Solving for the Walrasian Equilibrium: Two examples

Example 1:
Every consumer has the same utility function function $U(x) = -x_1^{-1} - x_2^{-1}$. There is an initial endowment of 30 units of commodity 1. Commodity 1 is both consumed and used as an input in the production of commodity 2. The production function is $q = 4z$.

The utility function is homothetic since $U(\theta x) = \frac{1}{\theta} U(x)$ . Then $U(x) \geq U(y)$ implies that $U(\theta x) \geq U(\theta y)$.

(a) Solve for the production plan that maximizes the utility of the representative consumer.
(b) What must be the WE price ratio?

Example 2:
Every consumer has the same utility function function $U(x) = x_1^4 x_2$. There is an initial endowment of 81 units of commodity 1. Commodity 2 is produced using commodity 1 as an input. The production function is $q = 6z^{1/2}$.

(a) Show that the utility function is homothetic.
(b) Solve for the production plan that maximizes the utility of the representative consumer.
(c) What must be the WE price ratio?
(d) What is the equilibrium profit of the firm?
Example 1

Step 1: Feasible outcomes

No production: \( x = (x_1, x_2) = (\omega_1, \omega_2) = (30, 0) \).

Use 1 unit of commodity 1 in production. \( q \leq 4z \). So the maximum output of commodity 2 is 4.

Use \( z \) units of commodity 1. The maximum output is \( q = 4z \).

Then \( x_1 = 30 - z \) and \( x_2 = 4z \).

The maximum output of commodity 2 is \( 4 \times 30 = 120 \).

The set of feasible alternatives is therefore as depicted.

Preferences

Instead of maximizing \( U \) we maximize \( u = \ln U = 4 \ln x_1 + \ln x_2 \) Note that

\[
MU_1 = \frac{\partial u}{\partial x_1} = 4x_1^{-1} = \frac{4}{x_1} \quad \text{and} \quad MU_2 = \frac{\partial u}{\partial x_2} = x_2^{-1} = \frac{1}{x_2}.
\]

These are both positive so utility is strictly increasing.

Note also that

\[
MRS(x_1, x_2) = \frac{MU_1}{MU_2} = \frac{x_2}{x_1}^2
\]
Consider moving from A to B on the level set depicted below. The ratio \( \frac{x_2}{x_1} \) increases. Therefore the \( MRS(x_1, x_2) \) increases. Thus the steepness of the level set is greater at \( B \). Thus without worrying about the exact shape, we can draw level sets as shown below.

Superimposing the level sets on the first figure, the optimum for the representative consumer is the point on the boundary of the feasible set tangential to the indifference curve.
Step 2: Solve for the optimum

For any \( z \) the output must satisfy \( q \leq 4z \). Since utility is increasing this must be an equality for a maximum. All the output of commodity 2 will be consumed therefore

\[ x_2 = q = 4z. \]

Commodity 1 is used both as an input in the production of commodity 2 and as consumption. Therefore if \( z \) units are used in production,

\[ x_1 = a_1 - z = 30 - z. \]

Substitute these into the utility function.

\[ U = -(30-z)^{-1} - (4z)^{-1} = -(30-z)^{-1} - \frac{1}{4} z^{-1}. \]

Look on the margin

\[
\frac{dU}{dz} = -(30-z)^{-2} + \frac{1}{4} z^{-2} = -\frac{1}{(30-z)^2} + \frac{1}{4z^2}
\]

This must be zero if the maximizer, \( z^* > 0 \). Then \((30-z)^2 = 4z^2\). Take the square root.

\[ 30-z = 2z \]

Therefore \( z^* = 10 \) and so \((x_1^*, x_2^*) = (20, 40)\)
**Step 3: Solve for prices that support the optimal production plan.**

In the model, firms are price takers. Consider any pair of prices \( p = (p_1, p_2) \) and a production plan \((z, q)\). That is, the firm purchases \( z \) units of commodity 1 and sells \( q \) units of commodity 2. The profit is

\[ \pi(z, q) = p_2q - p_1z. \]

For any \( z \) the firm will produce the maximum possible output to maximize profit. In this example the set of feasible plans is the set \( S = \{(z, q) \mid q \leq 4z\} \) so the firm will choose \( q = 4z \). Then profit is

\[ \pi = p_24z - p_1z. \]

The marginal profit to increasing the input is therefore

\[ \frac{d\pi}{dz} = 4p_2 - p_1. \]

**Supporting prices**

The price vector \( p \) is said to “support” the optimum if \((z^*, q^*)\) is profit-maximizing. Then marginal profit must be zero. Then the price vector is supporting if

\[ \frac{p_1}{p_2} = 4. \]

If the relative price of the input is higher then marginal profit is always negative so \(z^* = 10\) is not supported.

This is depicted in the figure. The level set of zero profit,

\[ p_2q - p_1z = 0, \]

passes through \((z, q) = (0, 0)\).

Profit is higher in the direction of the arrows (more output and less input).

The firm therefore maximizes profit by producing nothing.
If \( \frac{p_1}{p_2} \) is below 4 it is always strictly profitable to increase output so again the prices are not supporting. Therefore the only possible WE price ratio is 4. In this case the zero profit line is the green line (the boundary of the feasible set.)

Step 4: Explain why consumer demand is equal to supply at these prices

Since the endowment of commodity 2 is zero, \( x_2 = q \). Since the endowment of commodity 1 is \( \omega_1 = 30 \), \( x_1 = \omega_1 - z \).

Therefore

\[
p_2 x_2 + p_1 x_1 = p_2 q + p_1 (\omega_1 - z) = p_2 q - p_1 z + p_1 \omega_1 = \pi(z, q) + p_1 \omega_1
\]

In Step 3, we showed that for profit maximization, equilibrium profit must be zero. Therefore the budget constraint of the representative consumer is

\[
p_2 x_2 + p_1 x_1 = p_1 \omega_1
\]
In the figure below this is the green line through the endowment point.

Therefore \( x^* = (20, 40) \) is utility maximizing, i.e. the choice of the representative consumer.
Example 2

Step 1: Feasible outcomes

No production: \( x = (x_1, x_2) = (\omega_1, \omega_2) = (81, 0) \).

With an input of \( z \) units of commodity 1, the maximum output is \( q = 6z^{1/2} \).

Then \( x_1 = 81 - z \) and \( x_2 = q = 6z^{1/2} \).

The maximum output of commodity 2 is \( 6 \times (81)^{1/2} = 54 \).

The set of feasible alternatives is depicted below.

Preferences

It is simpler to maximize \( u = \ln U = 4 \ln x_1 + \ln x_2 \). Note that

\[
MU_1 = \frac{\partial u}{\partial x_1} = 4x_1^{-1} = \frac{4}{x_1} \quad \text{and} \quad MU_2 = \frac{\partial u}{\partial x_2} = x_2^{-1} = \frac{1}{x_2}.
\]

These are both positive so utility is strictly increasing.

Note also that

\[
MRS(x_1, x_2) = \frac{MU_1}{MU_2} = 4 \frac{x_2}{x_1}
\]
Arguing as in Example 1, moving from A to B on the level set depicted below, the $MRS(x_1, x_2)$ increases. Thus the steepness of the level set is greater at $B$. Thus without worrying about the exact shape, we can draw level sets as shown below.

Superimposing the level sets on the first figure, the optimum for the representative consumer is the point on the boundary of the feasible set tangential to the indifference curve.
**Step 2: Solve for the optimum**

For any \( z \) the output must satisfy \( q \leq 6z^{1/2} \). Since utility is increasing this must be an equality for a maximum. All the output of commodity 2 will be consumed therefore

\[
x_2 = q = 6z^{1/2}.
\]

Commodity 1 is both consumed and used as an input in the production of commodity 2. Therefore if \( z \) units are used in production,

\[
x_i = a_1 - z = 81 - z.
\]

Substitute these into the utility function.

\[
u = 4\ln(81-z) + \ln 6z^{1/2} = 4\ln(81-z) + \ln 6 + \frac{1}{2}\ln z.
\]

Look on the margin

\[
\frac{du}{dz} = -\frac{4}{81-z} + \frac{1}{2z}
\]

This must be zero if the maximizer, \( z^* > 0 \). Then \( 81 - z = 8z \).

Therefore \( z^* = 9 \) and so \((x_1^*, x_2^*) = (72,18)\). 

![Diagram](image-url)
Step 3: Solve for prices that support the optimal production plan.

In the model, firms are price takers. Consider any pair of prices \( p = (p_1, p_2) \) and a production plan \((z, q)\). That is, the firm purchases \( z \) units of commodity 1 and sells \( q \) units of commodity 2. The profit is

\[
\pi(z, q) = p_2 q - p_1 z .
\]

For any \( z \) the firm will produce the maximum possible output to maximize profit. In this example the set of feasible plans is the set \( S = \{ (z, q) | q \leq 6z^{1/2} \} \) so the firm will choose \( q = 6z^{1/2} \). Then profit is

\[
\pi = p_2 6z^{1/2} - p_1 z .
\]

The marginal profit to increasing the input is therefore

\[
\frac{d\pi}{dz} = 3p_2 z^{-1/2} - p_1 = \frac{3p_2}{z^{1/2}} - p_1 .
\]

Supporting prices

The price vector \( p \) is said to “support” the optimum if \((z^*, q^*)\) is profit-maximizing. Then marginal profit must be zero at \( z^* = 9 \). Then the price vector is supporting if

\[
\frac{p_1}{p_2} = \frac{(z^*)^{1/2}}{3} = 1 .
\]

The zero profit level set is the green line through the endowment point.

The maximum profit level set is the green line,

\[
\Pi = p_2 q - p_1 z ,
\]

tangential to the boundary of the production set at \((z^*, q^*)\)

So this must have a slope of 1.
Step 4: Explain why consumer demand is equal to supply at these prices

Since the endowment of commodity 2 is zero, \( x_2 = q \). Since the endowment of commodity 1 is \( \omega_1 = 81 \), \( x_1 = \omega_1 - z \).

The level set for maximized profit is

\[
p_2q - p_1z = \bar{\Pi}
\]

But \( q = x_2 \) and \( z = \omega_1 - x_1 \).

Therefore

\[
p_2x_2 - p_1(\omega_1 - x_1) = \bar{\Pi}
\]

Rearranging this equation,

\[
p_2x_2 + p_1x_1 = p_1\omega_1 + \bar{\Pi}
\]

Thus the maximum profit line is also the budget line for the representative consumer.

Therefore \( x^* = (72,18) \) is utility maximizing, i.e. the choice of the representative consumer.