Strategic equilibrium

UCLA Auction House http://games.jriley.sscnet.ucla.edu/

55 pages

A. Cournot Duopoly

Two firms produce related products

Demands are linear

$$q_1 = 60 - 20p_1 + 10p_2 \ , \ \ q_2 = 150 + 10p_1 - 20p_2$$

The demand price functions are as follows:

$$p_1 = 9 - \tfrac{1}{15}q_1 - \tfrac{1}{30}q_2 \ , \quad p_2 = 12 - \tfrac{1}{30}q_1 - \tfrac{1}{15}q_2 .$$

The cost of production in firm 1 is $C_1(q_1) = 4q_1$ and in firm 2 is $C_2(q_2) = 7q_2$

Generalization of Cournot's model (1842)

Firm 1: Computes its marginal revenue and adjusts its output until $MR_1(q_1, q_2) = MC_1 = 4$

$$R_1(q) = p_1(q)q_1 = 9q_1 - \tfrac{1}{15}q_1^{\,2} - \tfrac{1}{30}q_1q_2 \ , \ \ C_1(q) = 4q_1 \ , \ \pi_1(q) = R_1(q) - C_1(q_1)$$

$$MR_1 = \frac{\partial R_1}{\partial q_1} = 9 - \tfrac{2}{15}q_1 - \tfrac{1}{30}q_2$$

Firm 1 then chooses its best response $\bar{q}_1$ to satisfy

$$\frac{\partial \pi_1}{\partial q_1} = MR_1 - MC_1 = 9 - \tfrac{2}{15}q_1 - \tfrac{1}{30}q_2 - 4$$

$$= 5 - \tfrac{2}{15}q_1 - \tfrac{1}{30}q_2$$

$$= 0 \,.$$

Therefore

$$\bar{q}_1 = B_1(q_2) = 37\tfrac{1}{2} - \tfrac{1}{4}q_2$$

Best response function

$B_1(q_2)$ is firm 1's best (i.e. profit-maximizing) response for any choice by firm 2.

We can similarly derive firm 2's best response function $q_2 = B_2(q_1)$

Exercise: Show that $\bar{q}_2 = B_2(q_1) = 37\tfrac{1}{2} - \tfrac{1}{4}q_1$.

Exercise: Confirm that $(\bar{q}_1, \bar{q}_2) = (30, 30)$

The best response line for firm 1 is depicted.

Marginal profit,

$$\frac{\partial \pi_1}{\partial q_1} = MR_1 - MC_1 = 5 - \tfrac{2}{15} q_1 - \tfrac{1}{30} q_2$$

is decreasing in q_1. Therefore if the marker

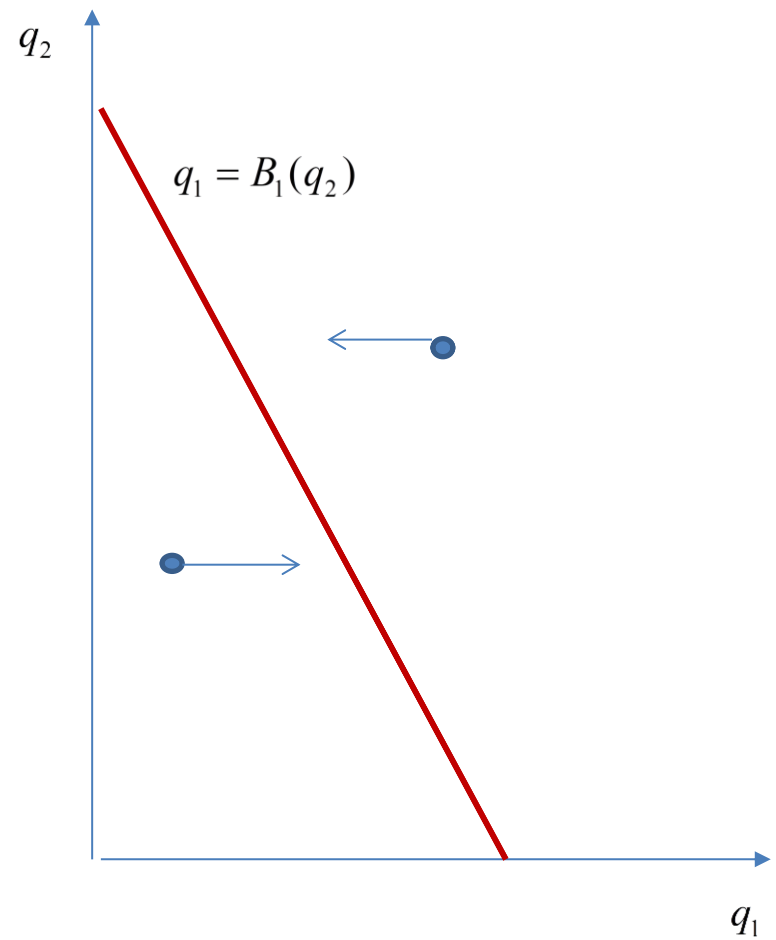
indicating the choices of the two firms is to the

right of the best response line, firm 1 will wish to

adjust by reducing its output.

If the marker is to the left, firm 1 will wish to

adjust by increasing its output.



Firm 1's profit-increasing response.

The best response line for firm 2 is depicted.

Marginal profit

$$\frac{\partial \pi_2}{\partial q_2} = MR_2 - MC_2 = 5 - \tfrac{1}{30} q_1 - \tfrac{2}{15} q_2$$

is decreasing in q_2.

Therefore if the marker

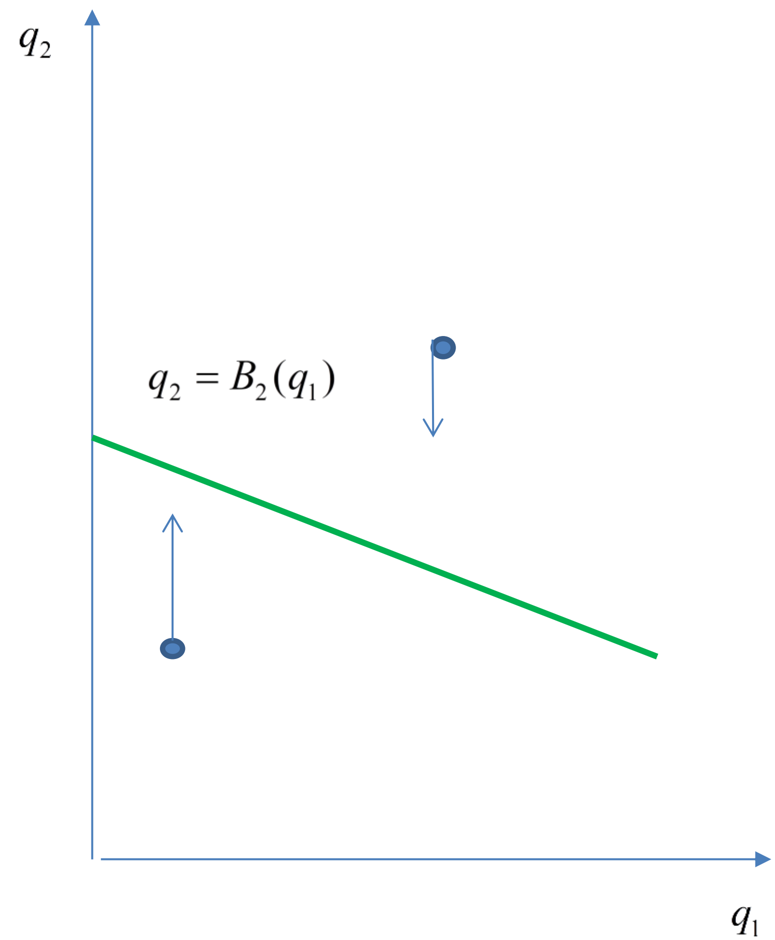
indicating the choices of the two firms is above

its best response line, firm 2 will wish to

adjust by reducing its output.

If the marker is below, firm 2 will wish to

adjust by increasing its output.



Firm 2's profit-increasing response.

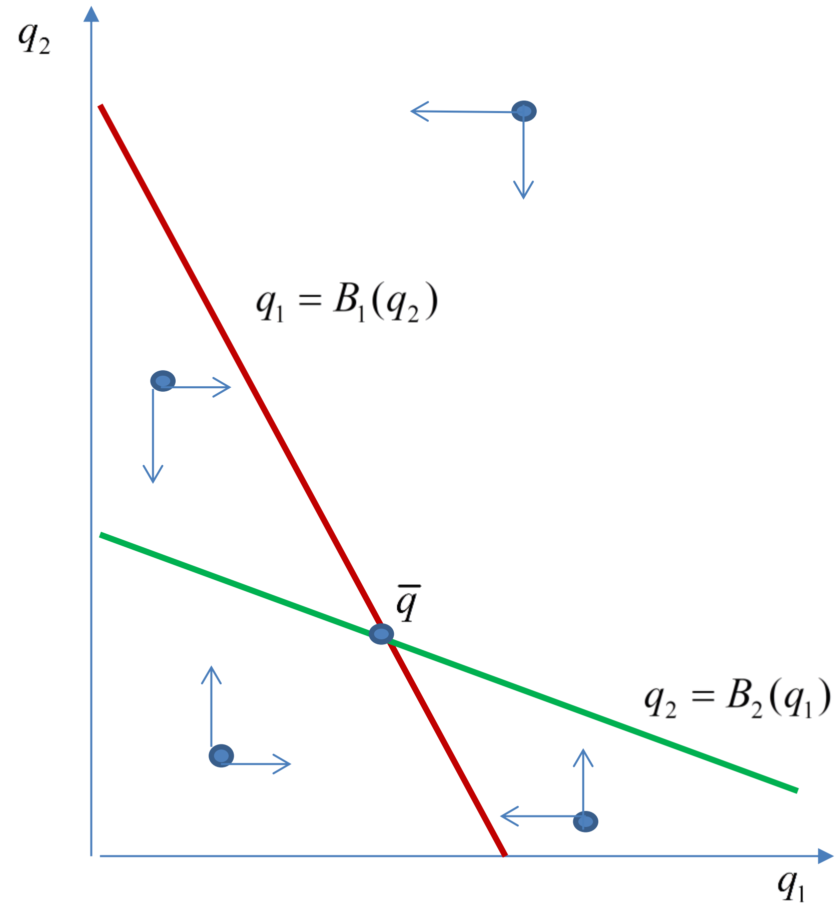
The two best response lines are depicted.

The arrows depict the directions of increasing profit for the two firms in each "zone".

Note that the adjustment process converges to $\bar{q}=(\bar{q}_1,\bar{q}_2)$ where

$B_1(\bar{q}_2)=\bar{q}_1$ and $B_2(\bar{q}_1)=\bar{q}_2$.

At this point the strategy if the firms are said to be mutual best responses.



Arrows show directions of increasing profit for each firm

Then the stable outcome of this adjustment process satisfies the two first order conditions

$$\frac{\partial \pi_1}{\partial q_1}(q_1,q_2)=0 \text{ and } \frac{\partial \pi_2}{\partial q_2}(q_1,q_2)=0.$$

B. Strategic (Nash) equilibrium

Consider two firms competing. In the Cournot model a firm does not try to understand how its opponent might respond. In modern economics we assume that firms have a deep understanding of their close competitors. In particular, each firm can figure out how the other firm's best respond to its action. Let $q_2 = B_2(q_1)$ be firm 2's best response to firm 1's output choice and let $q_1 = B_1(q_2)$ be firm 1's best response to firm 2's output choice.

The outputs $\bar{q} = (\bar{q}_1, \bar{q}_2)$ are called **mutual best responses** if each is a best response to the opponent's action.

*

B. Strategic (Nash) equilibrium.

Consider two firms competing. In the Cournot model a firm does not try to understand how its opponent might respond. In modern economics we assume that firms have a deep understanding of their close competitors. In particular, each firm can figure out how the other firm's best respond to its action. Let $q_2 = B_2(q_1)$ be firm 2's best response to firm 1's output choice and let $q_1 = B_1(q_2)$ be firm 1's best response to firm 2's output choice.

The outputs $\bar{q} = (\bar{q}_1, \bar{q}_2)$ are called **mutual best responses** if each is a best response to the opponent's action.

Definition: Nash equilibrium (n players)

Let $\Pi_j(q_1,...,q_n)$ be player j's payoff if the action vector is $q = (q_1,...,q_n)$. The action vector $\bar{q}$ is a Nash equilibrium action vector if each player's action is a best response.

Nash equilibrium (2 players)

$\bar{q}_1$ solves $\underset{q_1}{Max}\{\Pi_1(q_1, \bar{q}_2)$ and $\bar{q}_2$ solves $\underset{q_2}{Max}\{\Pi_2(\bar{q}_1, q_2)$

Consider the two player example discussed above. The market clearing price of each firm depends on the output vector $q=(q_1,q_2)$.

$$\Pi_1(q_1,q_2)=p_1(q_1,q_2)q_1-C(q_1)$$

$$\Pi_2(q_1,q_2)=p_2(q_1,q_2)q_2-C(q_2)$$

$\bar{q}_1$ is a best response to $\bar{q}_2$ if $\bar{q}_1$ solves $\underset{q_1}{Max}\{\Pi_1(q_1,\bar{q}_2)$

$\bar{q}_2$ is a best response to $\bar{q}_1$ if $\bar{q}_2$ solves $\underset{q_2}{Max}\{\Pi_2(\bar{q}_1,q_2)$

FOC (i) $\dfrac{\partial\Pi_1}{\partial q_1}(\bar{q}_1,\bar{q}_2)=0$ (ii) $\dfrac{\partial\Pi_2}{\partial q_2}(\bar{q}_1,\bar{q}_2)=0$

These conditions are the conditions for equilibrium in the Cournot model.

Note that the modern approach is silent about how each competing firm learns the best response of its opponents.

Group Exercise 1: Equilibrium when the firms produce the same product

In this case if the total output is $q_1 + q_2$ then the market clearing price is $p(q_1 + q_2)$.

Suppose $p(q_1 + q_2) = 60 - (q_1 + q_2)$, $C_1(q_1) = 12q_1$ and $C_2(q_2) = 12q_2$.

Firms choose outputs simultaneously.

(a) Solve for the Nash equilibrium outputs and show that the equilibrium price is 28.

(b) If the two firms were to collude what would they do?

Exercise 2

(c) Suppose you are the owner of firm 1. You have discussed collusion on the golf course with the owner of firm 2. If this simultaneous move game is going to be played 10 times what output would you choose in the first round.

Answer to (c)

Total profit is $\Pi = \Pi_1 + \Pi_2 = (p(q_1 + q_2) - 12)(q_1 + q_2)$.

Note that this is a function of the sum of the outputs $x = q_1 + q_2$

$$\Pi = (p(x) - 12)x = (60 - x - 12)x .$$

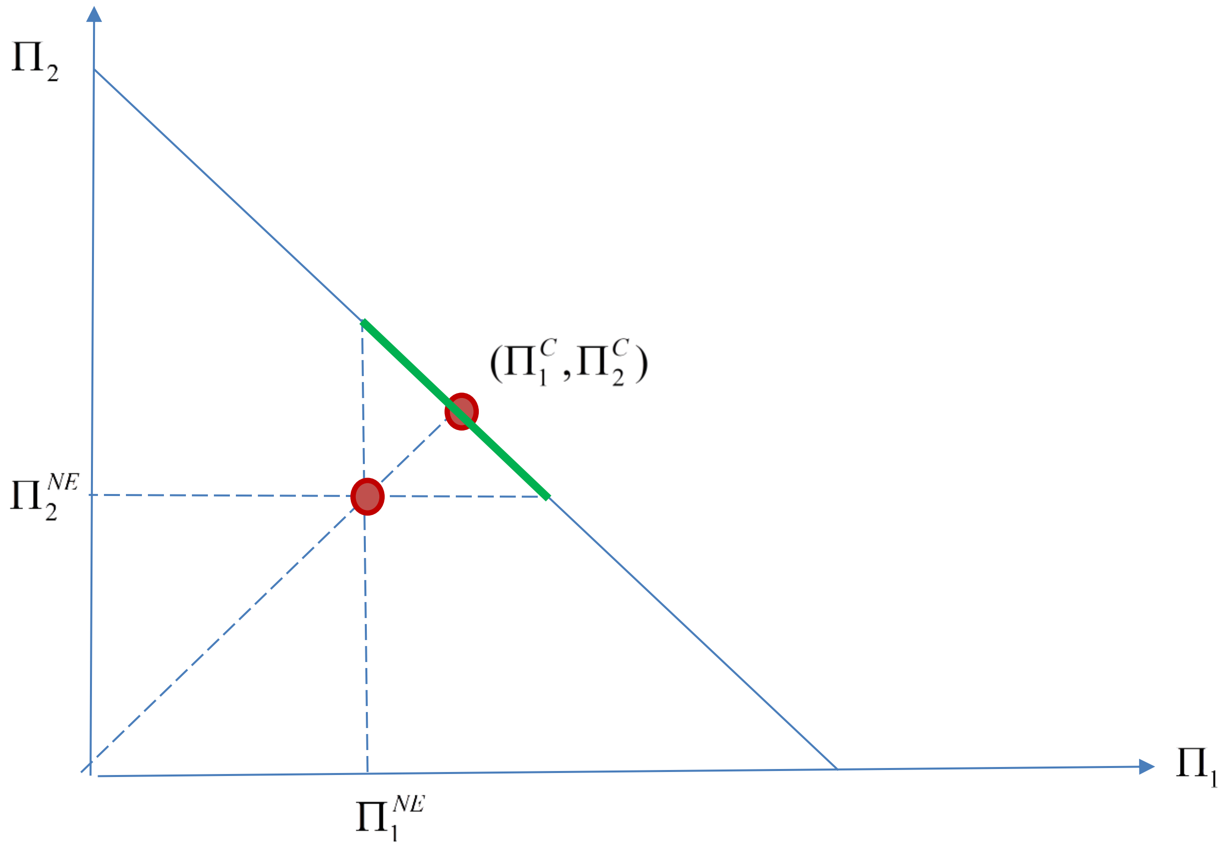
As is readily checked, this is maximized if $x^* = 24$. Then maximized total profit is 576. Note that any pair (q_1, q_2) satisfying $q_1 + q_2 = 24$ is profit maximizing. The possible profits can therefore be depicted as a line

$$\Pi_1 + \Pi_2 = 576$$

The total profit in the Nash Equilibrium is lower. Thus there are gains to colluding.

Give the symmetry of the problem it seems plausible that the two firms would agree to share profits equally.

Symmetric collusion



If the model is not symmetric both firms will only collude if both gain. But it is no longer so clear how the gains from collusion will be cleared.

The case of different costs (to be discussed in TA session)

Suppose $p(q_1+q_2)=60-(q_1+q_2)$, $C_1(q_1)=12q_1$ and $C_2(q_2)=18q_2$.

Firms choose outputs. The set of feasible profit levels is the shaded region $\mathbf{R}$ depicted below.

If you check you will find that the NE strategies are $q^{NE}=(18,12)$ and the NE profits are

$\Pi^{NE}=(324,144)$ Consider the following joint profit maximization problem.

$$\underset{q}{Max}\{\Pi_1(q)+\Pi_2(q)\,|\,(\Pi_1(q),\Pi_2(q))\in R\}$$

Since marginal cost is higher for firm 2,

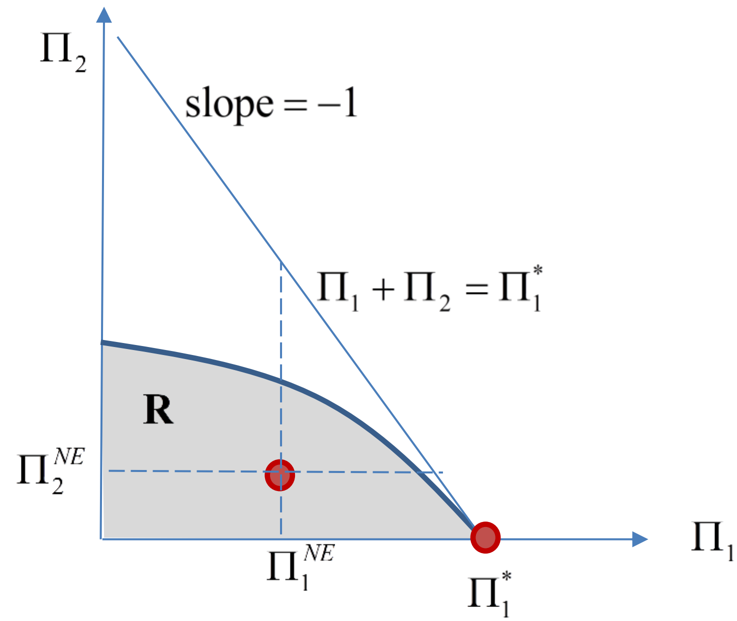
joint profit is maximized at $q^*=(24,0)$

and the profits are $(\Pi_1^*,\Pi_2^*)=(576,0)$

In the figure, the slope of the maximand is -1.

Points on the boundary with $\Pi_2>0$

have a lower slope.



Note that Firm 1 could then pay a “bribe” to firm 2 to produce nothing. But exactly how big a bribe?

Without a theory of cooperation it is not so clear what the payment would be.

If such profit transfers are illegal, then the collusive agreement must be on how much the two firms should produce.

Again it is not clear which point on the frontier to the North-East of the NE profit levels would be chosen.

C. First mover advantage

In the model of strategic competition (or “game”) used thus far, firms must make simultaneous decisions (equivalently, make decisions without knowing the decision of competitors).

Such games are called simultaneous move games.

In some environments (e.g. incumbent firm and potential entrant) the incumbent may have the opportunity to choose its action first. This is called an alternating move game.

An example

$$q_1 = 4 - \tfrac{2}{3}p_1 + \tfrac{1}{3}p_2\ ,\ \ q_2 = 4 + \tfrac{1}{3}p_1 - \tfrac{2}{3}p_2$$

The demand price functions are as follows:

$$p_1 = 12 - 2q_1 - q_2\ ,\quad p_2 = 12 - q_1 - 2q_2 .$$

The cost of production in firm 1 is $C_1(q_1) = 4q_1$ and in firm 2 is $C_2(q_2) = 4q_2$

Firm 1 moves first and chooses q_1 . Firm 2 observes q_1 and makes a best response.

$$\pi_2(q_1,q_2) = R_2 - C_2 = p_2 q_2 - 4q_2 = (12 - q_1 - 2q_2)q_2 - 4q_2$$

$$= 8q_2 - q_1 q_2 - 2q_2^{\,2} \ .$$

The best response $q_2 = B_2(q_1)$ satisfies the necessary condition for a maximum.

$\dfrac{\partial \pi_2}{\partial q_2} = 8 - q_1 - 4q_2 = 0$. Therefore $q_2 = B_2(q_1) = 2 - \frac{1}{4}q_1$

Remark: In a simultaneous move game, by a symmetric argument, $q_1 = B_1(q_2) = 2 - \frac{1}{4}q_2$.

As you may confirm, the Nash Equilibrium of this game is $(\bar{q}_1, \bar{q}_2) = (\frac{8}{5}, \frac{8}{5})$.

Firm 1's best response

In the alternating move game, firm 1 can predict what firm 2's best response will be. So firm 1 predicts that

$$q_2 = B_2(q_1) = 2 - \tfrac{1}{4}q_1 . \qquad (*)$$

Using this prediction, firm 1's profit is

$$\pi_1(q_1,q_2) = R_1 - C_1 = p_1 q_1 - 4q_1 = (12 - 2q_1 - q_2)q_1 - 4q_1 = (12 - 2q_1 - B_2(q_1))q_1 - 4q_1$$

$$= 8q_1 - 2q_1^2 - B_2(q_1)q_1$$

*

Firm 1's best response

In the alternating move game, firm 1 can predict what firm 2's best response will be. So firm 1 predicts that

$$q_2 = B_2(q_1) = 2 - \tfrac{1}{4}q_1. \qquad (*)$$

Using this prediction, firm 1's profit is

$$\pi_1(q_1,q_2) = R_1 - C_1 = p_1q_1 - 4q_1 = (12-2q_1-q_2)q_1 - 4q_1 = (12-2q_1-B_2(q_1))q_1 - 4q_1$$

$$= 8q_1 - 2q_1^2 - B_2(q_1)q_1$$

Necessary condition for a maximum

$$\frac{d\pi_1}{dq_1}(q_1,B_2(q_1)) = R_1 - C_1 = 8 - 4q_1 - B_2(q_1)) - B_2'(q_1)q_1 = 0$$

From (*) the last term is positive. $-B_2'(q_1) = \tfrac{1}{4}$. Therefore

$$\frac{d\pi_1}{dq_1}(q_1,B_2(q_1)) = R_1 - C_1 = 8 - 4q_1 - (2 - \tfrac{1}{4}q_1) + \tfrac{1}{4}q_1 = 6 - \tfrac{7}{2}q_1 = 0 \text{ so } q_1^* = \frac{12}{7} > \frac{8}{5}$$

Key insights:

1. If firm 1 were to choose the same output as in the simultaneous move game, then firm 2’s best response would also be the same. Thus the first mover can achieve the same profit as in the simultaneous move game.

2. But in contrast to the simultaneous move game, firm 1 correctly predicts that if it increases its output, then firm 1 will lower its output in response. This increases firm 1’s demand price function $p_1(q_1, q_2)$ and thus raises firm 1’s profit.

Pricing games

In many applications, short-run output choices are constrained by capacity constraints. So it is not possible to lower a price and then sell a much larger quantity.

Thus, when studying strategic competition, economists often model firms as quantity setters.

However if firms can adjust capacity quickly then price setting strategic completion is possible.

This case will be discussed in the TA session.

The example (firms choose prices)

$$q_1 = 4 - \tfrac{2}{3}p_1 + \tfrac{1}{3}p_2 \;,\;\; q_2 = 4 + \tfrac{1}{3}p_1 - \tfrac{2}{3}p_2$$

The demand price functions are as follows:

$$p_1 = 12 - 2q_1 - q_2 \;, \quad p_2 = 12 - q_1 - 2q_2 .$$

The cost of production in firm 1 is $C_1(q_1) = 4q_1$ and in firm 2 is $C_2(q_2) = 4q_2$

D. Bidding games

Sealed high-bid auction of a single item with I buyers

Buyer i has value v_i. Buyer i submits a sealed bid b_i . To win, buyer i must submit the high bid. If b_i is the winning bid, buyer i pays b_i and receives the item. Buyer i's payoff is therefore $u_i = v_i - b_i$. The other buyers' payoffs are zero.

Tie breaking rule: In the event of tying high bids the winner is determine randomly from among the tying high bidders.

Example 1: Two buyers with known values $v_1 = 200,\ v_2 = 200$.

Class exercise: We will run an auction.

What are the Nash Equilibrium strategies of this bidding game?

Example 2: Three buyers with known values $v_1 = 200,\ v_2 = 200,\ v_3 = 200$

Group exercise: Are the bids $b_1 = 200,\ b_2 = 200,\ b_3 = 200$ mutual best responses (i.e. Nash equilibrium bidding strategies? Are there other bids that are mutual best responses?

Example 3: Two buyers with known values $v_1 = 800$ and $v_2 = 700$

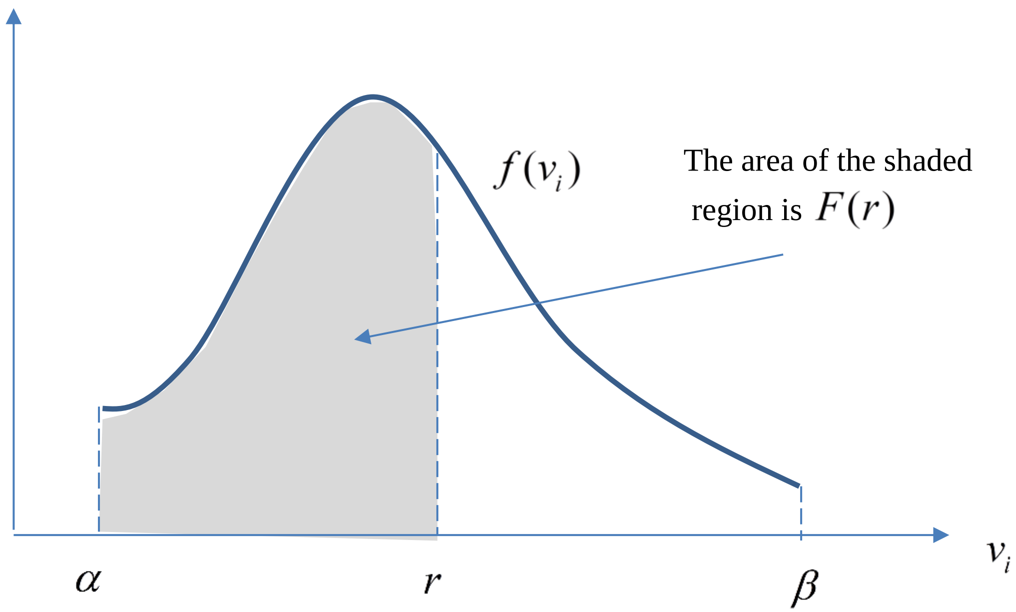
Bids restricted to be integers.

Group exercise: What are the Nash Equilibrium strategies of this bidding game?

E. Bidding games with private information

There are I bidders. Buyer i has a valuation $\tilde{v}_i$ that is private information. All that is known by other buyers is that $\tilde{v}_i$ is continuously distributed on $[\alpha,\beta]$. The probability distribution is also known: $\Pr\{\tilde{v}_i \le v_i\} = F(v_i) = \int_0^{v_i} f(x)dx$.

The probability density function is depicted opposite.



The probability density function

Bidding games with private information

There are I bidders. Each of the buyers may submit a non-negative sealed bid.

**

Bidding games with private information

There are I bidders. Each of the buyers may submit a non-negative sealed bid.

Allocation rule

Bidder i with bid b_i loses if another bid is higher. If there are m bidders who submit the tying high bid, the winner is selected randomly from one of these high bidders so that win probability of each such bidder is $1/m$.

*

Bidding games with private information

There are I bidders. Each of the buyers may submit a non-negative sealed bid.

Allocation rule

Bidder i with bid b_i loses if another bid is higher. If there are m bidders who submit the tying high bid, the winner is selected randomly from one of these high bidders so that win probability of each such bidder is $1/m$.

Payment rule

(i) Sealed high-bid auction

The winner pays his or her bid. Losers pay nothing.

(ii) Electronic ascending price auction

Asking price p rises steadily. A buyer exits the auction by switching his bidder light from green to red. The asking price stops when only one light is green and the remaining buyer pays the final ask.

(iii) Sealed second-bid auction

The winner pays the highest of the losing bids (the second highest bid).

(iv) English ascending price auction

An auctioneer calls out (usually) ascending asking prices seeking bidders. To accept an ask, a buyer raises a paddle with the buyer's bidder number on it. The auction ends when no one accepts the asking price and the auctioneer cries "Going once, going twice, sold!"

(v) Dutch or "clock" auction

An auctioneer starts an electronic clock at a high price. The clock then ticks down till a bidder raises his hand (or hits a button to stop the clock.) This is the successful bidder and the price paid is the price on the clock.

Equilibrium bidding in the electronic ascending price auction

If your light is still green when $p > v_i$, you incur a loss if you are the winner. Thus it is never profitable to keep your light on green when the asking price $p > v_i$.

*

Equilibrium bidding in the electronic ascending price auction

If your light is still green when $p > v_i$, you incur a loss if you are the winner so it is never profitable to keep your light on green when the asking price $p > v_i$.

If you switch your light to red when $p < v_i$, you win nothing. By leaving the light green, all the other buyers may drop out before $p = v_i$ so the final ask is below v_i. You have thus missed out on a possible profit. So it is never profitable to switch your light to red when $p < v_i$.

Thus buyer i's best response is to flip his switch when $p = v_i$.

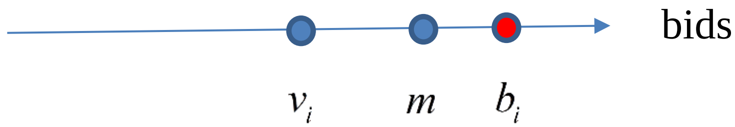
Note that this is true regardless of the strategies of the other buyers. When this is the case, the strategic equilibrium is called a **dominant strategy equilibrium.**

Equilibrium bidding in the sealed second price auction

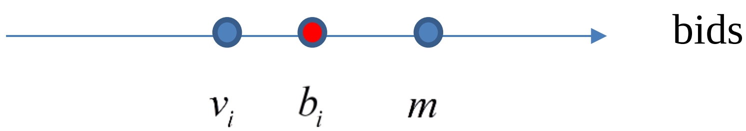
Proposition: In the sealed second price auction it is a dominant strategy for buyer i to bid his value v_i .

Proof: Let m be the maximum of the bids of the opposing buyers.

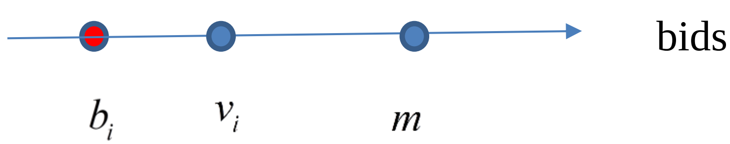
Case 1: Change in payoff if $v_i < m$ and buyer i does not bid v_i



Buyer i makes a loss as he pays $m > v_i$

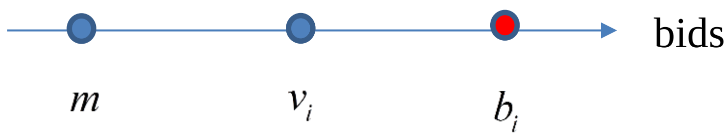


Buyer i's payoff is still zero

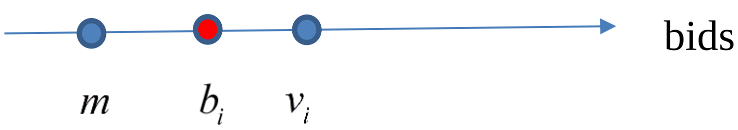


Buyer i's payoff is still zero

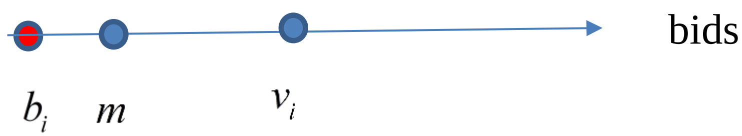
Case 2: Change in payoff if $m < v_i$ and buyer i does not bid v_i



Buyer i still wins and pays m



Buyer i still wins and pays m



Buyer i loses out on a profit

Thus in every eventuality, buyer i either has the same payoff or a strictly lower payoff.

QED

F. Reserve prices

As a preliminary, consider the sale of an item to one buyer.

That buyer must win so the seller must set a minimum price (the “reserve “price) r . The single buyer wins the item if and only if $b_1 \geq r$.

Buyer 1’s best response is to bid the minimum acceptable bid so $b_1 = r$.

The probability of a sale

$$\Pr\{\tilde{v}_1 \geq r\} = 1 - \Pr\{\tilde{v}_1 \leq r\} = 1 - F(r) .$$

Therefore the expected profit of the seller is

$$\Pi = r \Pr\{\text{item is sold}\} = r(1 - F(r)$$

$$\Pi'(r) = 1 - F(r) + -rF'(r) = 1 - F(r) - rf(r)$$

Example: Uniform distribution $[\alpha, \beta] = [0, \beta]$ $F(v) = \dfrac{v}{\beta}$, hence $f(v) = \dfrac{1}{\beta}$

$$\Pi'(r) = 1 - F(r) + rf(r) = 1 - \frac{r}{\beta} - \frac{r}{\beta} = 1 - \frac{2r}{\beta}.$$

Then $r^* = \frac{1}{2}\beta$

Alternative derivation: Consider the first order effects of raising the reserve price by Δr .

$p = \Pr\{\tilde{v}_1 \le r\} = F(r)$

$\Delta p = \Pr\{r \le \tilde{v}_1 \le r + \Delta r\}$

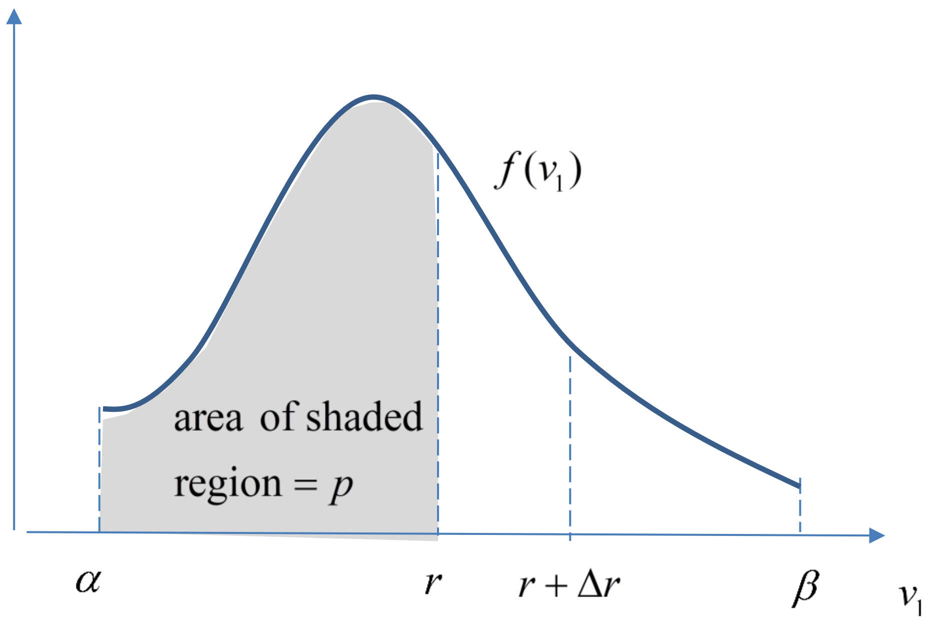
$1 - p - \Delta p = \Pr\{\tilde{v}_1 \ge r + \Delta r\}$

If $r \le v_1 \le r + \Delta r$ there is no sale

so the change in profit is $-r$.

If $v_1 \ge r + \Delta r$ the profit

rises by Δr .



The probability density function

The change in expected profit is therefore $\Delta\Pi = -r\Delta p + \Delta r(1 - p - \Delta p)$

$$= \underbrace{-r\Delta p + \Delta r(1 - p)}_{\text{first order effects}} \; - \; \underbrace{\Delta r \Delta p}_{\text{second order effect}}$$

The change in expected profit is

$$\Delta\Pi = -r\Delta p + \Delta r(1-p) \;-\; \Delta r\Delta p$$

Therefore

$$\frac{\Delta\Pi}{\Delta r} = -r\frac{\Delta p}{\Delta r} + 1 - p - \Delta p.$$

Ignoring the second order effect,

$$\frac{\Delta\Pi}{\Delta r} = -r\frac{\Delta p}{\Delta r} + 1 - p$$

In the limit the second order effect vanishes and

$$\frac{d\Pi}{dr} = -r\frac{dp}{dr} + 1 - p.$$

Since $p = F(r)$, $\frac{dp}{dr} = F'(r) = f(r)$.

$\frac{d\Pi}{dr} = -rf(r) + 1 - F(r)$ = 0 for a maximum.

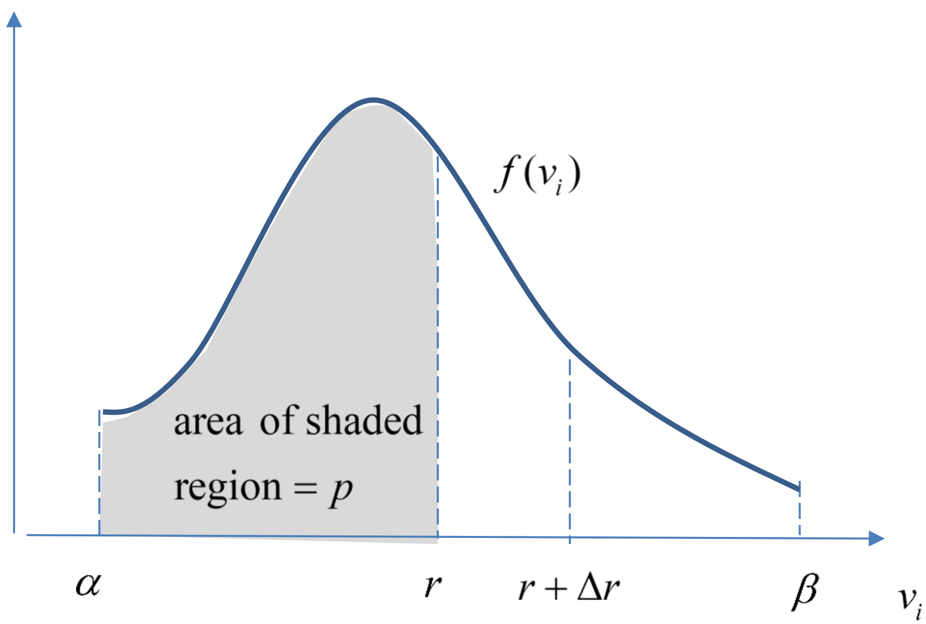
Reserve price in a two buyer sealed second price auction (so buyers bid their values)

Reminder: Probabilities

$$p = \Pr\{\tilde{v}_i \le r\} = F(r)$$

$$\Delta p = \Pr\{r \le \tilde{v}_i \le r + \Delta r\}$$

$$1 - p - \Delta p = \Pr\{\tilde{v}_i \ge r + \Delta r\}$$

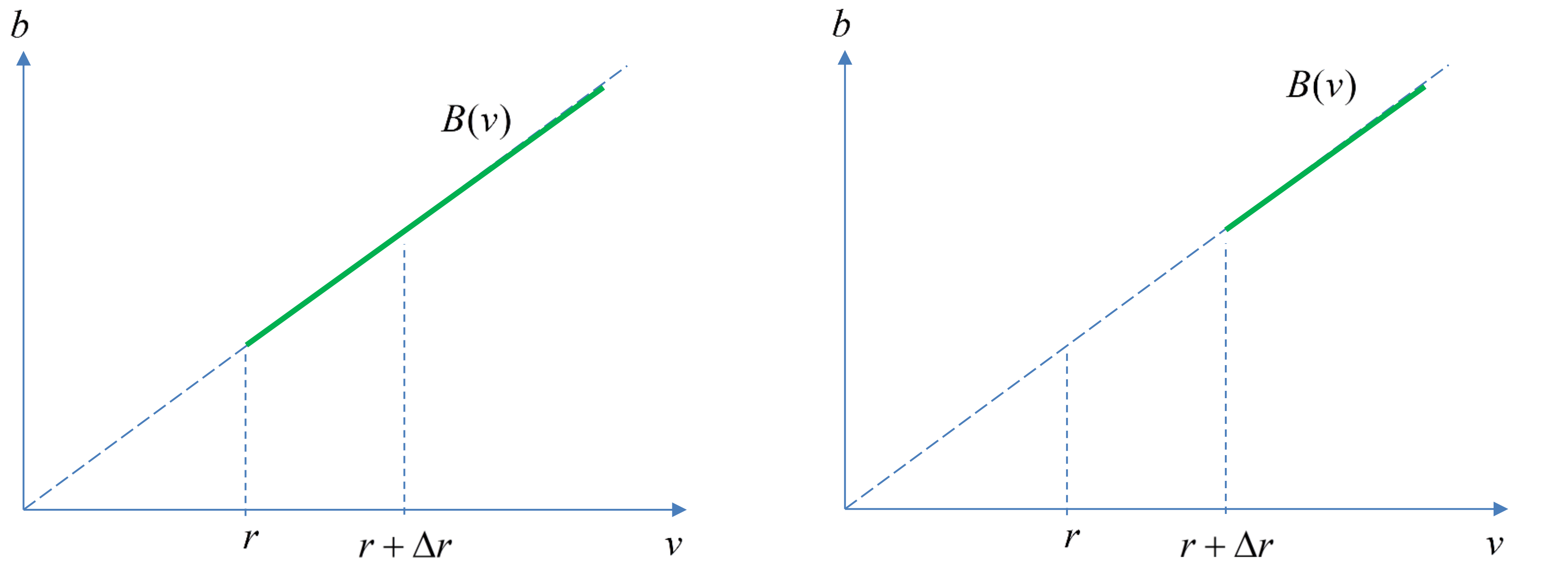


The probability density function

We consider the effects on profit when the reserve price is raised from r to $r+\Delta r$

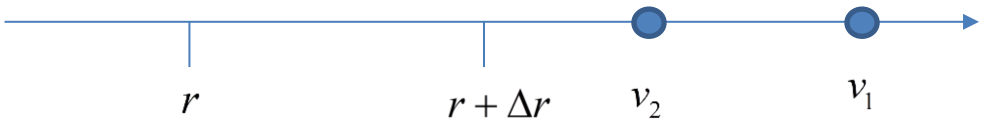
when $v_1 > v_2$. Given symmetry, the effects of raising the reserve price when $v_2 > v_1$ are the same.

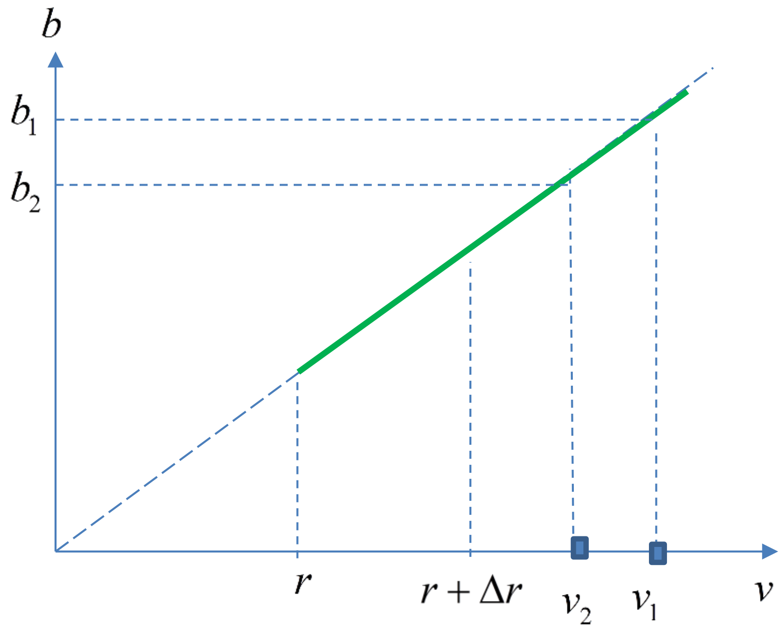
Thus the total effect is doubled.

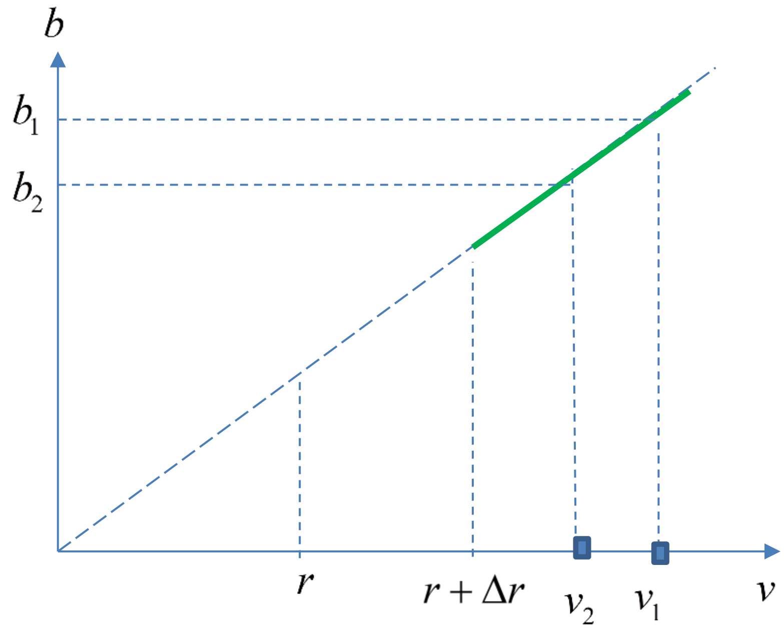


The equilibrium bid functions

Case 1:

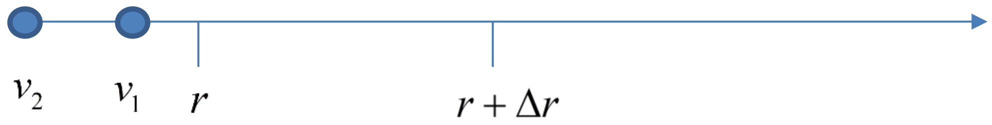






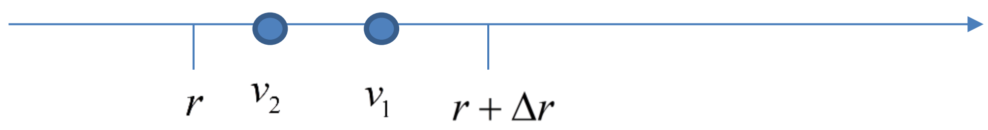
No effect.

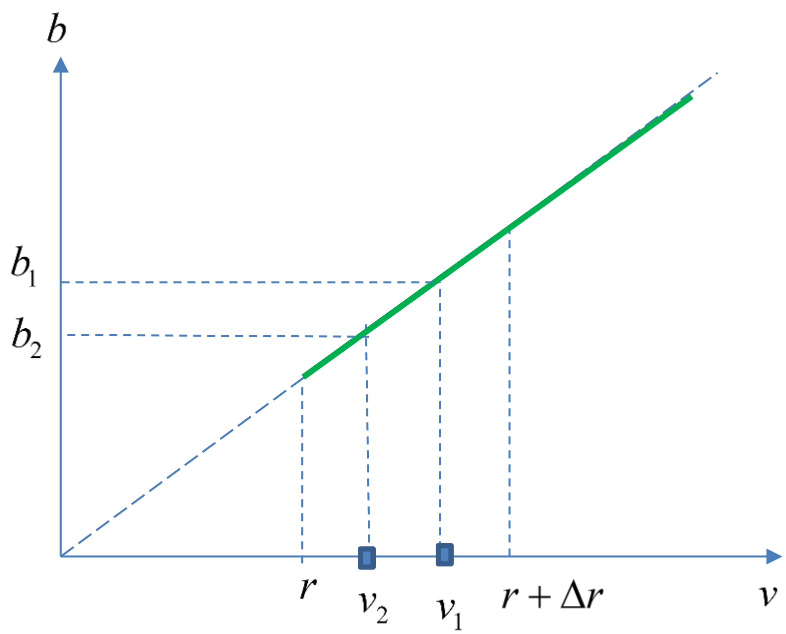
Case 2:

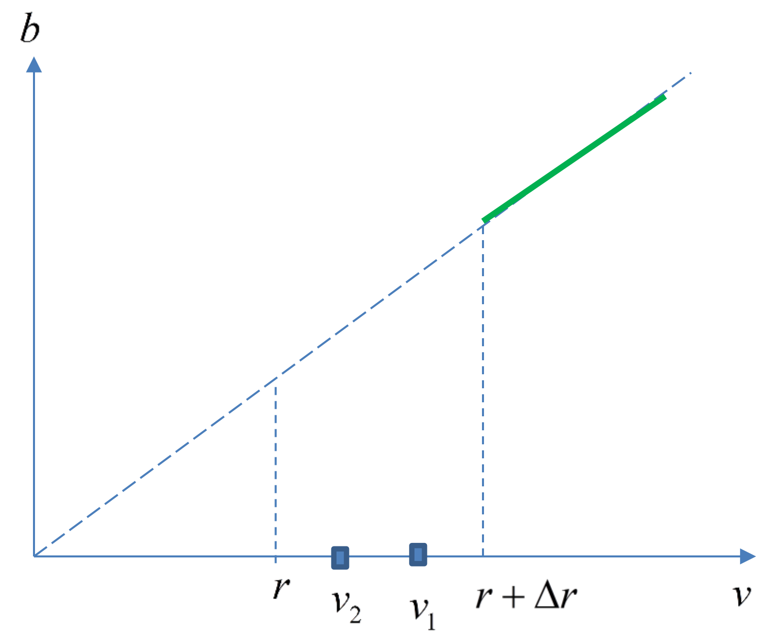


There are no bids so no effect.

Case 3:



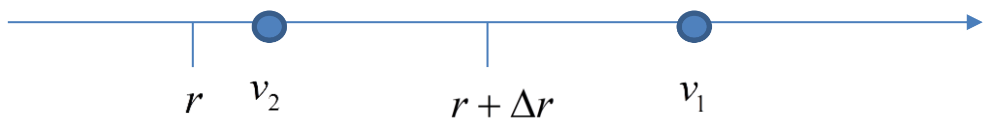


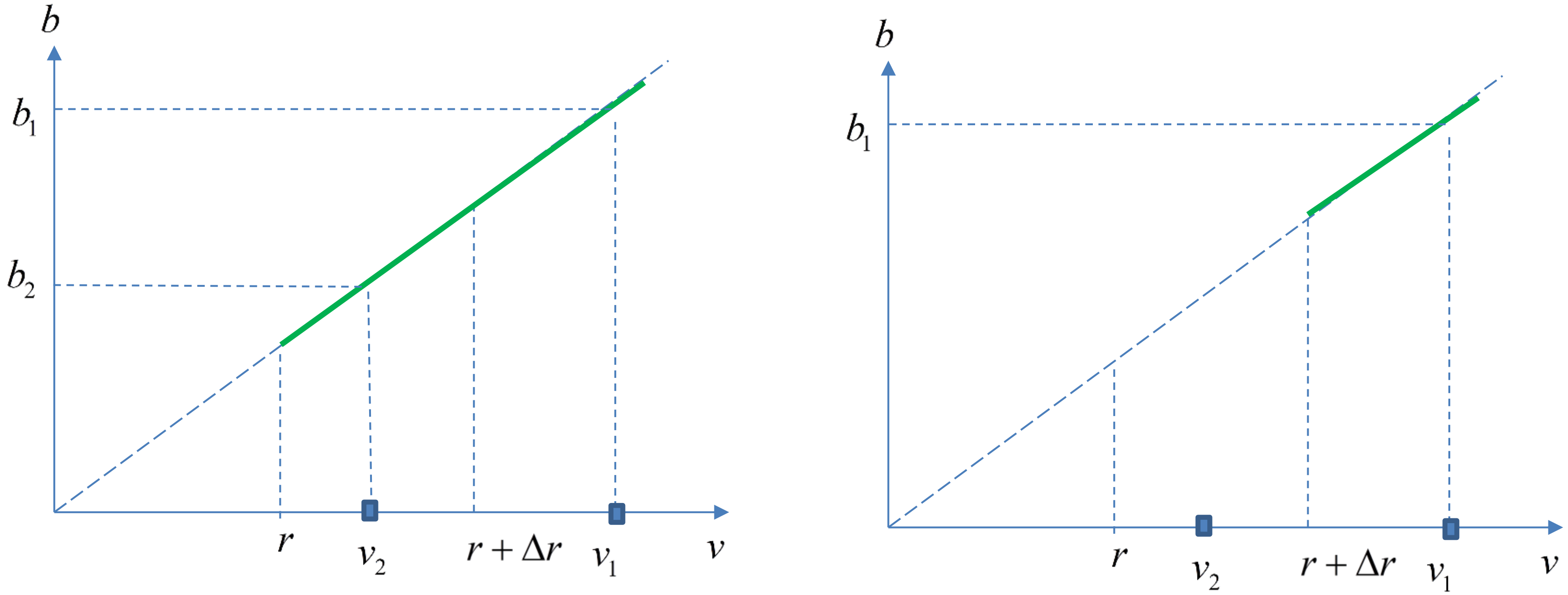


The probability of this event is $(\Delta p)^2$.

So this is a second order effect.

Case 4:





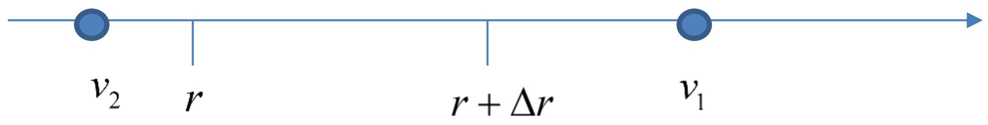
The probability of this event is $\Delta p(1-p-\Delta p)$

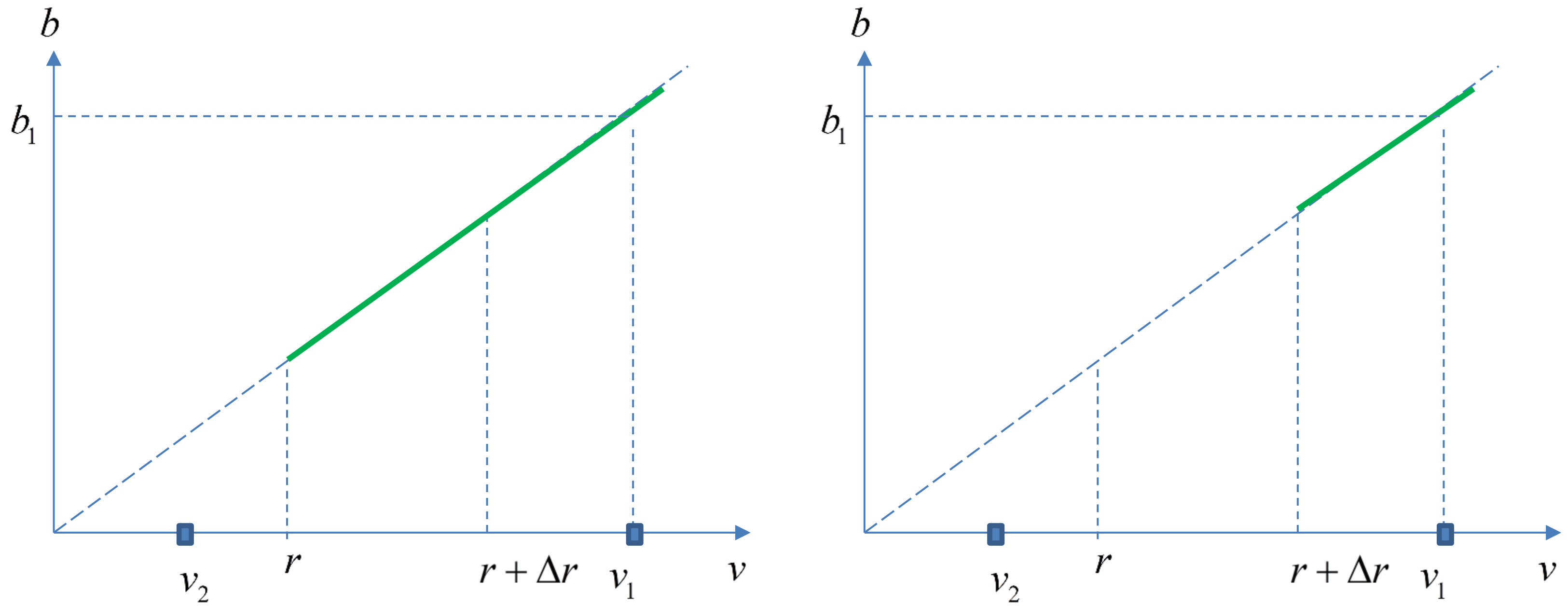
The price paid rises from v_2 to $r+\Delta r$.

Thus the change in payment is $r+\Delta r-v_2$ and this lies between 0 and Δr .

Combining the first order probability and first order change in payment, this is a second order effect.

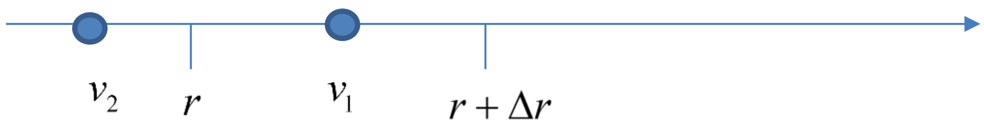
Case 5:

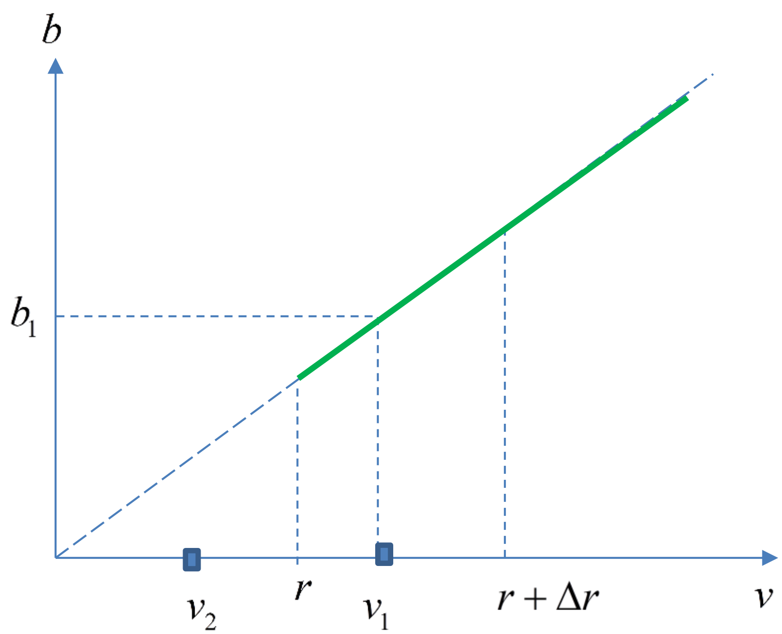


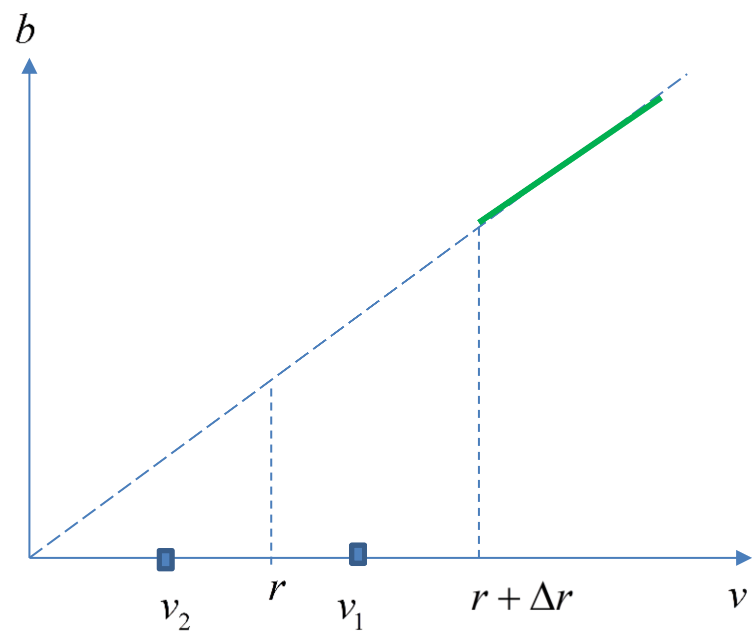


The probability is $p(1-p-\Delta p)$. The change in payment is Δr .

Case 6:







The probability is $p\Delta p$. The change in revenue is $-r$

Case 5: The probability is $p(1-p-\Delta p)$. The change in payment is Δr .

Case 6: The probability is $p\Delta p$. The change in revenue is $-r$

The change in expected profit is therefore $\Delta\Pi = p(1-p-\Delta p)\Delta r - p\Delta pr$

Ignoring the second order effect,

$$\Delta\Pi = p(1-p)\Delta r - p\Delta pr .$$

Therefore

$$\frac{\Delta\Pi}{\Delta r} = p[(1-p) - r\frac{\Delta p}{\Delta r}]$$

The bracketed expression is the marginal profit from raising r with one buyer.

Thus if r^* is the unique solution to the necessary condition for a maximum with one buyer it is also the maximizing reserve price for two buyers.

Remark: A very similar argument holds when there are three or more buyers.

F. Sealed high bid auction model

Private information: Each buyer's value is private information.

**

Sealed high bid auction model

Private information: Each buyer's value is private information.

Common knowledge: It is common knowledge that buyer i's value is an independent random draw from a continuous distribution . We define

$$F(\theta) = \Pr\{v_i \leq \theta\}.$$

This is called the cumulative distribution function (c.d.f.).

*

Sealed high bid auction model

Private information: Each buyer's value is private information.

Common knowledge: It is common knowledge that buyer i's value is an independent random draw from a continuous distribution . We define

$$F(\theta) = \Pr\{v_i \leq \theta\}.$$

This is called the cumulative distribution function (c.d.f.).

The values: The values lie on an interval $\Theta = [0, \beta]$.

Strategies

With private information a player's action depends upon his private information. In the sealed high-bid auction, a player's private information is the value θ_i that he places on the item for sale. His bid is then some mapping $b_i = B_i(\theta_i)$ from every possible value (i.e. every $\theta_i \in \Theta$) into a non-negative bid.

This mapping is the player's bidding strategy.

Buyers with higher values have more to lose by not winning so it is natural to assume that buyers with higher values will bid more so that $B_i(\theta_i)$ is a strictly increasing function.

**

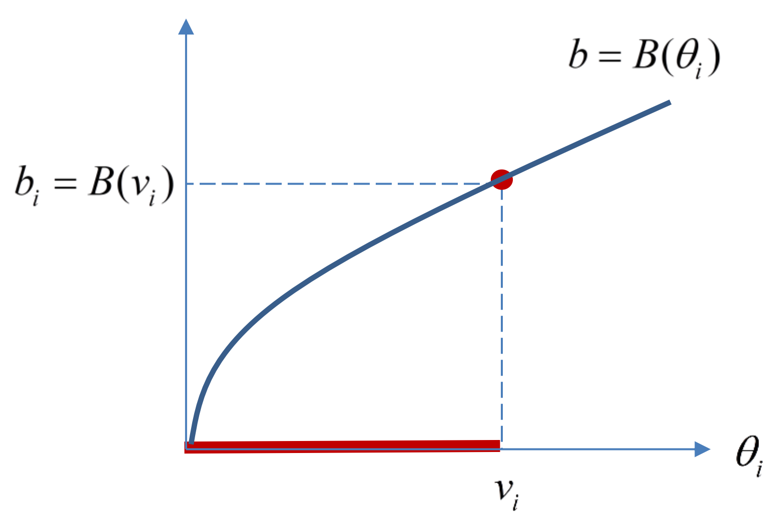
Strategies

With private information a player's action depends upon his private information. In the sealed high-bid auction, a player's private information is the value θ_i that he places on the item for sale. His bid is then some mapping $b_i = B_i(\theta_i)$ from every possible value (i.e. every $\theta_i \in \Theta$) into a non-negative bid.

This mapping is the player's bidding strategy.

Buyers with higher values have more to lose by not winning so it is natural to assume that buyers with higher values will bid more so that $B_i(\theta_i)$ is a strictly increasing function.

Since we assume that each buyer's value is a draw from the same distribution it is natural to assume that the equilibrium is symmetric. $B_i(\theta_i) = B(\theta_i)$



Equilibrium Strategies

Bayesian Nash Equilibrium (BNE) strategies: With private information mutual best response strategies are called Bayesian Nash Equilibrium strategies.

Symmetric BNE of the sealed high bid auction

If all other buyers other then buyer i use the bidding strategy $b_j = B(\theta_j)$ then buyer i's best response is to use the same strategy, i.e. $b_i = B(\theta_i)$.

An example: Two buyers with values uniformly distributed on $[0,100]$.

For the uniform distribution values are equally likely.

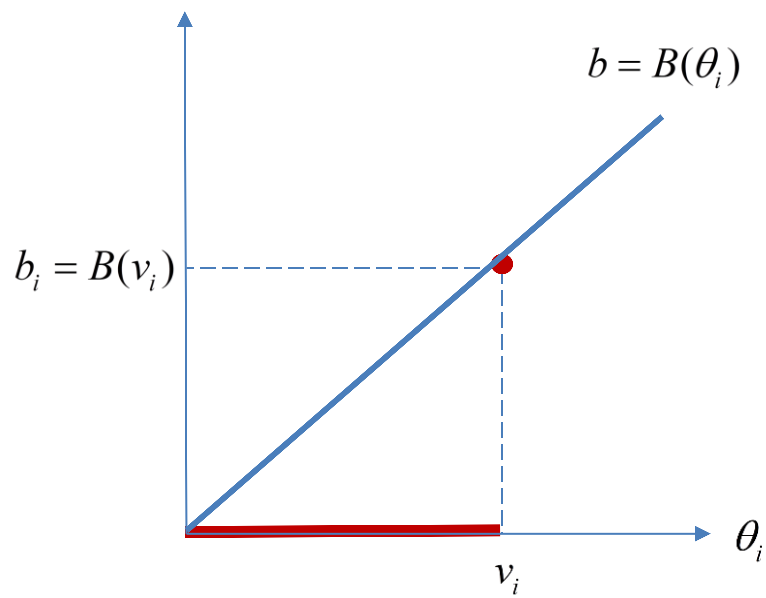
Therefore $\Pr\{v_i \le 25\} = \dfrac{25}{100}$, $\Pr\{v_i \le 50\} = \dfrac{50}{100}$, $\Pr\{v_i \le 80\} = \dfrac{80}{100}$ .....

Thus the c.d.f. is $F(\theta_i) = \Pr(v_i \le \theta_i) = \dfrac{\theta_i}{100}$

For any guess as to the equilibrium strategy,

We can check to see if the guess is correct.

*

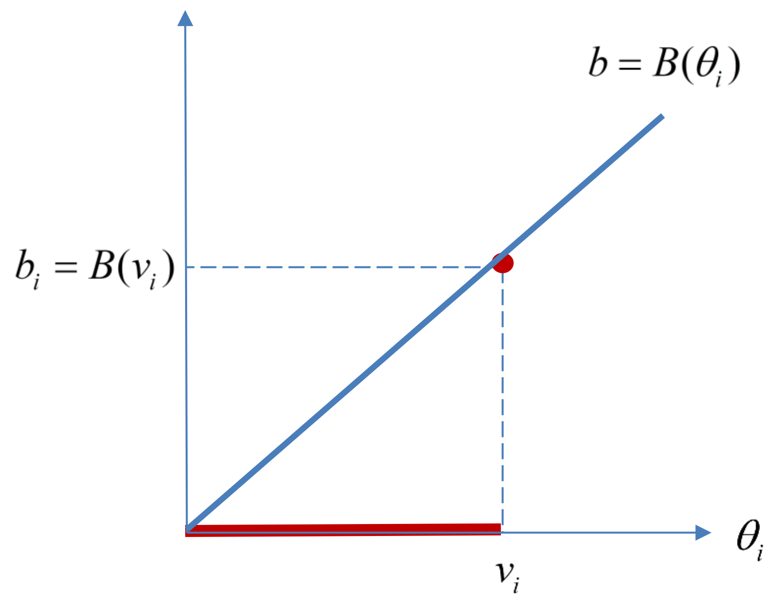


An example: Two buyers with values uniformly distributed on $[0,100]$.

For the uniform distribution values are equally likely.

Therefore $\Pr\{\theta_i \le 25\} = \frac{25}{100}$, $\Pr\{\theta_i \le 50\} = \frac{50}{100}$, $\Pr\{\theta_i \le 80\} = \frac{80}{100}$.....

Thus the c.d.f. is $F(\theta) = \Pr(v_i \le \theta) = \frac{\theta_i}{100}$



For any guess as to the equilibrium strategy,

We can check to see if the guess is correct.

There are two buyers. Suppose that

buyer 2 bids according to the strategy

$$B(\theta_2) = \tfrac{1}{2}\theta_2 .$$

We need to show that buyer 1's best response is to bid $b_1 = \frac{1}{2}\theta_1$.

Then these strategies are mutual best responses.

Solving for buyer 1's best response when his value is v_1

If buyer 1 bids b he has the high bid if $B(\theta_2) = \frac{1}{2}\theta_2 \le b$, i.e. $\theta_2 \le 2b$.

Buyer 1's win probability is therefore $w(b) = \Pr\{\theta_2 \le 2b\} = \dfrac{2b}{100}$.

Buyer 1's expected payoff is therefore

$$U_1(v_1, b) = (v_1 - b)w(b) = (v_1 - b)\frac{2b}{100} = \frac{2}{100}(v_1 b - b^2)$$

$$\frac{\partial U_1}{\partial b}(v_1, b) = \frac{2}{100}(v_1 - 2b)$$

Therefore buyer 1's expected gain is maximized if $b_1 = \frac{1}{2}v_1$.

Then $b_j = B(\theta_j) = \frac{1}{2}\theta_j$ is the equilibrium bidding strategy.

Exercise: Three buyers with values uniformly distributed

(a) Show that if buyer 2 and buyer 3 bid according to $b_j = \frac{1}{2}\theta_j$, then buyer 1's best response is to bid $b_1 > \frac{1}{2}v_1$ when his value is v_1

(b) Show that for some $\alpha > \frac{1}{2}$, $b_j = B(\theta_j) = \alpha\theta_j$ is the equilibrium bidding strategy

(c) What is the equilibrium bidding strategy with 4 buyers?

Answer to (b)

The probability that buyer 1 wins with a bid of b is the joint probability that $b_2 \leq b$ and $b_3 \leq b$, ie.

$$w_1(b) = \Pr\{b_2 \leq b\} \times \Pr\{b_3 \leq b\}$$

$$= \Pr\{\alpha v_2 \leq b\} \times \Pr\{\alpha v_3 \leq b\}$$

$$= \Pr\{v_2 \leq \frac{b}{\alpha}\} \times \Pr\{v_3 \leq \frac{b}{\alpha}\} = F^2(\frac{b}{\alpha}) = (\frac{b}{\alpha})^2$$

$$U_1(v_1, b) = (v_1 - b)w(b) = (v_1 - b)(\frac{b}{\alpha})^2 = \frac{1}{\alpha^2}(v_1 b^2 - b^3)$$

$\dfrac{\partial U_1}{\partial b} = \dfrac{1}{\alpha^2}(2v_1 b - 3b^2) = 0$ for a maximum.

Therefore buyer 1's best response is $B_1(v_1) = \frac{2}{3}v_1$.

Note that this is true if $\alpha = \frac{2}{3}$. Thus if the other buyers bid $B_j(\theta_j) = \frac{2}{3}\theta_j$, then buyer 1's best reponse is to do so as well.

The problem with this approach is that it requires an inspired guess.

