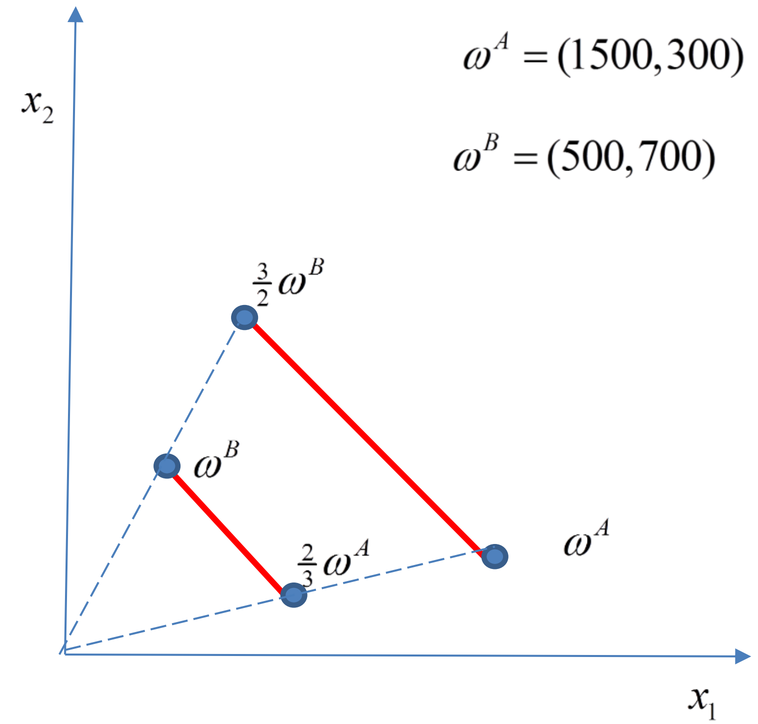


Undiversified portfolio returns for Bev

$P=(P_A,P_B)=(1800,1200)$

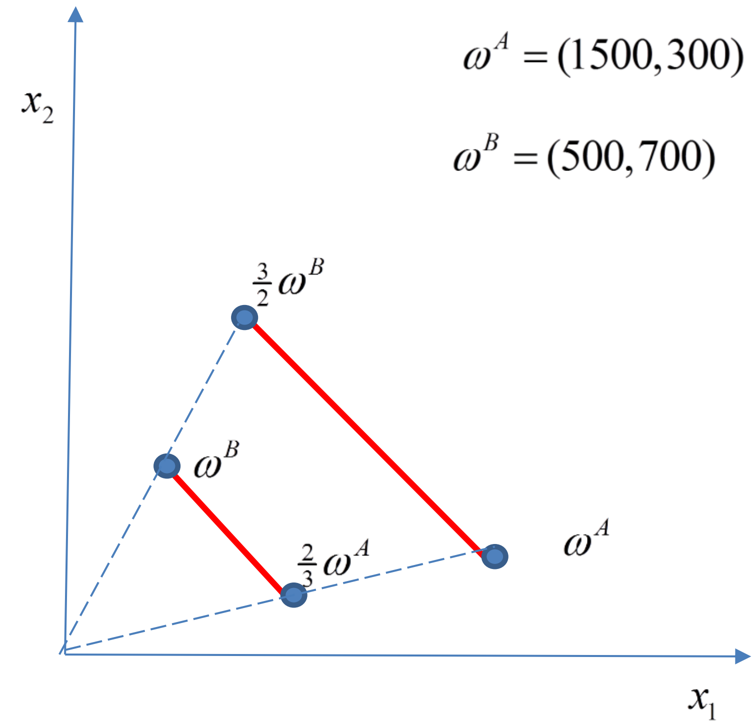
$\hat{q}=(0,1),\ \hat{x}=\omega^A,$ Market value P_B

$\hat{\hat{q}}=(\frac{2}{3},0),\ \hat{\hat{x}}=\frac{2}{3}\omega^B$, Market value $\frac{2}{3}P_A=P_B$

$P\cdot\hat{q}=P_B \quad \times\ (1-\lambda)$

$P\cdot\hat{\hat{q}}=P_B \quad \times\ \lambda$

$P\cdot((1-\lambda)\hat{\hat{q}}+\lambda\hat{q})=P_B \quad \times\ \lambda$



All convex combinations of the non-diversified portfolios have the same market value so are also feasible.

Thus the red lines are on the boundary of the feasible outcomes for Alex and Bev.

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Undivided portfolio returns for Bev

$P=(P_A,P_B)=(1800,1200)$

$\hat{q}=(0,1),\ \hat{x}=\omega^A,$ Market value $P_B=1200$

$\hat{\hat{q}}=(\frac{2}{3},0),\ \hat{\hat{x}}=\frac{2}{3}\omega^B$, Market value $\frac{2}{3}P_A=P_B$

$P\cdot\hat{q}=P_B \quad \times\ (1-\lambda)$

$P\cdot\hat{\hat{q}}=P_B \quad \times\ \lambda$

$P\cdot((1-\lambda)\hat{\hat{q}}+\lambda\hat{q})=P_B \quad \times\ \lambda$

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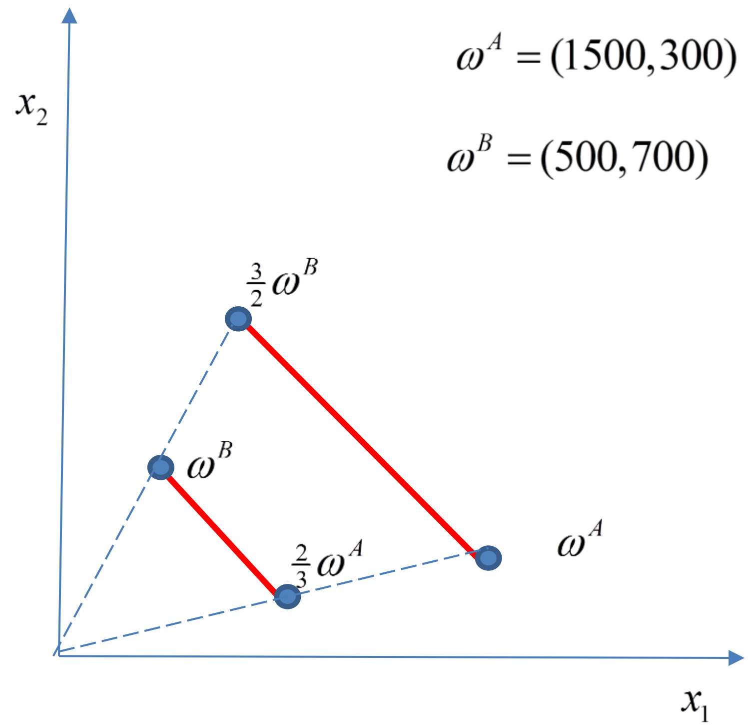

Thus the red lines are on the boundary of the feasible outcomes for Alex and Bev.

Remark: Short selling extends these lines to the boundary.

It follows that trading in asset markets can replicate trading in state claims markets.

General 2 state 2 asset model

Asset returns $z^A = (z_1^A, z_2^A)$ **,** $z^B = (z_1^B, z_2^B)$

Case 1: $\dfrac{z_2^A}{z_1^A} = \dfrac{z_2^B}{z_1^B}$ so $z_s^A = \theta z_s^B$ and so $P_A = \theta P_B$

Trading assets that differ only in scale does not change a consumer's state dependent consumption.

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Portfolios that yield a return in only one state

$q_A z_1^A + q_B z_1^B = 1$

$q_A z_2^A + q_B z_2^B = 0$

Equivalent to a claim to state 1

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Equivalent to a claim to state 2

*

With a little work we can solve the system of equations

$$\hat{q}_A = \frac{z_2^B}{\Delta},\ \hat{q}_B = \frac{-z_1^B}{\Delta}$$

$$\hat{\hat{q}}_A = -\frac{z_2^A}{\Delta},\ \hat{\hat{q}}_B = \frac{z_1^A}{\Delta}$$

Market value of these portfolios

$$\hat{q}_A = \frac{z_2^B}{\Delta},\ \hat{q}_B = \frac{-z_1^B}{\Delta} \qquad \hat{\hat{q}}_A = -\frac{z_2^A}{\Delta},\ \hat{\hat{q}}_B = \frac{z_1^A}{\Delta}$$

The cost of each of these portfolios is the price of a state claim

$$P_A\hat{q}_A + P_B\hat{q}_B = p_1 \qquad P_A\hat{\hat{q}}_A + P_B\hat{\hat{q}}_B = p_1$$

Proposition: Equivalence of asset markets and state claims markets

Let the $p = (p_1,..., p_S)$ be a WE price vector in a world with S states. Suppose there are S assets.

Asset a, $a = 1,...,S$ has returns $z^a = (z_1^a,..., z_S^a)$. Let $P_a = p \cdot z^a$ be the WE price of asset a, $a = 1,...,S$. Note that this mapping yields a unique asset price vector, $P = (P_1,..., P_S)$ for every state claims price vector p.

If the inverse mapping from asset prices to state claims prices is unique, then the state claims market equilibrium can be replicated by trading in asset markets.

Exercise: Pareto Efficiency with a risk neutral consumer

Alex is risk averse while Bev is risk neutral. They have the same beliefs.

Then Bev's utility function is linear

$$u(x) = a + bx \ .$$

$$U_B(x,\lambda) = \pi_1(a + bx_1) + \pi_2(a + bx_2)$$

$$= (\pi_1 + \pi_2)a + b(\pi_1 x_1 + \pi_2 x_2)$$

$$= a + b(\pi_1 x_1 + \pi_2 x_2) = a + b\mathbb{E}[x]$$

Such an individual cares only about the expected outcome.

Since the parameters a and b play no role in determining choice we choose $a = 0$ and $b = 1$. Then, for a risk neutral individual

$$u(x) = x$$

Characterize the PE allocations in the Edgeworth box given that Alex and Bev have the same beliefs.

Hence explain why the equilibrium price ratio is equal to the odds if Bev's endowment is sufficiently large relative to Alex's endowment.

E. Pareto Efficiency with a risk-neutral consumer

Review: An individual has preferences over prospects

We write this "consumption prospect" as follows:

$$(x;\pi) = (x_1, x_2; \pi_1, \pi_2)$$

If we make the usual assumptions about preferences, but now on prospects, it follows that preferences over prospects can be represented by a continuous utility function

$$U(x_1, x_2, \pi_1, \pi_2)$$

Given the independence axiom, one representation of these preferences is the expected utility representation, i.e. the utilities of each possible outcomes weighted by their probabilities

$$U(x_1, x_2, \pi_1, \pi_2) = \pi_1 u(x_1) + \pi_2 u(x_2) = \mathbb{E}[u(x)]$$

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With S rather than 2 outcomes the prospect is

$$(x;\pi) = (x_1, ..., x_S; \pi_1, ..., \pi_S)$$

And so expected utility is

$$U(x,\pi) = \pi_1 u(x_1) + ... + \pi_S u(x_S) = \mathbb{E}[u(x)]$$

We can think of a prospect as a random variable with S possible realizations. Each such realization is called a "state of the world" or "state".

Risk averse individual

If the utility function $u(x)$ is strictly concave then expected utility is strictly concave as it is the sum of strictly concave functions. Then

$$u((1-\lambda)x_1 + \lambda x_2) > (1-\lambda)u(x_1) + \lambda u(x_2) \text{ for all } \lambda \text{ between zero and one.}$$

Set $\lambda = \pi_2$. Then $1-\lambda = \pi_1$ and so

$$u(\pi_1 x_1 + \pi_2 x_2) > \pi_1 u(x_1) + \pi_2 u(x_2) \ .$$

Thus the individual is strictly better off if instead of facing an uncertain prospect, he receives its expectation with probability 1. That is, such an individual dislikes risk. We say that the individual exhibits risk aversion.

Risk neutral individual.

Suppose that the utility function is linear

$$u(x) = a + bx \ .$$

Then

$$U(x,\lambda) = \pi_1(a + bx_1) + \pi_2(a + bx_2)$$

$$= (\pi_1 + \pi_2)a + b(\pi_1 x_1 + \pi_2 x_2)$$

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$$u(x) = x$$

Level sets of a risk averse individual

In the figure the green line is a level set

for the individual's expected payoff

$$\mathbb{E}[x] = \pi_1 x_1 + \pi_2 x_2$$

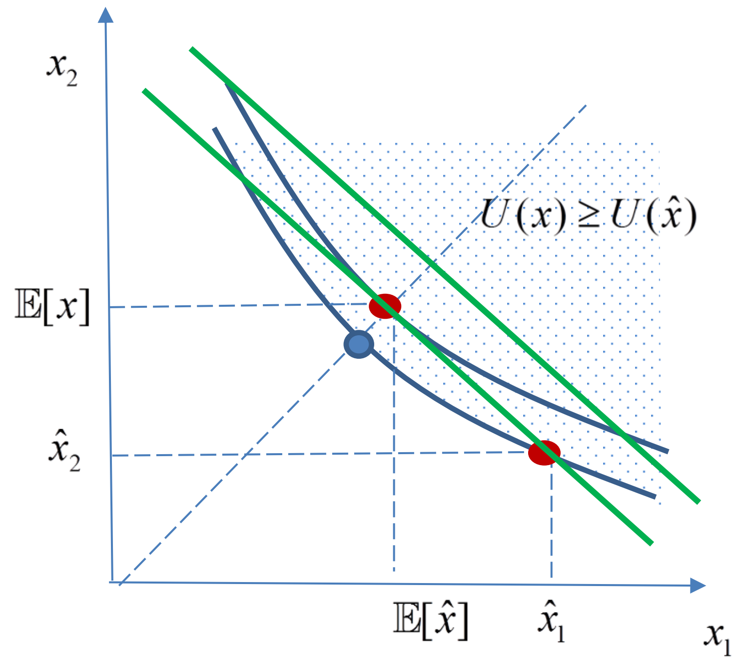
Given the concavity of $u(x)$, the individual

strictly prefers the expected outcome

$$\mathbb{E}[\hat{x}] = \pi_1 \hat{x}_1 + \pi_2 \hat{x}_2$$

to the risky prospect

$$(\hat{x}_1, \hat{x}_2; \pi_1, \pi_2)$$



Consider moving from the blue marker

around the level set by increasing x_1 .

At the blue marker the MRS set Is $\dfrac{\pi_1}{\pi_2}$.

The change in expected value is

$$\frac{d}{dx_1}(\pi_1 x_1 + \pi_2 x_2) = \pi_2 \frac{d}{dx_1}(\frac{\pi_1}{\pi_2} x_1 + x_2)$$

$$= \pi_2 (\frac{\pi_1}{\pi_2} + \frac{dx_2}{dx_1})$$

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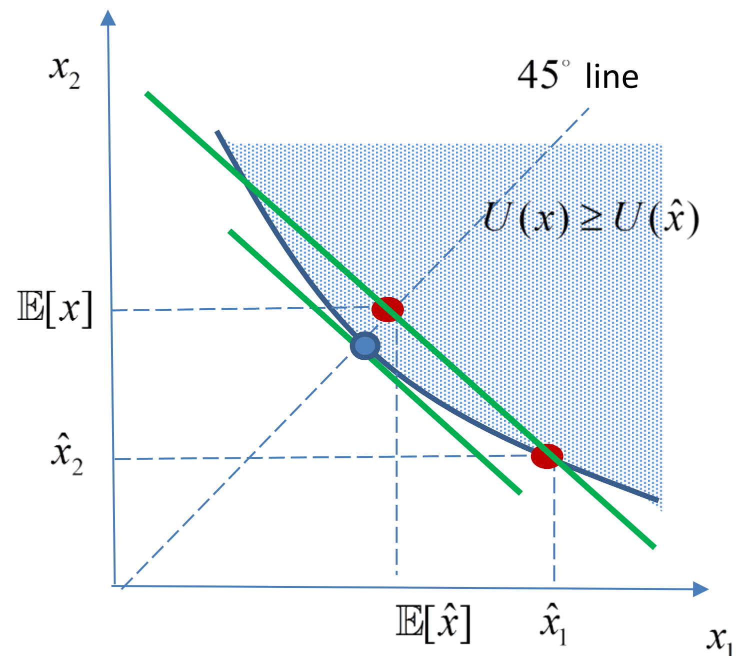

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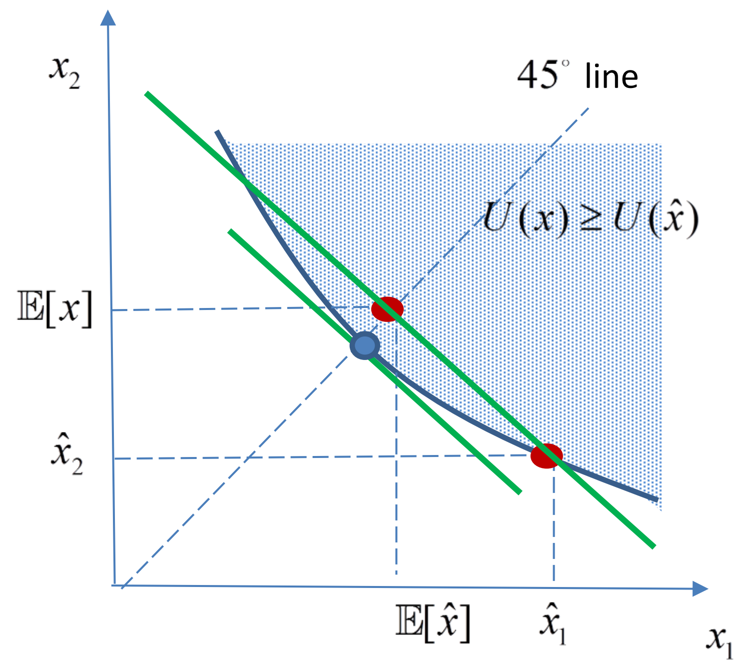

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$$\frac{d}{dx_1}(\pi_1 x_1 + \pi_2 x_2) = \pi_2 \frac{d}{dx_1}\left(\frac{\pi_1}{\pi_2} x_1 + x_2\right)$$

$$= \pi_2\left(\frac{\pi_1}{\pi_2} + \frac{dx_2}{dx_1}\right)$$

$$= \pi_2\left(\frac{\pi_1}{\pi_2} - MRS(x)\right)$$

Below the certainty line $MRS(x) < \dfrac{\pi_1(a_2)}{\pi_2(a_2)}$.

Therefore the expected payoff to A increases around the level set.

Hence the expected payoff to B decreases.

So along the level set as the size of the risk ($x_1 - x_2$) increases, the expected return must increase.

Level sets of a risk neutral individual

The risk neutral individual maximizes his expected payoff,

$$\mathbb{E}[x] \equiv \pi x_1 + \pi_2 x_2$$

In the figure the green line is a level set

through $\hat{x}$.

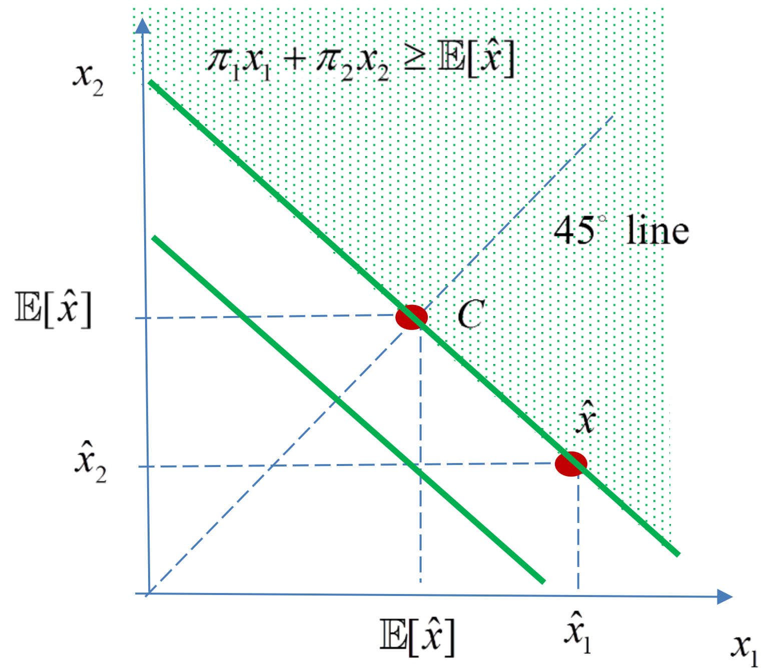
$$\pi_1 x_1 + \pi_2 x_2 = \mathbb{E}[\hat{x}] = U(\hat{x})$$

Note that the level set goes through the

point C on the 45° line.

The individual is indifferent between

$(\hat{x}_1, \hat{x}_2)$ and $(\mathbb{E}[\hat{x}], \mathbb{E}[\hat{x}])$



Thus the expected payoff of a risk averse individual is given by the coordinates of the point C on the 45° line (the no risk line).

Pareto Efficient Allocations when Alex is risk averse and Bev is risk neutral

The dotted region is the set of outcomes preferred to $\hat{x}^A$

All the allocations which have the same expected value for Alex as $\hat{x}^A$ also have the same expected value for Bev. These are on the blue line

$$\pi_1 x_1^A + \pi_2 x_2^A = \pi_1 \hat{x}_1^A + \pi_2 \hat{x}_2^A$$

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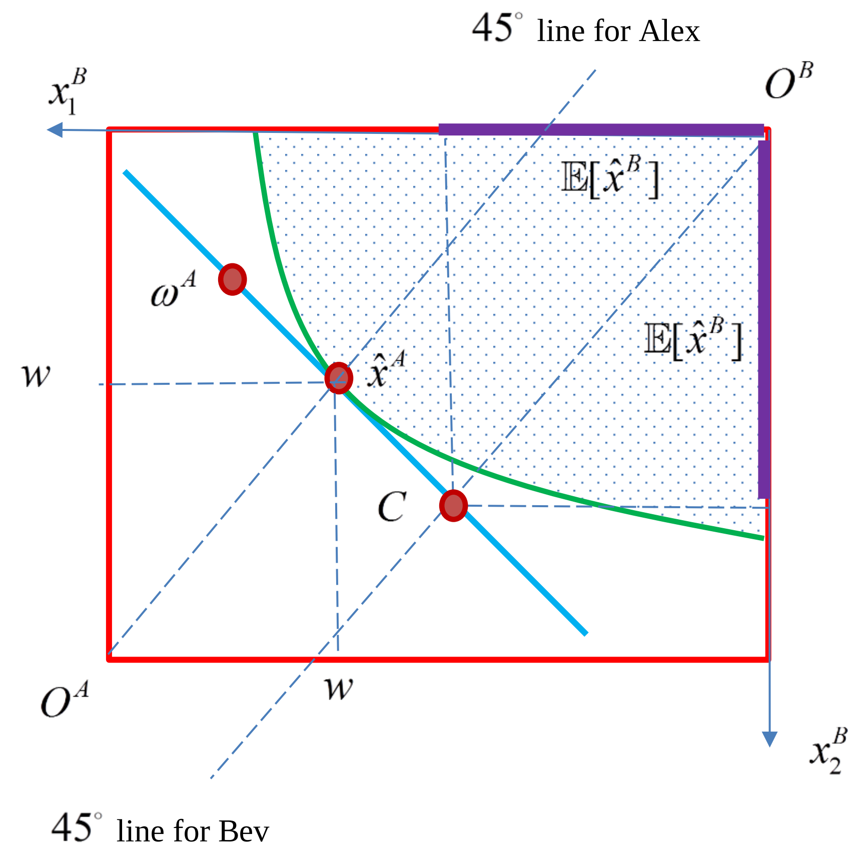

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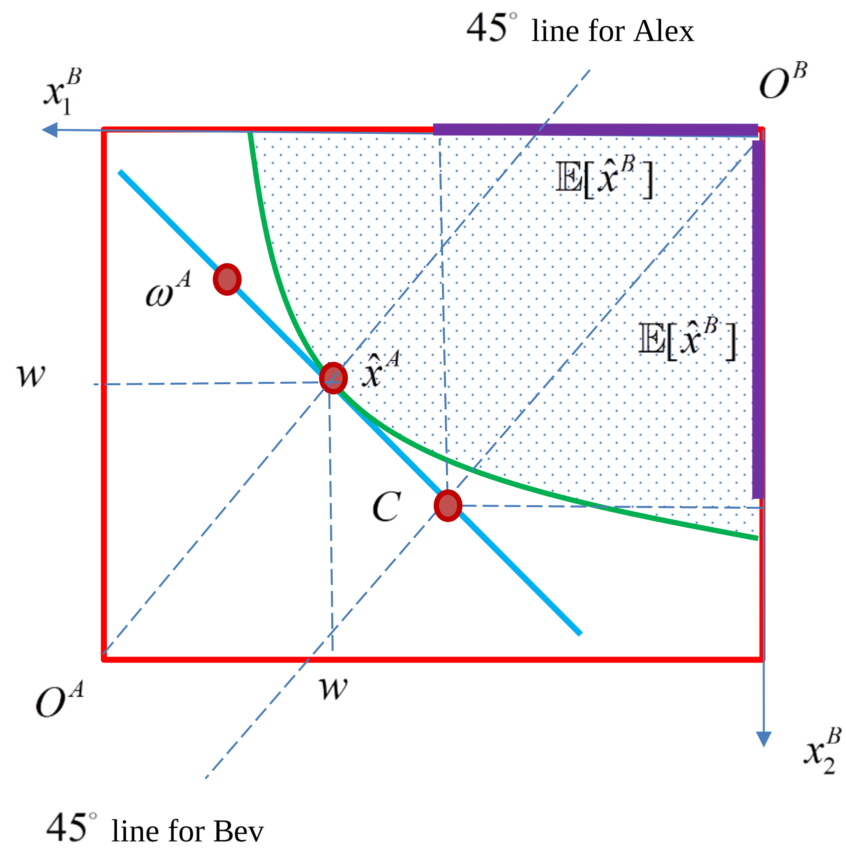

For $\hat{x}^A$ to be Pareto efficient,

$$MRS(x^A) = \frac{\pi_1 u'(x_1^A)}{\pi_1 u'(x_2^A)} = \frac{\pi_1}{\pi_2}.$$

Thus the PE allocations are on the 45° line for Alex where $\hat{x}_2^A = \hat{x}_1^A$

Alex's contract is therefore a fixed wage contract: $\hat{x}_2^A = \hat{x}_1^A = w$.

E. Contract Design: the model

Individual A (Alex) is a risk averse operator of a small (one person) business. Individual B (Bev) is a risk neutral owner

Outcomes

There are S possible revenue levels, $y_1 < y_2 < ... < y_S$ i.e. S states.

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Initially we will assume that both the action and outcomes are observable and verifiable

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Information

Initially we assume that both the action and outcomes are observable and verifiable. Then, following Holmstrom, we consider contracts when only the outcome is observable.

The contract

An action and a payment in each state $x^A = (x_1^A, ..., x_S^S)$.

We will consider the simplest case of two states and two actions

Payoffs

If the worker takes action a and is paid x_s^A in state s his utility in that state is $u(x_s, a)$. Therefore his expected utility is

$$U(x;\pi(a),a)) = \pi_1(a)u(x_1,a) + \pi_2(a)u(x_2,a) \ .$$

The remaining revenue $x_s^B = y_S - x_s^A$ accrues to the risk neutral owner who has an expected

$$\mathbb{E}[\hat{x}^B] = \mathbb{E}[y] - w$$

$$= \pi_1(a)(y_1 - \hat{x}_1^A) + \pi_2(a)(y_2 - \hat{x}_2^A)$$

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*

Pareto Efficient outcome

Solve for Bev's best contract (highest expected payoff) give that Alex has a utility of $\bar{U}_O$. i.e.

$$U(x^A, \pi(a), a) = \bar{U}_O$$

Stage 1: Fix an action and solve for the contract that maximizes Bev's expected payoff given Alex's expected utility.

Note that this is a PE contract so we can appeal to our earlier analysis.

Alex is paid the same in both states.

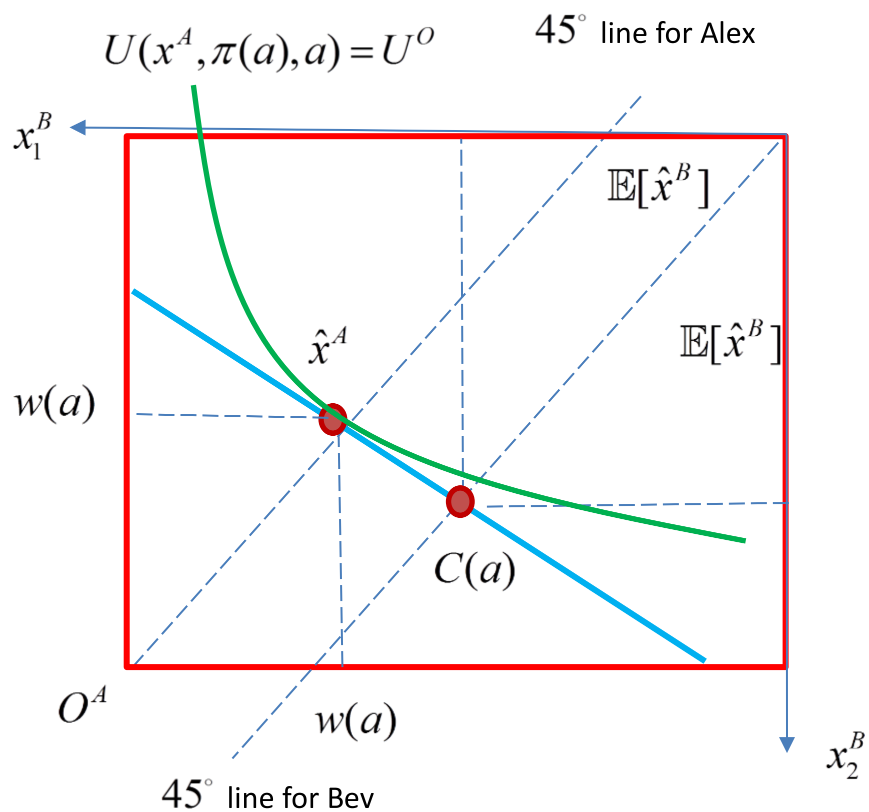
Bev accepts all the risk.

The wage $w(a)$ is chosen so that

$$u(w,a)=U^O$$

Bev's expected payoff is

$\mathbb{E}[\hat{x}^B(a)]=\mathbb{E}[y]-w(a)$.



Comparison of the two contracts

Bev's expected payoff given action a is

$$\mathbb{E}[\hat{x}^B(a)] = \pi_1(a)y_1 + \pi_2(a)y_1 - w(a) \quad \text{where } u(w(a),a) = U^O$$

The difference in expected payoffs is therefore

$$[\pi_1(a_2)y_1 + \pi_2(a_2)y_2 - w(a_2)] - [\pi_1(a_1)y_1 + \pi_2(a_1)y_2 - w(a_1)]$$

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$$= [\pi_1(a_2) - \pi_1(a_1)]y_1 + [\pi_2(a_2) - \pi_2(a_1)]y_2 - [w(a_2)] - w(a_1)]$$

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$$= [\pi_1(a_2) - \pi_1(a_2)](y_1 - y_2) - [w(a_2) - w(a_1)]$$

since probabilities sum to 1 and so

$$\pi_2(a_2) - \pi_2(a_1) = -(\pi_1(a_2) - \pi_1(a_1)$$

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The increase in expected revenue arise from the incremental probability of high revenue times the revenue difference. The increase in expected cost is the wage difference needed to compensate Alex for taking action a_2

G. Contract design with unobservable actions and moral hazard

Henceforth we will assume that the costly action is Pareto Efficient with full information.

Suppose that only the worker knows the action that he took. If he is completely moral he will always tell the truth so the own can simply ask him what action he took and pay him accordingly. The opportunity to take advantage of the fact that an action is private is called “moral hazard”.

We have seen that the optimal contract with no moral hazard is a fixed payment regardless of the state (a fixed wage contract). To compensate for the most costly action the worker is payed a higher wage.

Then unless he is completely moral, the worker will say he is choosing the more costly action and instead choose the less costly action.

What contact must the owner off to induce the worker to choose the high action.

This is the question addressed by Holmstrom.

We characterize the best contract given that Alex's expected utility is U^O

At $\hat{x}(a_2)$ the slope of the level set

is $\dfrac{\pi_1}{\pi_2}$. Consider moving around the

level set increasing x_1

The change in expected value is

$$\frac{d}{dx_1}(\pi_1(a_2)x_1+\pi_2(a_2)x_2)$$

$$=\pi_2(a_2)\frac{d}{dx_1}(\frac{\pi_1(a_2)}{\pi_2(a_2)}x_1+x_2)$$

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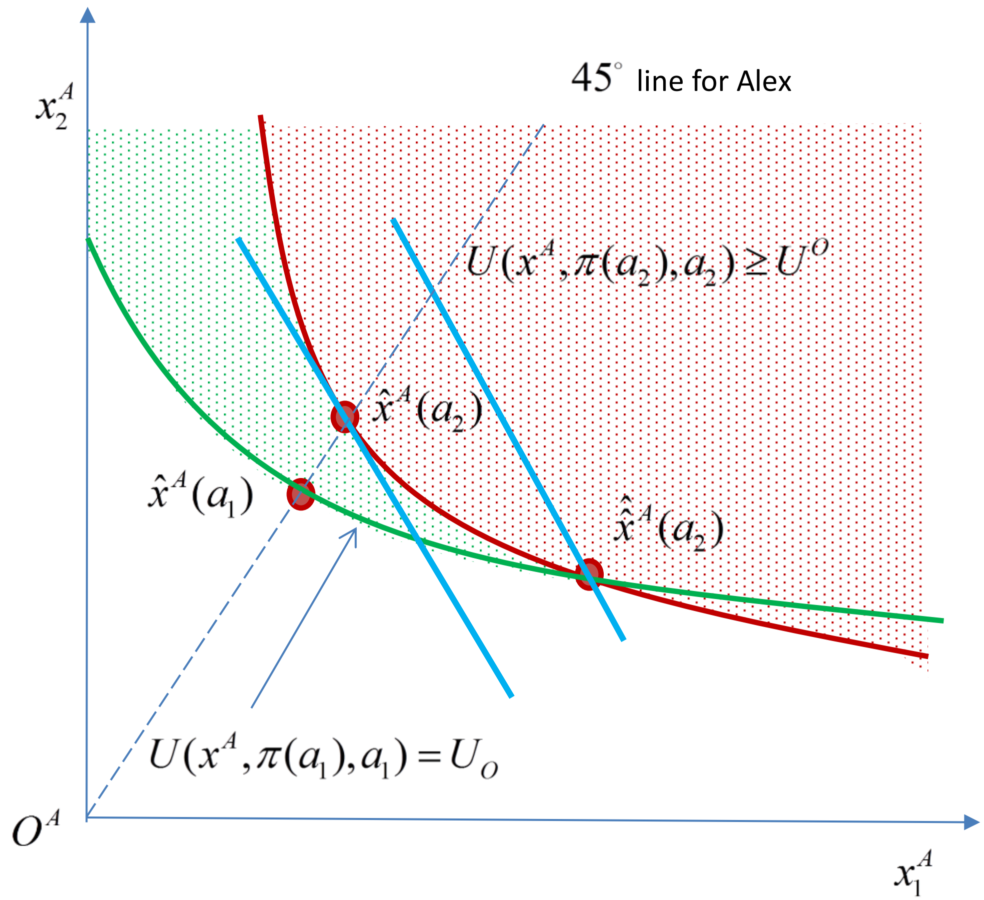

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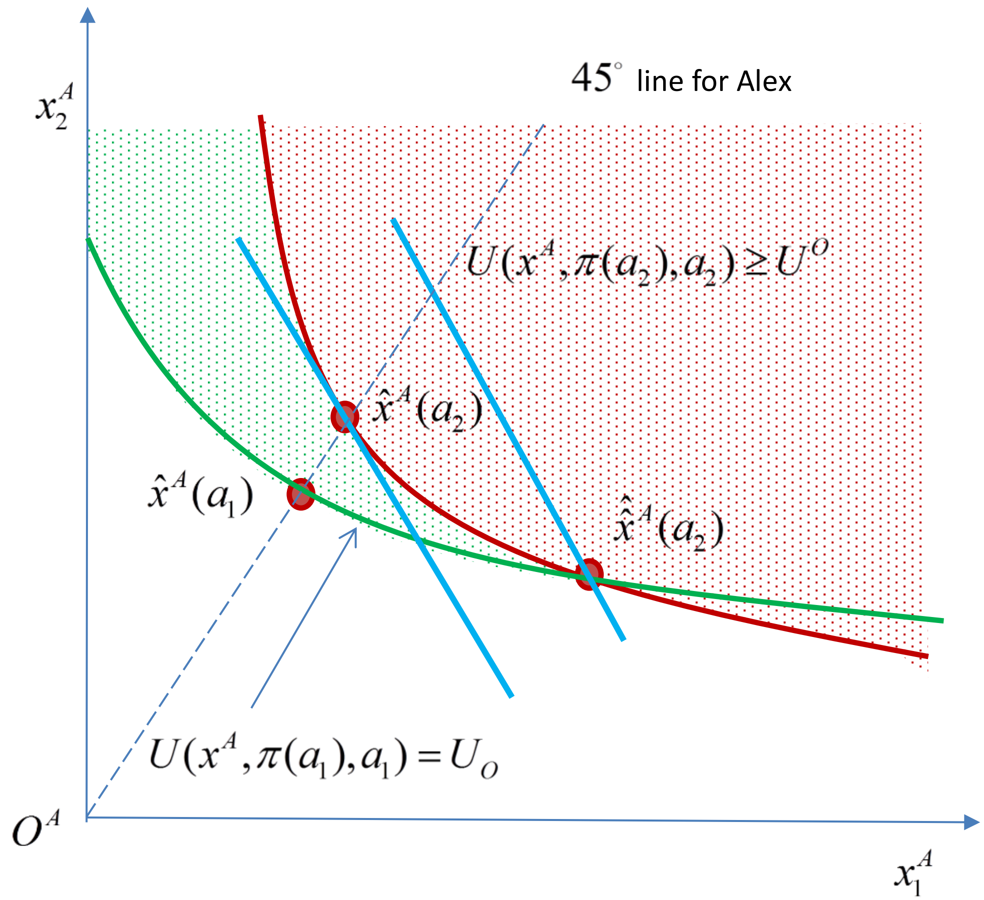

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$$= \pi_2(a_2)(\frac{\pi_1(a_2)}{\pi_2(a_2)} - MRS(x))$$

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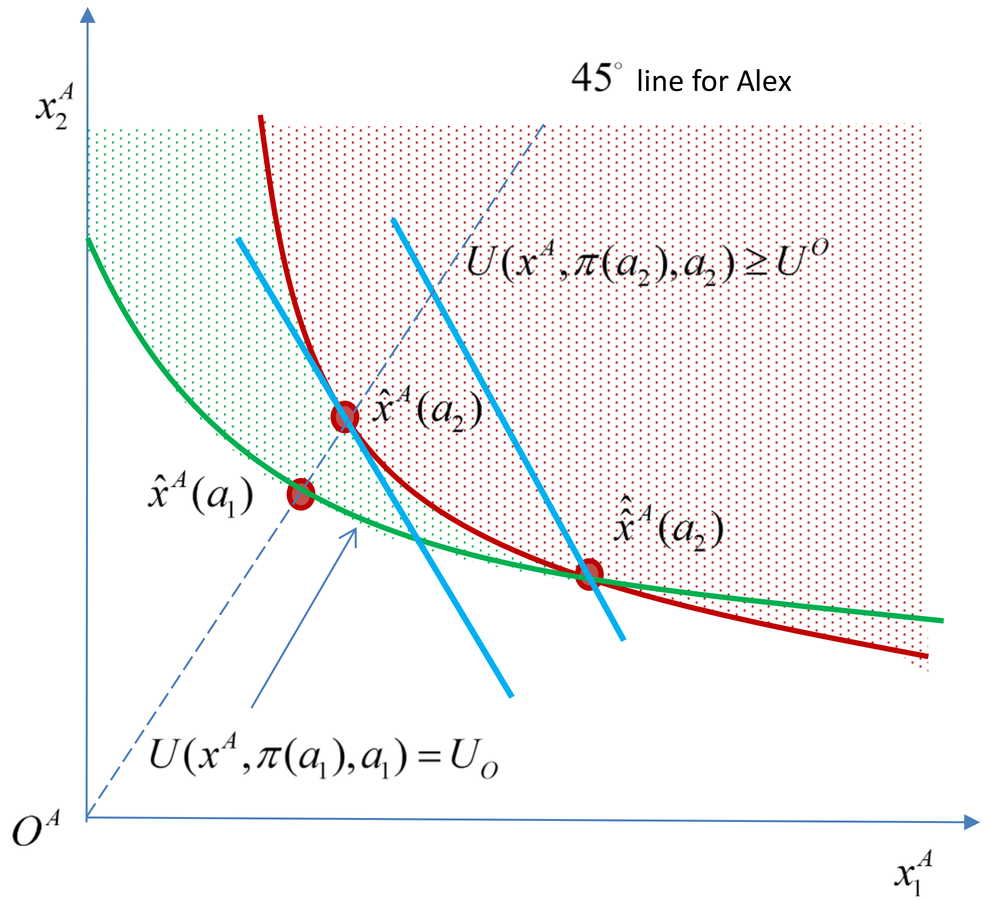

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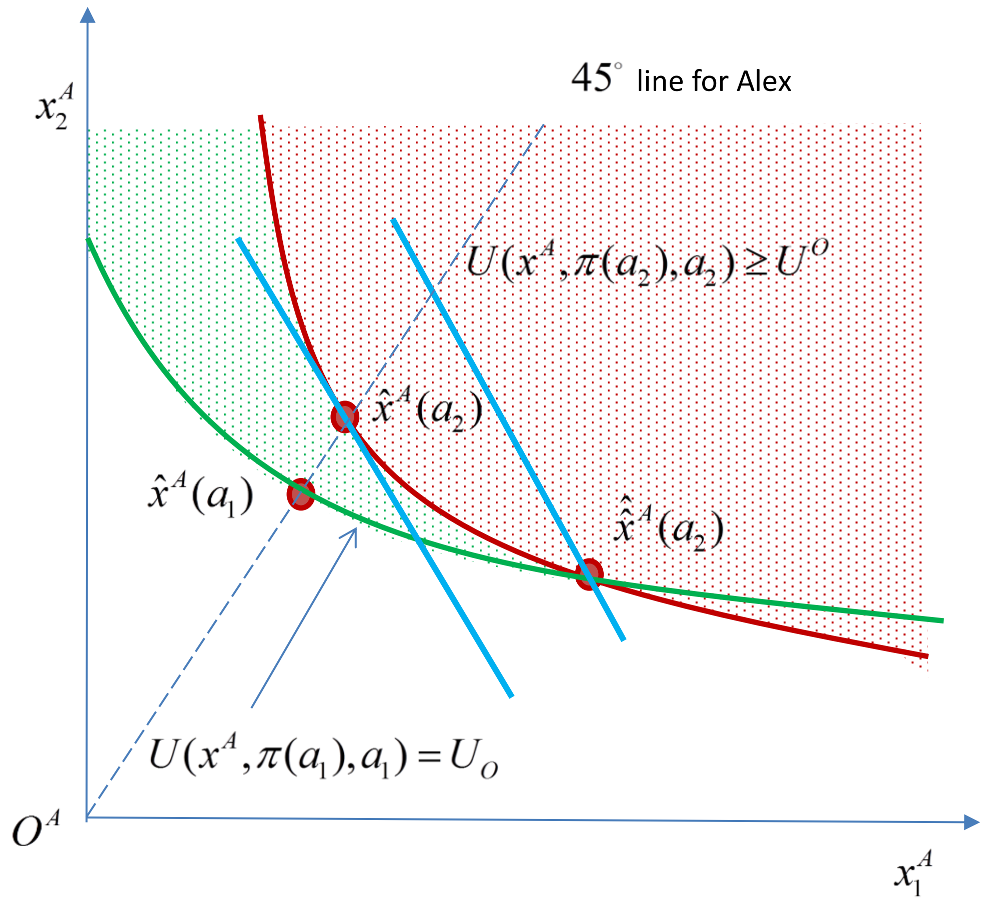
$$=\pi_2(a_2)\frac{d}{dx_1}(\frac{\pi_1(a_2)}{\pi_2(a_2)}x_1+x_2)$$

$$=\pi_2(a_2)(\frac{\pi_1(a_2)}{\pi_2(a_2)}+\frac{dx_2}{dx_1})$$

$$=\pi_2(a_2)(\frac{\pi_1(a_2)}{\pi_2(a_2)}-MRS(x))$$



Below the certainty line $MRS(x)<\dfrac{\pi_1(a_2)}{\pi_2(a_2)}$. Therefore the expected payoff to A increases around the level set. Hence the expected payoff to B decreases.

To yield Alex a utility of U^O when taking

The high action we require that

$$U(\hat{x}(a_2),\pi(a_2),a_2)=U^O$$

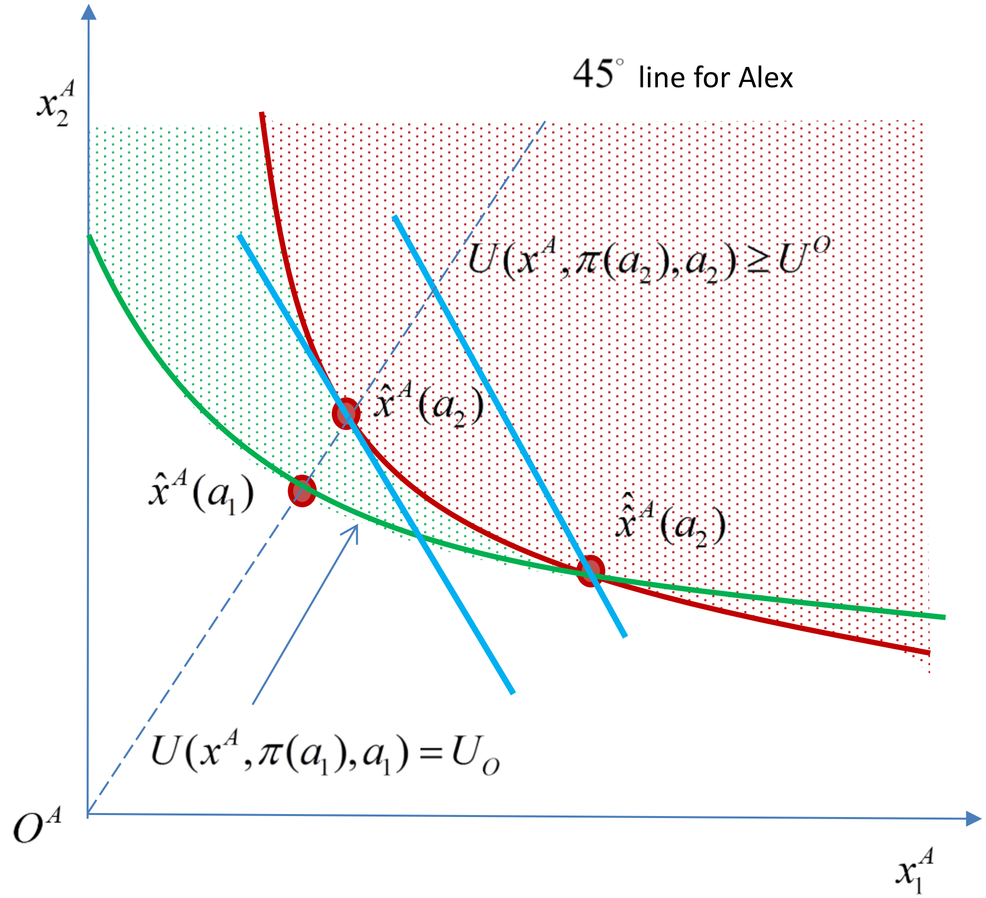
The contract is therefore on the red level set

The boundary of the superlevel set

$$U(x^A,\pi(a_1),a_1)\geq U^O$$

is the green curve.

***



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The optimal contract is therefore $\hat{\hat{x}}^A(a_2)$

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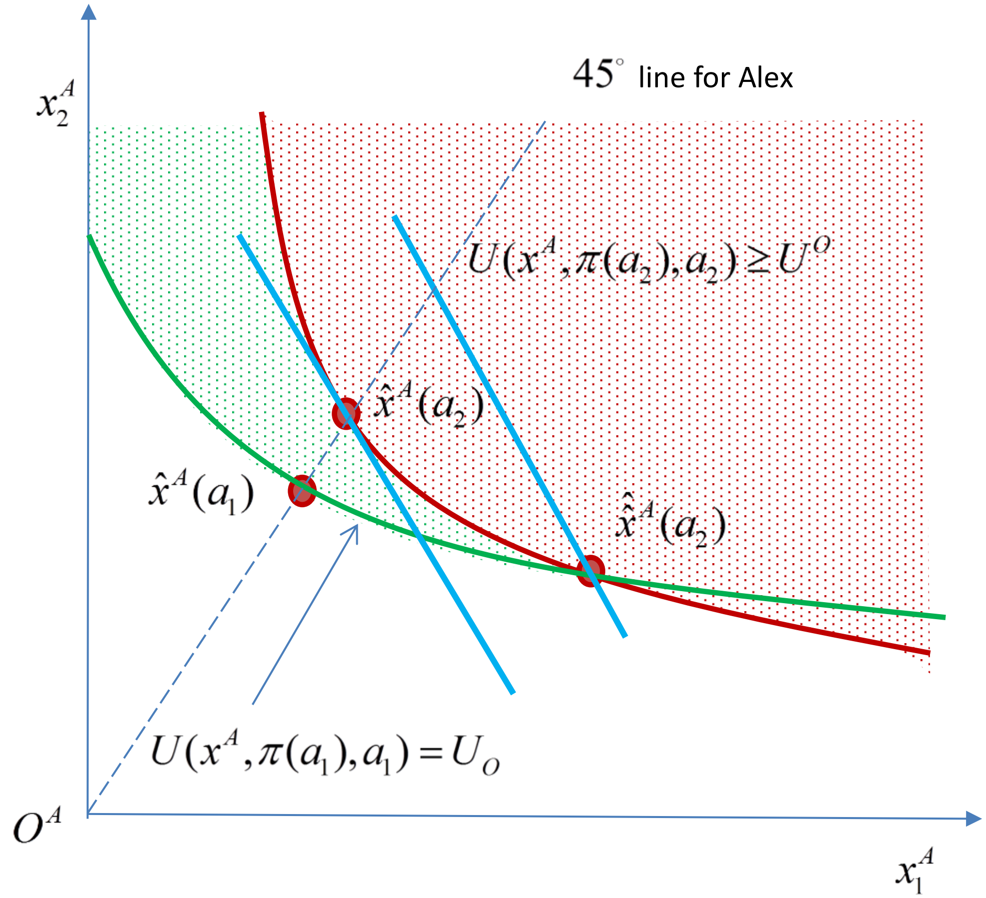

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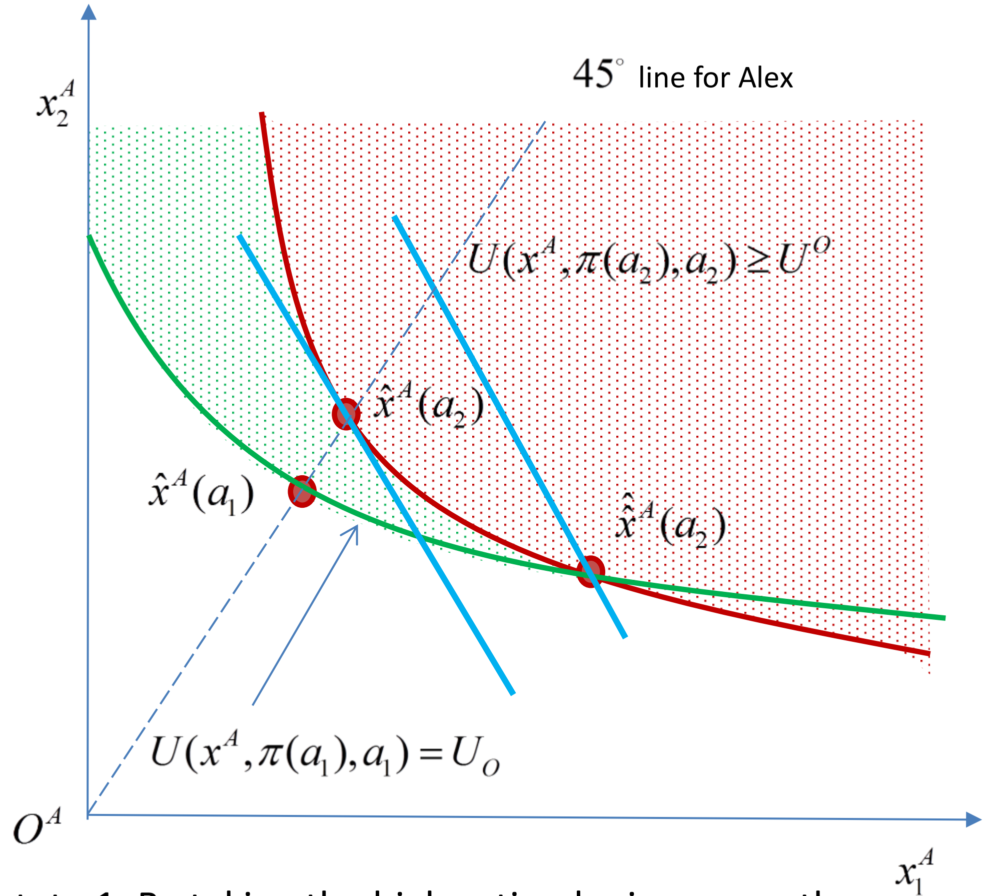

Alex is offered a contract with a higher payoff in state 1. By taking the high action he increases the probability of receiving the higher payoff. If the difference in payoffs is sufficiently large Alex is incentivized to take the more costly action a_2.

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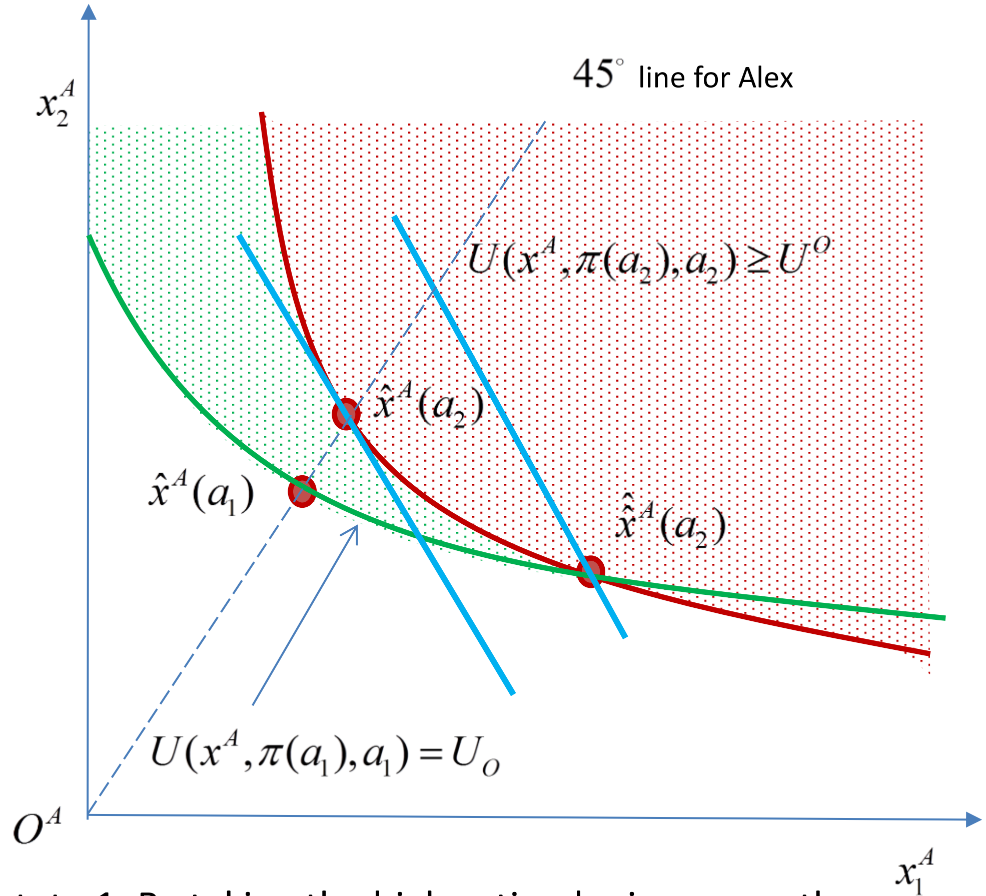

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Holmstrom showed that, under sensible assumptions, with S states the efficient contract is still monotonic. For a contract to be efficient, it must give the worker a higher payoff in the higher revenue states.

H. Expected utility theorem

Reference lottery

$$(\bar{x},\underline{x};u,1-u)$$

We argue that for any certain outcome x a consumer must be indifferent between x and a "reference lottery" in which the two outcomes are $\bar{x}$ and $\underline{x}$, i.e. the most and least preferred.

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Step 1:

If $x\sim(\overline{x},\underline{x};1,0)$ then $u(x)=1$ and we are done. Similarly, if $x\sim(\overline{x},\underline{x};0,1)$ then $u(x)=0$ and we are done.

Step 2:

If not consider $u^2=\frac{1}{2}$. If $x\sim(\overline{x},\underline{x};\frac{1}{2},\frac{1}{2})$ then $u(x)=\frac{1}{2}$ and we are done.

If $x\succ(\overline{x},\underline{x};\frac{1}{2},\frac{1}{2})$ then if there is a u it must be in $(\frac{1}{2},1)$. Then consider $u^3=\frac{3}{4}$.

If $x\prec(\overline{x},\underline{x};\frac{1}{2},\frac{1}{2})$ then if there is a u it must be in $(0,\frac{1}{2})$. Then consider $u^3=\frac{1}{4}$.

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Step 2:

If not consider $u^2 = \frac{1}{2}$. If $x \sim (\bar{x}, \underline{x}; \frac{1}{2}, \frac{1}{2})$ then $u(x) = \frac{1}{2}$ and we are done.

If $x \succ (\bar{x}, \underline{x}; \frac{1}{2}, \frac{1}{2})$ then if there is a u it must be in $(\frac{1}{2}, 1)$. Then consider $u^3 = \frac{3}{4}$.

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Continue this process. Either there is some t-th step such that $x \sim (\bar{x}, \underline{x}; u^t, 1-u^t)$ or there is an infinite sequence sequence $\{u^t\}$ converging to some u^0.

In the latter case, given the continuity axiom, $x \sim (\bar{x}, \underline{x}; u^0, 1-u^0)$

We have established the existence of a utility function $u(x)$ over the certain outcomes.

Extending this to prospects requires a further assumption:

Independence Axiom

Suppose that $L^1 = (x,\pi)$, $L^2 = (\hat{x},\hat{\pi})$ and $L^1 \succsim L^2$ i.e. a consumer prefers $L^1 = (x,\pi)$ over $L^2 = (\hat{x},\hat{\pi})$. Let L^3 be any other lottery. Then

$$(L^1, L^3; p, 1-p) \succsim (L^2, L^3; p, 1-p)$$

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Note that if $L^1 \sim L^2$ $L^2 = (\hat{x},\hat{\pi})$ it follows from two applications of the Axiom that

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Let L^1 be the reference lottery for x_1 and let L^2 be the reference lottery for x_2

i.e. $x_1 \sim L^1 = (\overline{x}, \underline{x}, u(x_1), 1-u(x_1))$ and $x_2 \sim L^2 = (\overline{x}, \underline{x}, u(x_2), 1-u(x_2))$.

By the Independence axiom, since $x_1 \sim L^1$,

$$(x_1, x_2; p, 1-p) \sim (L^1, x_2; p, 1-p) \; .$$

***

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Again by the Independence Axiom, since $x_2 \sim L^2$

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Therefore

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Therefore

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Consider the lottery on the right hand side. The two possible outcomes are $\overline{x}$ and $\underline{x}$.

With probability p the consumer plays lottery 1 and receives the favorable outcome with probability $u(x_1)$.

With probability $1-p$ the consumer plays lottery 2 where the probability of the favorable outcome is $u(x_2)$.

Thus the joint probability of the favorable outcome is

$$U(x, p, 1-p) = pu(x_1) + (1-p)u(x_2) \ .$$

With a little work this argument can be extended to the following lottery over *S* outcomes.

$$(x,\pi)=(x_1,...,x_S;\pi_1,...,\pi_S)\ .$$

The joint probability of winning the reference lottery is

$$U(x,\pi)=\pi_1 u(x_1)+...+\pi_S u(x_S)$$