uni1219a/d.doc

UNIQUENESS IN SEALED HIGH BID AUCTIONS

by

Eric Maskin and John Riley

Last Revision

December 14, 1996**

* Department of Economics, Harvard University and UCLA

** A much earlier version of this paper focussed on the symmetric case and the 2 bidder case under asymmetry. Comments by Bernard LeBrun are gratefully acknowledged.

While much has been written on the theory of auctions, almost all of this work focusses exclusively on the symmetric equilibrium of an auction in which bidders are symmetric. That is, two bidders with the same private information have exactly the same beliefs about all of the opposing bidders.

In a companion paper (Maskin and Riley (1994), we have examined the question of existence of equilibrium in a sealed high bid auction in the absence of the symmetry assumption. There we show that under quite weak assumptions there exists an equilibrium in which bids increase monotonically with bidders' reservation prices for the item.

In this paper we turn to the question of uniqueness. Under the symmetry assumption it is well known that there is a unique symmetric equilibrium (Milgrom and Weber, 1982, Maskin and Riley, 1984). However, it is not unreasonable to suppose that a particular buyer might establish a reputation as an aggressive bidder if it is in his interest to do so. Riley (1980) provides an example of the "war of attrition" in which this is indeed the case. In fact there is a continuum of asymmetric equilibria in which one buyer bids "aggressively" and the other "passively". Furthermore, the greater the degree of aggression, the larger is the equilibrium expected gain of the aggressive buyer.

A second example of a continuum of equilibria occurs in the common value auction, if the item is sold by open ascending bid. As first noted by Milgrom (1981) there is always a continuum of equilibria in the two buyer case. Bikchandani and Riley (1991) also present an example in which, with n bidders, there is a continuum of equilibria.

For the symmetric sealed high bid auction, however, we show that there can be no asymmetric equilibrium under the assumption of independence. Thus equilibrium is unique.

When we drop the symmetry assumption we have a very general uniqueness result if there are only 2 bidders. All we require is a mild additional restriction on preferences which ensures monotonicity.

For the general case of n bidders, our results are more limited. If differences can be parametrized simply as variations in the distribution of reservation prices, we have a further quite strong uniqueness result. But when buyers also have different preferences all we are only able to establish uniqueness when these differences are small.

We describe the model in section 1. In section 2 we present characterization results. We use these in section 3 to derive our main theorems. Section 4 considers the possibility that buyers might sometimes "overbid", that is, bid more than their reservation prices. Concluding remarks are in section 5.

1. THE MODEL

Throughout the paper we shall make the following assumptions about the auction and those participating in it. A single item is to be sold to the buyer who makes the highest non-negative bid. If two or more tie, the winner is selected at random from among the high bidders. There are n potential buyers. Buyer i has a utility of $u_i(b,s_i)$ if he wins with a bid of b and is of type s_i. Buyer i's type has support $[a_i, \bar{s}_i]$ and is distributed with c.d.f. $F_i(\cdot)$. We assume that F_i is continuously differentiable and that the density is strictly positive on $[a_i, \bar{s}_i]$. Without loss of generality we normalize so that the utility of buyer i is zero if his bid is unsuccessful. We further assume that $u_i(b,s_i)$ is continuously differentiable,

with $\frac{\partial u_i}{\partial b} < 0$ and $\frac{\partial u_i}{\partial s} > 0$, $i = 1, \ldots, n.$

Let $\bar{b}_i(s_i)$ be the reservation price of buyer *i* if his type is s_i, that is $u_i(\bar{b}_i(s_i), s_i) = 0$. We assume that for all *i*, buyer *i*'s highest reservation price, $\bar{b}_i(\bar{s}_i)$, is strictly positive (otherwise buyer *i* never has an incentive to bid). Clearly buyers cannot gain by bidding more than their reservation prices. In fact we will begin by assuming that they do not.

Assumption 1: No buyer bids ever bids more than his reservation price.

Next, define $\underline{s}_i$ to be the lowest type with a non-negative reservation price. If $\underline{s}_i > a_i$ so that $F_i(\underline{s}_i) > 0$, all those types $s_i < \underline{s}_i$ have a negative reservation price and are therefore strictly worse off submitting a winning bid than staying out of the auction. Throughout we assume that such types never bid and therefore focus on types drawn from the interval $S_i \equiv [\underline{s}_i, \bar{s}_i]$.

We next introduce the usual "single crossing property" which underlies so much of incentive theory. Let $M_i(b, s_i)$ be the rate at which a buyer is willing to increase his bid in return for a greater probability of winning. As is easily confirmed, Assumption 2 is the requirement that M_i increases with s_i.

Assumption 2: **Single Crossing Property**

For all $u_i(b, s_i) > 0$, $\frac{\partial}{\partial b} \ln u_i$ is strictly increasing with s.

We have argued elsewhere (Maskin and Riley, 1984) that this is a weak assumption. Indeed if $u_i(b, s_i) = V_i(s_i - b)$ so that s_i is buyer *i*'s reservation price, Assumption 2 holds if buyer i is risk neutral or risk averse.

For some of our results we will also explicitly introduce the assumption of risk aversion.

Assumption 3: Buyers are (weakly) risk averse.

$u_i(b, s_i) > 0 \Rightarrow u_i(b, s_i)$ is concave in *b*.

2. CHARACTERIZING THE EQUILIBRIUM BID FUNCTIONS

Above we defined $\bar{b}_i(s_i)$ to be the reservation price of buyer i if his type is s_i, that is, $u_i(\bar{b}_i(s_i), s_i) = 0$. It will be useful below to define the inverse function

(2.1) $\bar{f}_i(b) \equiv \bar{b}_i^{-1}(b)$.

That is, $\bar{f}_i(b)$ is the smallest type i buyer willing to pay b for the item. From Maskin and Riley (1994) we have the following results.

Lemma 1: If Assumptions 1 and 2 hold, the distribution of winning bids has support $[b_*, b^*]$ and c.d.f. which is continuous on $(b_*, b^*]$.

Lemma 2: **Monotonicity**

Suppose that a realization of i's equilibrium strategy is b' if his type is s' and b'' if his type is $s'' > s'$. Suppose, moreover that the expected return to bidding b' is strictly positive. Then $b' \leq b''$

As our first preliminary here, we characterize b_*, the lower support of the distribution of winning bids.

Lemma 3: **Characterization of the minimum bid**

Let $\bar{b}_i(\underline{s}_i)$ be the lowest nonnegative reservation price of buyer i, $i = 1,\ldots,n$. Without loss of generality we suppose that $\bar{b}_n(\underline{s}_n) \leq \ldots \leq \bar{b}_2(\underline{s}_2) \leq \bar{b}_1(\underline{s}_1)$

If Assumption 2 holds, the minimum bid satisfies

(2-2) $\bar{b}_2(\underline{s}_2) \leq b_* \leq \bar{b}_1(\underline{s}_1)$

Moreover, either both of these are equalities or both are strict inequalities. If the latter, (2-3) $$b_* = Max \arg \underset{b}{Max} \mathop{\times}_{i=2}^{n} F_i(\bar{f}_i(b)u_i(b,\underline{s}_i)$$

where $\bar{f}_i(b) \equiv \bar{b}_i(\underline{s}_i), i = 1,\ldots,n$

<u>Proof</u>: Suppose that $b_* < \bar{b}_2(\underline{s}_2)$. By Lemma 1 there are no mass points on $(b_*, b^*]$. Then both buyer 1 and buyer 2, regardless of type, have a strictly positive expected payoff from bidding b_l, where

$$b_l \equiv lb_* + (1-l)\bar{b}_2(\underline{s}_2),\ 0 < l < 1.$$

Let $p_i, i = 1,2$ be the probability that buyer i bids b_*. If both p_1 and p_2 are strictly positive, bidding b_* results in a tie with positive probability. Then buyer 1, regardless of his type, is strictly better off bidding slightly above b_* since this breaks the tie and increases his win probability by a finite amount.

Hence p_1 and p_2 cannot both be strictly positive. Suppose then that p_1 is zero. Consider a bid b_1 by buyer 2 in the neighborhood of b_*. Since $p_1 = 0$, buyer 2's probability of winning and hence his expected utility declines towards zero as $l \to 1$. But we have already argued that buyer 2's equilibrium expected utility is strictly positive so again we have a contradiction.

Next suppose that $b_* > \bar{b}_1(\underline{s}_1)$. Any buyer type who submits a bid must have a reservation price of at least b_*. Then any such type must have a strictly positive expected utility since a bid in the interval $(\bar{b}_1(\underline{s}_1), b_*)$ wins with positive probability. It follows that any buyer type who submits a bid has a reservation price exceeding b_*. But then at most one buyer bids b_* with positive probability. (For otherwise it would pay to break the tie by bidding slightly more.) Again we have a contradiction, hence b_* satisfies (2.2).

Suppose next that $\bar{b}_2(\underline{s}_2) < \bar{b}_1(\underline{s}_1)$. Any buyer $i > 1$ is better off bidding above b_* if his reservation price exceeds b_*. Given Assumption 1, buyer i bids less than b_* if his reservation price is less than b_*. If buyer 1 bids b_* with positive probability and buyer i's reservation price exceeds b_* he is strictly better off responding with a bid just above b_* since his probability of winning rises discontinuously at b_*. If buyer 1 bids b_* with zero probability, buyer i's expected payoff is zero if he bids b_*. Thus again he is strictly better off responding with a bid greater than b_*. Combining these results it follows that for all $i > 1$, buyer 1 outbids buyer i with probability $F_i(\bar{f}_i(b_*))$ when he bids b_*. The expected payoff to buyer 1 of type $\underline{s}_1$ if he makes his equilibrium bid of b_* is therefore $\prod_{i=2}^{n} F_i(\bar{f}_i(b_*)u_i(b_*,\underline{s}_1)$.

Since we have assumed that no buyer ever bids more than his reservation price, if buyer 1 bids $b \neq b_*$ his expected payoff is at least $\prod_{i=2}^{n} F_i(\bar{f}_i(b)u_i(b,\underline{s}_1)$. It follows that for b_* to be a best response for type $\underline{s}_1$,

$$\prod_{i=2}^{n} F_i(\bar{f}_i(b)u_i(b,\underline{s}_1) \leq \prod_{i=2}^{n} F_i(\bar{f}_i(b_*)u_i(b_*,\underline{s}_1)$$

Thus

$$b_* \in \arg \underset{b}{Max} \prod_{i=2}^{n} F_i(\bar{f}_i(b))u_i(b,s_1)$$

Finally, suppose that both b' and b'' solve this maximization problem and that $b' < b''$. Buyer 1 of type $\underline{s}_1$ is at least indifferent between bidding b'' and any lower bid. Given Assumption 2, all other buyer 1 types strictly prefer b'' over any lower bid. Thus the minimum bid for all types $s_1 > \underline{s}_1$ is at least b''. But then b' is not the lower support of the equilibrium distribution of winning bids.

Q.E.D.

Lemma 4: **Strict Monotonicity of the probability of winning**:

Let $G_w(b)$ be the c.d.f. of the distribution of winning bids. Suppose $b' < b''$ and $0 < G_w(b') < G_w(b'') < 1$. Then at least two bidders bid in the interval (b', b'') with positive probability.

Proof: Suppose first that there exists some such interval over which no one bids with positive probability. Define $\hat{b} \equiv \inf\{b \mid G_w(b) < G_w(b')\}$. With no one bidding in (b', b'') with positive probability it follows that $\hat{b} > b''$. By Lemma 1, any buyer bidding ties with probability zero. Then such a buyer can lower his bid towards b' and so raise his gain to winning without lowering his probability of winning. But then bidding $\hat{b}$ cannot be a best response.

Suppose then that only buyer 1 bids in the interval (b', b'') with positive probability. In this case buyer 1 will never bid in the interval $(\frac{1}{2}b' + \frac{1}{2}b'', b'')$ with positive probability since he can lower his bid to just above b' without lowering his win probability. Thus no buyer bids in the interval $(\frac{1}{2}b' + \frac{1}{2}b'', b'')$ with positive probability. But this contradicts our earlier conclusion.

Q.E.D.

Let $(\tilde{b}_1(s_1), \ldots, \tilde{b}_n(s_n))$ be equilibrium bidding strategies (possibly mixed strategies.) Any deterministic selection $b_1(s_1)$ from $\tilde{b}_1(s_1)$ is strictly increasing for all $s_i \in S_i$. It follows that

$$y_i(\cdot) = \tilde{b}_i^{-1}(\cdot)$$

is an increasing function that is well defined at all b for which there exists s_i with $b \in \operatorname{supp} \tilde{\mathrm{b}}_\mathrm{i}(s_i)$. Then, for all bids exceeding the minimum bid b_* we can define

(2-4) $$f_i(b) = \sup\{y_i(\hat{b}) \mid \hat{b} \le b, y_i(\hat{b}) \text{ defined}\}$$

Because $y_i(\cdot)$ is increasing, $f_i(\cdot)$ is nondecreasing and continuous for all $b > b_*$. Note, furthermore, that the probability of winning can be written as

$$G_i(b) \equiv \underset{j \ne i}{\times} F_i(f_i(b))$$

Since $f_j(b)$ is continuous for all j, so is $G_i(b)$.

As a preliminary to proving uniqueness we now derive properties of $f_i(\cdot)$ and $G_i(b)$. Proofs can be found in the Appendix.

Lemma 5: **Strict monotonicity property of bid distributions.**

Let $G_i(b)$ be the c.d.f. of the maximum bid of all i's opponents. Then for any $\hat{b} = b_i(\hat{s}_i)$ such that $0 < G_i(\hat{b}) < 1$, and for any $e > 0$, $G_i(\hat{b} - e) < G_i(\hat{b})$.

Lemma 6: If $f_i(b)$ is strictly increasing to the right (from the left) at $b = b$, then b is a best response for $\hat{s}_i = f_i(b)$.

Lemma 7: If $f_i(b)$ is strictly increasing to the right (from the left) at $b = b > b_*$, $G_i(b)$ is right (left) differentiable at b. Moreover, the right (left) derivative satisfies

(2-5) $$G_i^{'}(b)u_i(b, f_i(b)) + G_i(b)\frac{\P}{\P b}u_i(b, f_i(b)) = 0$$

Lemma 8: $f_i(b)$ is right (left) differentiable for all $b > b_*$ and all i.

Suppose $f(b) \equiv (f_1(b),\ldots, f_n(b))$ is strictly increasing at b. It follows from Lemmas 7 and 8 that $f(b)$ satisfies

$$\text{(2-6)} \quad \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{F_j^{'}(f_j)}{F_j(f_j)} \frac{d f_j}{db} = \frac{\frac{\partial}{\partial b} u_i(b, f_i)}{u_i(b, f_i)}$$

We can rewrite this in matrix form as follows.

$$\text{(2-7)} \quad \mathbf{A}\,[\frac{F_j^{'}(f_j)}{F_j(f_j)} \frac{d f_j}{db}] = [\frac{\frac{\partial}{\partial b} u_i(b, f_i)}{u_i(b, f_i)}] \text{ where } \mathbf{A} = \begin{bmatrix} 0 & 1 & 1. & . & 1 \\ 1 & 0 & 1 & . & 1 \\ 1 & 1 & 0 & . & 1 \\ . & . & . & . & . \\ 1 & 1 & . & 1 & 0 \end{bmatrix}.$$

Lemma 9: Endpoint condition if no bidder has a positive probability of winning at the minimum bid.

Suppose that $F_i(\underline{s}_i) = 0$ and $u_i(b_*, \underline{s}_i) = 0$, $i = 1,\ldots,n$. Define

$$e_i \equiv \frac{\underline{s}_i F_i^{'}(\underline{s}_i)}{F_i(\underline{s}_i)}$$

Then if the vector of equilibrium inverse bid functions $f(b) \equiv (f_1(b),\ldots, f_n(b))$ satisfies the endpoint condition

$$f_i(b_*) = \underline{s}_i,\ i = 1,\ldots,n,$$

and is strictly increasing at b_*,

$$\text{(2-7)} \quad f_i^{'}(b_*) = (1 + \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^{n} e_j}) \frac{\frac{\partial}{\partial b} u_i(b_*, \underline{s}_i)}{u_i(b_*, \underline{s}_i)}$$

3. UNIQUENESS

It is now helpful to transform variables and define

$$\text{(3-1)} \qquad z_i = \ln F_i(s_i),\ s_i \in [\underline{s}_i, \bar{s}_i].$$

Since this function is strictly increasing over its domain we can invert and define the strictly increasing function

$$\text{(3-2)} \qquad h_i(z) = F_i^{-1}(e^z), \quad z_i \in [\underline{z}_i, 0], \text{ where } \underline{z}_i \equiv \ln F_i(\underline{s}_i).$$

Also define $v_i(b, z_i) \equiv \ln u_i(b, h_i(z_i))$.

By Lemmas 7 and 8, if $f_i(\cdot)$ is increasing at b, then b is the solution to the following maximization problem:

$$\underset{x}{Max}\ U_i(x, f_i(b)) = \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(f_j(x)) u_i(x, f_i(b))$$

Moreover the first order condition

$$\text{(3-3)} \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{F_j^{'}(f_j)}{F_j(f_j)} \frac{d f_j}{db} = \frac{\frac{\partial}{\partial b} u_i(b, f_i)}{u_i(b, f_i)}$$

must be satisfied, where it is understood that the derivatives are either left or right derivatives.

Then after transforming the variables, b is the solution to the maximization problem

$$\underset{x}{Max}\ V_i(x, z_i(b)) = \sum_{\substack{j=1 \\ j \neq i}}^{n} z_j(x) + v_i(x, z_i(b))$$

and must satisfy the first order conditions:

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{dz_j}{db} + \frac{\partial}{\partial b} v_i(x, z_i(b)) = 0, \; i = 1, \ldots, n.$$

To simplify notation we also define

$$P_i(b, z_i) = -\frac{\partial}{\partial b} v_i(x, z_i), \quad i = 1, \ldots, n.$$

If Assumptions 2 and 3 hold,

(3-4) $\quad v_i > 0 \Rightarrow \dfrac{\partial P_i}{\partial z_i} < 0$ and $\dfrac{\partial P_i}{\partial b} > 0$.

The first order conditions can then be rewritten as:

(3-5) $$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{dz_j}{db} = P_i(b, z_i), \; i = 1, \ldots, n.$$

Lemma 10: Consider solutions $(z_1(b), \ldots, z_k(b))$ and $(\hat{z}_1(b), \ldots, \hat{z}_k(b))$ to the system of differential equations

(3-6) $$\sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{dz_j}{db} = P_i(b, z_i), i = 1, \ldots, k.$$

on some interval $[b', b'']$ over which, for all $i=1,..,k$, $z_i(b) < 0$ and $P_i(b, z_i) > 0$. Suppose that $\hat{z}_j(b'') - z_j(b'') > 0$ for all $j = 1,...,k$. Then $\hat{z}_j(b') - z_j(b') > 0$, $j = 1, \ldots, k$

Moreover,

$$\frac{d}{db}[\sum_{j=1}^{k} \hat{z}_j(b) - z_j(b)] < 0, \quad b \in [b', b''].$$

Proof:

Let $z_j(b, a)$, $j = 1, \ldots, k$ be a solution to the system of differential equations satisfying the endpoint condition

$$z_j(b'', a) = (1 - a) z_j(b'') + a \hat{z}_j(b'')$$

Then $\frac{\P}{\P a} z_j(b'',a) = \hat{z}_j(b'') - z_j(b'') > 0, \quad j = 1,\ldots,k.$

Rewriting (3-6) in matrix form we have

(3-7) $\mathbf{A}[z_j^{'}(b)] = [P_i(b,z_i)]$

where $\mathbf{A}$ is as defined in (2-7) except that it is now a $k \times k$ matrix. It is readily confirmed that $\mathbf{A}$ is invertible and that

$$\mathbf{B} \equiv \mathbf{A}^{-1} = \frac{1}{k-1}\begin{bmatrix} g & 1 & 1 & . & . \\ 1 & g & 1 & . & . \\ 1 & 1 & g & . & . \\ . & . & . & . & 1 \\ 1 & 1 & 1 & . & g \end{bmatrix} \text{ where } g = -(k-2)$$

Inverting (3-7) we obtain

(3-8) $[\frac{dz_j}{db}] = \mathbf{B}[P_i(b,z_i)]$

In particular,

(3-9) $\frac{dz_j}{db} = \mathbf{B}_j[P_i(b,z_i)] = \frac{1}{k-1}(\sum_{\substack{i=1\\ i\neq j}}^{k} P_i - (k-2)P_j)$ 1

where $\mathbf{B}_j$ is the jth row of $\mathbf{B}$.

Summing over j,

$$\sum_{j=1}^{n} \frac{dz_j}{db} = \frac{1}{k-1}\sum_{j=1}^{k} P_j$$

Differentiating by a ,

(3-10) $\frac{d}{db}\sum_{j=1}^{k} \frac{\P z_j}{\P a} = \frac{1}{k-1}\sum_{j=1}^{k} \frac{\P P_j}{\P z}\frac{\P z_j}{\P a}$ 2

By construction $\frac{\P z_j}{\P a} > 0$ at b'', $j=1,\ldots,k$.

Define $\hat{b} = \inf\{b | \frac{\P z_j}{\P a} > 0, \text{ for all } j = 1,\ldots,k\}$

Then, from (3-10) $\frac{d}{db}\sum_{j=1}^{k}\frac{\partial z_j}{\partial a}<0$ on $(\hat{b},b'']$.

Hence for some $i=1,\ldots,k$

(3-11) $\frac{\partial z_i(\hat{b})}{\partial a}>\frac{\partial z_i(b'')}{\partial a}>0.$

Differentiating (3-9) by a

(3-12) $\frac{d}{db}\frac{\partial z_j}{\partial b}=\frac{1}{k-1}\left(\sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\partial P_i}{\partial z}\frac{\partial z_i}{\partial b}_i-(k-2)\frac{\partial P_j}{\partial z}\frac{\partial z_j}{\partial b}\right)$

By construction, there must be some j such that $\frac{\partial}{\partial a}z_j(\hat{b},a)=0.$ Thus, for this j, the final term on the right hand side of (3-12) approaches zero as $b\downarrow\hat{b}$. Moreover, from (3-11) $\frac{\partial}{\partial a}z_i(\hat{b},a)>0$ for at least one other $i\neq j$. Then, since $\frac{\partial P_i}{\partial z}<0$, the right hand side of (3-12) is strictly less than zero in some right neighborhood of $\hat{b}$. Hence $\frac{\partial z_j}{\partial a}$ is strictly decreasing in this right neighborhood of . But then $\frac{\partial}{\partial a}z_j(\hat{b},a)$ cannot be zero after all. We conclude that for all *j =1,...,k*, and all $b<b''$, $\frac{\partial z_j}{\partial a}>0.$
This proves the first claim. The second claim follows immediately from (3-10).

Q.E.D.

The proof of uniqueness for the case of two buyers is now relatively straightforward.

Proposition 1: **Uniqueness with two buyers**[1]

If Assumption 1 holds, equilibrium is unique.

<u>Proof</u>: Lemma 3 uniquely defines the lower support of each buyer's bid distribution, b_*. By

Lemma 4 the support must be an interval, $[b_*,b^*]$.

[1]With only a little further work, the proof of uniqueness also provides an alternative proof of existence for the 2 buyer case.

Case (i): For some i, $F_i(\bar{f}_i(b_*)) > 0.$

In this case, for some i, the lower support for z_i, $\underline{z}_i = \ln F_i(\underline{s}_i)$ is bounded from below. By Lemma 8, buyers' equilibrium inverse bid functions satisfy (3-3) and hence (3-6) must also hold. Then by Lemma 10, there is a unique b^* such that the pair of differential equations $(z_1(b), z_2(b))$ satisfying $z_i(b^*) = 0, i = 1,2,$ also satisfies the lower boundary condition $z_i(0) = \underline{z}_i, i = 1,2.$

Case (ii): For all i, $F_i(\bar{f}_i(b_*)) = 0.$

Since both equilibrium inverse bid functions must be strictly increasing we can apply Lemma 9. That is, any equilibrium bid functions for buyer i must have the same slope at b^*.

Let g be the maximum bid in one equilibrium and let $\hat{g} < g$ be the maximum bid in another. Let $f_i(b, g)$ and $f_i(b, \hat{g})$ be corresponding equilibrium inverse bid functions. Then

$$F_i(f_i(\hat{g}, \hat{g})) = 1 > F_i(f_i(\hat{g}, g), i = 1,2.$$

By Lemma 10, $f_i(b, \hat{g}) > f_i(b, g)$ for all $b > b_*$.

From Lemma 9,

$$F_i(f_i(b)) - F_i(\hat{f}_i(b)) = O((b - b_*)^2)$$

Since $F'(\cdot)$ is strictly positive and $F_i(f_i(b_*)) = F_i(\bar{f}_i(b_*)) = 0,$ it also follows from Lemma 9 that

$$F_i(f_i(b)) = O(b - b_*).$$

Thus

$$\frac{F_i(\hat{f}_i(b)) - F_i(f_i(b_*))}{F_i(f_i(b_*))} = O(b - b_*)$$

It follows that for any $e > 0$, there exists $d > 0$ such that

$$\frac{F_i(\hat{f}_i(b)) - F_i(f_i(b))}{F_i(f_i(b))} < e, \quad \text{for all } b \in [b_*, b_* + d],$$

Rearranging and taking logs

$$\ln F_i(\hat{f}_i(b) < \ln F_i(f_i(b) + \ln(1+e)$$

Summing over i,

$$\sum_{i=1}^{2} \ln F_i(\hat{f}_i(b) - \ln F_i(f_i(b) < 2\ln(1+e), \quad b \in [b_*, b_* + d].$$

Moreover, by Lemma 9, this difference is decreasing in b. Then for all b,

$$\sum_{i=1}^{2} \ln F_i(\hat{f}_i(b) - \ln F_i(f_i(b) < 2\ln(1+e).$$

By construction the first sum is zero at b^*. Then

$$\sum_{i=1}^{2} \ln F_i(f_i(b) > \ln(1+e)^{-2}.$$

We have therefore shown that

$$1 > \prod_{i=1}^{2} F_i(f_i(\hat{b}, g) > \frac{1}{(1+e)^2}$$

But this must hold for all $e > 0$. Thus $g = \hat{g}$ and so again equilibrium is unique.

Q.E.D.

For more than 2 buyers, establishing uniqueness is significantly more complicated since it is no longer necessarily the case that all buyers have equilibrium bid distributions with the same support. It is intuitively clear that buyers may not have the same maximum bid. For if buyer 3's maximum reservation price is far lower than that of buyer 1 and buyer 2, it is likely that competition between the latter buyers will push the maximum bid above anything buyer 3 is willing to pay. While this complication can be dealt with, there is a further problem. In general there is no reason to suppose that the support of each buyer's equilibrium bid distribution is an interval. Instead there may be "gaps", that is, intervals over which a buyer does not bid.

As we shall see, such possibilities cannot arise if the following assumption on preferences also holds.

Assumption 4:

$$\frac{\frac{\partial}{\partial b}u_i(b,s_i)}{u_i(b,s_i)} > \frac{\frac{\partial}{\partial b}u_j(b,s_j)}{u_j(b,s_j)} \Rightarrow \frac{\partial}{\partial b}\ln\left(\frac{\frac{\partial}{\partial b}u_i(b,s_i)}{u_i(b,s_i)}\right) > \frac{\partial}{\partial b}\ln\left(\frac{\frac{\partial}{\partial b}u_j(b,s_j)}{u_j(b,s_j)}\right)$$

It can be readily confirmed that Assumption 4 holds if

$$u_i(b,s_i) = 1 - e^{-A(s_i - b)}$$

or

$$u_i(b,s_i) = (w + s_i - b)^q - (w)^q,\ 0 < q \le 1,$$

that is, all buyers are risk neutral or all have the <u>same</u> constant degree of absolute or relative risk aversion. Thus, in the case of identical preferences, the assumption is relatively mild. On the other hand, Assumption 4 fails generically if preferences differ.

Note also that after transforming variables, Assumption 4 becomes

$$P_i(b,z_i) > P_j(b,z_j) \Rightarrow \frac{\partial}{\partial b}\ln P_i(b,z_i) > \frac{\partial}{\partial b}\ln P_j(b,z_j)$$

Lemma 11: Suppose equilibrium inverse bid functions are differentiable on $[b',b'']$.

If the logarithm of buyer r's expected payoff, $V_r(b,z_r), r > k,$ is non-increasing at b' and Assumptions 2-4 hold, $V_r(b,z_r)$ is decreasing on $[b',b'']$.

<u>Proof</u>: Suppose it is buyers *1,...,k* who have strictly increasing inverse bid functions, $f_1(b),\ldots, f_k(b)$. By Lemma 8, these inverse bid functions are differentiable on $[b',b'']$. From

(3-6) it follows that

$$\text{(3-13)} \qquad \sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{dz_j}{db} - P_i(b, z_i) = 0, \;\; i = 1, \ldots, k.$$

Totally differentiating by b,

$$\text{(3-14)} \qquad \sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{d^2 z_j}{db^2} - \frac{\partial P_i}{\partial b} = \frac{\partial P_i}{\partial z_i}\frac{\partial z_i}{\partial b} < 0, \;\; i = 1, \ldots, k.$$

Summing (3-13) over i from 1 to k,

$$\text{(3-15)} \qquad (k-1)\sum_{j=1}^{k} \frac{dz_j}{db} - \sum_{j=1}^{k} P_j(b, z_j) = 0 .$$

Similarly, summing (3-14) over i from 1 to k,

$$\text{(3-16)} \qquad (k-1)\sum_{j=1}^{k} \frac{d^2 z_j}{db^2} - \sum_{j=1}^{k} \frac{\partial P_i}{\partial b} < 0.$$

Consider the logarithm of buyer *r*'s expected payoff, $V_r = \ln U_r(b, f_r(b))$.

$$\frac{\partial}{\partial b} V_r(b, s_r) = \sum_{j=1}^{k} \frac{dz_j}{db} - P_r .$$

Hence from (3-15)

$$\text{(3-17)} \qquad \frac{\partial}{\partial b} V_r(b, s_r) = \frac{1}{k-1}\sum_{j=1}^{k} P_j - P_r$$

Also

$$\frac{\partial^2}{\partial b^2} V_r(b, s_r) = \sum_{j=1}^{k} \frac{d^2 z_j}{db^2} - \frac{\partial P_r}{\partial b} < \frac{1}{k-1}\sum_{j=1}^{k} \frac{\partial P_j}{\partial b} - \frac{\partial P_r}{\partial b}$$

Hence

$$\text{(3-18)} \qquad \frac{\partial^2}{\partial b^2} V_r(b, s_r) = \frac{\partial P_r}{\partial b}\,\frac{1}{k-1}\sum_{j=1}^{k} \frac{\frac{\partial P_j}{\partial b}}{\frac{\partial P_r}{\partial b}} - (k-1)$$

Suppose $\frac{\partial}{\partial b} V_r(b, s_r) \leq 0.$ From (3-17)

$$\sum_{j=1}^{k} \frac{P_j}{P_r} - (k-1) \leq 0.$$

It follows from (3-18) that

$$\text{(3-19)} \qquad \frac{\partial^2}{\partial b^2} V_r(b, s_r) < \frac{\partial P_r}{\partial b} \frac{1}{k-1} \sum_{j=1}^{k} \frac{\dfrac{\partial P_j}{\partial b}}{\dfrac{\partial P_r}{\partial b}} - \frac{P_j}{P_r}$$

From (3-13) and (3-15)

$$(k-1) P_i < \sum_{i=1}^{k} P_j, \; i = 1, \ldots, m.$$

Then, from (3-17) if $V_r = \frac{\partial}{\partial b} \ln U_r(b, s_r) \leq 0$, then $P_j < P_r$, $j = 1, \ldots, m.$ Appealing to Assumption 2 we obtain

$$\frac{\partial^2}{\partial b^2} V_r(b, s_r) < 0.$$

Q.E.D.

Appealing to Lemma 11 we have the following important result.

Lemma 12: **No Gaps**

If Assumption 4 holds, the support of buyer *i*'s equilibrium bid distribution is an interval

$[b_*, b_i^*]$, $i = 1, \ldots, n.$

<u>Proof</u>: Suppose that only $z_i, \ldots, z_k$ are strictly increasing (from the left) at $\hat{b}$. Then,

from (3-13),

$$\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{dz_j}{db} - P_i(b, z_i) = 0, \; i = 1, \ldots, k.$$

In matrix form,

$$\text{(3-20)} \qquad \mathbf{A}^{k \times k} [\frac{dz_j}{db}]^k = [P_i]^k.$$

Let $\mathbf{A}^{m\times m}$ be a matrix composed of the first m rows and columns of $\mathbf{A}^{k\times k}$. From (3-20),

(3-21) $\mathbf{A}^{m\times m}[\frac{dz_j}{db}]^m + a[1]^m = [P_i]^m.$

where $a = \sum_{j=m+1}^{k} \frac{dz_j}{db} > 0,$ unless $\frac{dz_j}{db} = 0,\ j = m+1,\ldots,k.$

Rearranging we obtain:

$$\mathbf{A}^{m\times m}[\frac{dz_j}{db}]^m = [P_i - a]^m.$$

Inverting this expression, we obtain

$$[\frac{dz_j}{db}]^m = \mathbf{B}^{m\times m}[P_i - a]^m = \mathbf{B}^{m\times m}[P_i]^m - \frac{a}{m-1}.$$

Suppose some subset of the k buyers bid on the interval $(\hat{b}, b')$. Without loss of generality we may relabel these buyers $m+1,...,k$. Then, from (3-20), the right derivatives of $z_1(b),\ldots,z_m(b)$ satisfy

$$\mathbf{A}^{m\times m}[\frac{dz_j}{db}]^m = [P_i]^m.$$

Comparing this with (3-21) it follows immediately that the right derivatives are strictly larger than the left derivatives unless $\frac{dz_j(\hat{b})}{db} = 0,\ j = m+1,\ldots,k.$3 Then $z_1(\cdot),\ldots,z_k(\cdot)$ are all differentiable at $\hat{b}$.

Let $[b_i^{'}, b_i^{''}]$ be the first gap for buyer i, $i=1,...,n$. Suppose $b_m^{'} = \underset{j}{Min}\, b_j^{''}$. From the above argument, it follows that $z_1(b),\ldots,z_n(b)$ is differentiable on $[b_*, b_m^{''}]$. Since equilibrium expected utility increases continuously with type, and type $z_m(b_m^{'})$ chooses $b_m^{'}$, it must be the case that

(3-22) $V_m(b_m^{'}, z_m(b_m^{''})) = V_m(b_m^{''}, z_m(b_m^{''})) \geq V_m(b, z_m(b_m^{''})),\ b \neq b'$

By Lemma 11, since V_m is nonincreasing at $b_m^{'}$ it must be the case that V_m is decreasing on $[b_m^{'}, b_m^{''}]$. But this contradicts (3-22). Then there can be no such interval.

Q.E.D.

We then have the following uniqueness result for n buyers.

Proposition 2: **Uniqueness with identical preferences and supports**

Suppose that all n buyers have the same payoff function $u_i(b, s_i) = u(b, s_i)$ and buyer types are all draws from distributions with the same support $[a, \bar{s}]$. Then if Assumptions 1-4 hold, the equilibrium bid functions are unique.

Proof: Since preferences are identical, the lowest type willing to pay *b* for the item

$\overline{f}_i(b) = \overline{f}(b), i = 1, \ldots, n$. We will consider only the case in which, for some *i*, $F_i(\overline{f}_i(b_*)) > 0$.[2] Then, for some *i*, the lower support for z, $\underline{z}$ is bounded from below. By Lemma 1, the lower support of each buyer's equilibrium bid distribution is $b_* = \overline{f}(\underline{s})$. We now show that under our hypotheses, the upper support of each buyer's bid distribution is the same. Suppose these upper supports are $b_1^* \geq b_2^* \geq \ldots \geq b_n^*$. Since at least 2 buyers must bid in any subinterval of $[b_*, b^*]$, $b_1^* = b_2^*$. Suppose then for some k > 2,

$$b^* = b_1^* = b_2^* \geq \ldots \geq b_{k-1}^* > b_k^*$$

Since b^* is optimal for $\bar{s}_1$,

$$\mathop{\times}_{i=2}^{n} \text{Prob}\{\text{i bids less than } b_k^*\} u(b_k^*, \bar{s}) \leq u(b^*, \bar{s}).$$

Hence

[2] The proof for the case in which, for all *i=1,...,n,* $F_i(\overline{f}_i(b_*)) = 0$ follows very closely case (ii) in Proposition 1.

$$\underset{i=1}{\overset{n}{\times}} \text{Prob}\{\text{i bids less than } \mathrm{b}_{\mathrm{k}}^*\} u(b_k^*, \bar{s}) < u(b^*, \bar{s}).$$

But then buyer k is better off bidding b^* than b_k^* when his type is $\bar{s}$. Thus b_k^* cannot, after all, be less than b_*.

By Lemma 11, it follows that equilibrium inverse bid functions $f_i(b)$ are strictly increasing on $[b_*, b^*]$. By Lemmas 6-8, the inverse bid functions are continuously differentiable on $[b_*, b^*]$, hence must satisfy

(3-3). After transforming variables, it follows that (3-6) must hold, that is,

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{dz_j}{db} = P_i(b, z_i), \quad i=1,\ldots,n.$$

Then appealing to Lemma 10, there is a unique solution to this differential equation satisfying the endpoint conditions.

Q.E.D.

We next show that this result can be extended to the case of different supports. Consider any sequence $0 \leq P_1 \leq \ldots \leq P_n$. Suppose that for some k,

(3-27) $$(k-1)P_{k+1} > \sum_{j=1}^{k} P_j$$

Adding P_{k+1} to both sides

$$kP_{k+1} > \sum_{j=1}^{k+1} P_j$$

Then, since $P_{k+2} \geq P_{k+1}$,

$$kP_{k+2} > \sum_{j=1}^{k+1} P_j \; 4$$

Thus if (3-27) holds for $k=m$ it holds for all $k > m$. Clearly (3-27) does not hold for $k=2$. Thus there is a unique m, $2 \leq m \leq n$ such that

(3-28) $(k-1)P_{k+1} \;\leq\; \sum_{j=1}^{k} P_j\,, \quad k \leq m$

(3-29) $(k-1)P_{k+1} \;>\; \sum_{j=1}^{k} P_j\,, \quad k > m\,.$

Lemma 13: **Maximum equilibrium bids**

Suppose that b^* is the upper support of the equilibrium bid distribution. Suppose that buyers are labelled so that $P_1(b^*,0) \leq \ldots \leq P_n(b^*,0)$. Define m to satisfy (3-28) and (3-29). Then $b = b^*$ if and only if $i \leq m$. Moreover, after appropriate relabelling so that $b^*_{m+1} \geq \ldots \geq b^*_n$,

(3-30) $(r-1)P_r(b_r^*,0) = \sum_{j=1}^{r-1} P_j(b_r^*, z_j(b_r^*))\,, \quad r > m.$

<u>Proof</u>:

Suppose first that for $k < m$

$$b^*_m \leq \ldots \leq b^*_{k+1} < b^*_k = b^*$$

Then, from (3-5), over the interval $[b_k^*, b^*]$,

$$\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{dz_j}{db} - P_i \;=\; 0 \text{ 5, } i=1,\ldots,k$$

Summing over i, from 1 to k,

$$(k-1)\sum_{j=1}^{k} \frac{dz_j}{db} - \sum_{j=1}^{k} P_j = 0$$

Hence

$$\frac{\partial}{\partial b}V_{k+1}(b,0) = \sum_{j=1}^{k} \frac{dz_j}{db} - P_{k+1}$$

$$= \frac{1}{k-1}\sum_{j=1}^{k} P_j - P_{k+1}$$

$$\geq 0 \text{ by (3-15)}$$

By Lemma 11 it follows that

$$\frac{\partial}{\partial b}V_{k+1}(b,0) > 0 \text{ on } [b^*_{k+1}, b^*].$$

But then $V_{k+1}(b,0)$ does not take on its maximum at b^*_{k+1}. Hence $b^*_k = b^*$ after all.

Suppose next that buyers 1,...,r have $b^*_i = b^*$ while all other buyers have lower maximum bids. Arguing as above,

$$\sum_{\substack{j=1\\ j\neq i}}^{k} \frac{dz_j}{db} - P_i \ = \ 0, \ i=1,...,r$$

Summing over i, from 1 to r,

$$(r-1)\sum_{j=1}^{r}\frac{dz_j}{db} - \sum_{j=1}^{r} P_j = 0$$

Hence

$$(r-1)P_r - \sum_{j=1}^{r} P_j = 0$$

But this contradicts (3-29).

Suppose we relabel buyers so that

$$b^*_n \leq ... \leq^*_{m+1} < b^*_m = b^*.$$

$V_{m+1}(b,0)$ must take on its maximum at b^*_{m+1}. Hence

$$\begin{aligned}\frac{\partial}{\partial b} V_{m+1}(b,0) \ &= \ \sum_{j=1}^{m} \frac{dz_j}{db} - P_{m+1} \\ &= \ \frac{1}{m-1}\sum_{j=1}^{m} P_j - P_{m+1} \\ &= 0, \ \text{at} \ \ b = b^*_{m+1}.\end{aligned}$$

By Lemma 11, given Assumption 3, there can be at most one such turning point. Proceeding in exactly the same manner we conclude that (3-27) uniquely defines $b^*_{m+1},\dots,b^*_n$.

Q.E.D.

We now note that for any b^*, (3-28)-(3-30) uniquely define $b^*_1,\dots,b^*_n$ as functions of the maximum bid b^*. Thus, for any b^* there is a unique solution to the system of differential equations through the endpoints $z_i(b^*_i) = 0$. Appealing once again to Lemma 10, it follows that this solution is a strictly decreasing function of the endpoint b^*. Thus once again the equilibrium bid functions are unique.

To summarize, we have proved:

Proposition 3: **Uniqueness with differing supports for each buyer's equilibrium bid distribution**

Suppose that each buyer has the same utility function $u_i = u(b,s_i)$. Then if Assumptions 1-4 hold, the equilibrium bid functions are unique.

We conclude with one result that does allow for differences in preferences.

Proposition 4: **Unique equilibrium with increasing bid shading**

Suppose Assumptions 1 and 2 hold and $u_i(b,s_i) = u_i(s_i - b)$, i=1,..,n, where $u_i(\cdot)$ is concave. Then there is a unique equilibrium for which the payoff to the winner $u_i(s_i\text{-}b_i(s_i))$ is strictly increasing for all s_i.

Proof:

Transforming variables,

$$P_i(b,z_i) \;=\; \frac{u_i'(h_i(z_i)-b)}{u_i(h_i(z_i)-b)}$$

Since $u_i(\cdot)$ is concave, it follows from the hypothesis of the Proposition that

$\frac{d}{db}P_i(b,z_i(b)) \;<\; 0$. Hence, from (3-6), $\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{d^2 z_j}{db^2} \;<\; 0,\quad i = 1,...,\text{n}$. Then

the first order conditions are sufficient for a maximum. Thus we can argue exactly as in the proof of the previous theorem.

Q.E.D.

What Proposition 4 tells us is that there is at most one equilibrium in which higher types always shade their bids more. That is,

(3-31) $\frac{d}{ds_i}(s_i - b_i(s_i)) \geq 0, \quad i = 1,...,n$

As we shall see, under a mild additional assumption, this is the case if there are no asymmetries. Then if the asymmetries are not too large, it is indeed plausible that (3-31) will hold.

Lemma 14: Bid shading in a symmetric equilibrium

Suppose $u_i(b,s_i) = u\,(s_i - b)$, that is s_i is buyer i's reservation price. Suppose also that each buyer's reservation price is a draw from the same distribution with c.d.f. $F(\cdot)$. Then if

(3-32) $\frac{d}{ds}\left(\frac{F(s)}{F'(s)}\right) > 0$

bidders with higher reservation prices shade their bids more.

<u>Proof</u>: In the symmetric case the first order conditions, (3-3) become

$$(n-1)\frac{F'(f)}{F(f)}\frac{df}{db} = \frac{u'(f-b)}{n(f-b)}$$

Transforming variables, the symmetric equilibrium bid function must satisfy

(3-33) $(n-1)\frac{dz}{db} = P(h(z)-b)$ where $h'(z) = \frac{F(f)}{F'(f)}$

Differentiating by b,

(3-34) $(n-1)\frac{d^2z}{db^2} = P'(h(z)-b)(h'(z)\frac{dz}{db} - 1)$

Since $\lim_{z \downarrow \underline{z}} P(z-b) = \infty$, $\frac{d^2 z}{db^2} < 0$ for all b sufficiently close to b_*. Let $\hat{b}$ be the smallest b such that $\frac{d^2 z}{db^2} = 0$ and $\frac{d^2 z}{db^2}$ is strictly increasing at $\hat{b}$. Differentiating (3-34) by b we have at $b = \hat{b}$,

$$(3\text{-}35) \qquad (n-1)\frac{d^3}{db^3} z(\hat{b}) \;=\; P'(\cdot)h''(z)z'(\hat{b})^2$$

$P'(\cdot)$ is negative by Assumption 2, and by hypothesis (3.32) holds and so *h(z)* is convex. Therefore the right hand side of (3-35) is strictly less than zero. But this contradicts our initial hypothesis. It follows that $\frac{d^2 z}{db^2}$ must be negative everywhere.

Next note that $f(b) = h(z(b))$, where $z = \ln F(h)$

Differentiating by b,

$$f'(b) = h'(z)z'(b) \text{ and } h'(z) = \frac{F(f)}{F'(f)}$$

Since $\frac{d^2 z}{db^2}$ is negative it follows from (3-34) that $h'(z)z'(b) > 1$. Hence $f'(b) > 1$ and so

$b'(s_i) < 1$.

Q.E.D.

4. Equilibrium with "Overbidding"

Throughout the previous sections we have assumed that no buyer ever bids more than his reservation price (Assumption 1). This is not quite as innocuous an assumption as it may seem. The reason is that if a buyer bids above his reservation price over some range of types, in equilibrium his opponents may always bid higher. If they do so, the overbidder never actually wins when he bids above his valuation. While we illustrate the point with a simple example, the analysis is quite general.

Suppose that there are just two buyers, each with a utility function $u_i(b,s_i) = s_i - b$. Thus buyer i's type is also his reservation price. For an overbidding equilibrium it is necessary that minimum reservation prices differ. Suppose therefore that

buyer 1's reservation prices are uniformly distributed over [1,2] buyer 2's over [0,1].

By Lemma 2, the minimum bid by buyer 1 is 0.5 if there is no overbidding. Suppose that the seller sets a minimum price $r \in [0.5,1)$. Again by Lemma 2, the minimum price is r. Let $b_1(s_1;r)$, $b_2(s_2;r)$ be the unique equilibrium of this auction with no overbidding. (For $s_2 < r$, buyer 2 stays out of the bidding.) Now suppose that the seller drops his minimum price and the new bid functions are

$$b_1^*(s) = b_1(s;r)$$

$$b_2^*(s) = \begin{cases} as + (1-a)r, & s < r \\ b_2(s;r), & s \geq r \end{cases}, \quad \text{where } 0 < a < 1$$

That is, buyer 2 overbids if his reservation price is less than r. Since $b_1^*(1) = r = b_2^*(r)$, buyer 2 wins with zero probability if his type is less than r. Thus overbidding by buyer 2 is a best response. For all s_1, and $b < r$,

$$U_1(b,s_1) = (s_1 - b)\,\text{Prob}\{b_2 < b\} = (s_1 - b)\,\text{Prob}\{as + (1-a)r < b\}$$

$$= (s_1 - b)(r + \frac{b-r}{a})$$

Hence

$$\frac{\P}{\P b}U_1(b,s_1) = \frac{1}{a}[s_1 + (1-a)r - 2b]$$

$$\geq \frac{1}{a}[1-(1+a)r], \quad \text{since } b < r$$

$$\geq \frac{1}{a}[1-(1+a)r], \quad \text{since } s_1 \geq 1$$

$$\geq 0,\ \ if\ a < \frac{1-r}{r} \leq 1,\ \text{ since } r \geq \tfrac{1}{2}$$

Thus for all r, $\tfrac{1}{2} < r < 1$, there exist continuous strictly monotonic equilibrium bid functions with the property that buyer 1's minimum bid is r and buyer 2 overbids if and only if his reservation price is less than r.

There are two reasons why such equilibria are considerably less interesting than the unique equilibrium without overbidding. First, as long as there is a positive probability that buyers 1 will make a mistake and bid less than r with positive probability, buyer 2 is strictly worse off bidding above his reservation price. That is, overbidding equilibria are not trembling hand perfect. Second, it is not difficult to show that equilibrium payoffs of all the buyers are lower when there is overbidding. Thus the buyers Pareto prefer the no-overbidding equilibrium.

5. **Concluding Remarks**

We have established a general uniqueness result for the case of two buyers. With more than 2 bidders, if differences among buyers can be expressed purely as differences in beliefs, we have a further strong uniqueness result. Finally, with differences in both preferences and beliefs, we have shown that there can be at most one equilibrium with the property that buyers shade their bids more when they have higher reservation prices. We also argue that this "monotonic shading" assumption is mild if asymmetries are sufficiently small.

When differences in utility functions and distributions of types are large, the analysis is considerably more complicated since it is no longer necessarily the case that the support of each buyer's equilibrium bid distribution is an interval. Our conjecture is that equilibrium bid functions are at least generically unique.

The strongest assumption made in the paper is that buyers' reservation values are independently distributed. In Maskin and Riley (1994) we establish existence of monotonically increasing equilibrium bidding strategies under the weaker assumption that buyers' reservation values are affiliated. It remains open as to whether there exist non-monotonic equilibria or whether there is a (generically) unique monotonic equilibrium.

REFERENCES

Bikhchandani, S. and J.G Riley, "Equilibrium in Open Common Value Auctions," Journal of Economic Theory, 49 (1991).

Cox, J. C., V. L. Smith and J.M. Walker, "Theory and Behavior of First Price Auctions," Journal of Risk and Uncertainty, 1 (1988) 61-99.

Dasgupta, P. and E.S. Maskin, "Existence of Equilibrium in Discontinuous Games, I: Theory, II: Applications," Review of Economic Studies, 53 (1986) 1-26, 27-41.

Maskin, E.S. and J.G. Riley, "Optimal Auctions with Risk Average Buyers," Econometrica, 52 (1984): 1473-1518.

______, "Asymmetric Auctions," revised (1993).

______, "Equilibrium in Sealed high Bid Auctions, revised (1994).

Milgrom, P., "Rational Expectations, Information Acquisition and Competitive Bidding," Econometrica, 50 (1982): 1089-1122.

Milgrom, P. and R.J. Weber, "A Theory of Auctions and Competitive Bidding," Econometrica, 50 (September, 1982): 1081-1122.

Nalebuff, B.J., "Prizes and Incentives," unpublished doctoral dissertation, Nuffield College, Oxford, 1982.

______, and J.G. Riley, "Asymmetric Equilibria in the War of Attrition," Journal of Theoretical Biology, 113 (1985): 517-527.

Riley, John G., "Strong Evolutionary Equilibrium and the War of Attrition," Journal of Theoretical Biology, 82 (1980): 383-400.

Vickrey, W., "Counterspeculation, Auctions and Competitive Sealed Tenders," Journal of Finance, 16 (1961): 8-37.

Wilson, R., "Bidding," in J. Eatwell, M. Milgate and . Newman (eds.), The New Palgrave

APPENDIX

Lemma 5: **Strict monotonicity property of bid distributions.**

Let $G_i(b)$ be the c.d.f. of the maximum bid of all i's opponents. Then for any $\hat{b} = b_i(\hat{s}_i)$ such that $0 < G_i(\hat{b}) < 1$, and for any $e > 0$, $G_i(\hat{b} - e) < G_i(\hat{b})$.

Proof: Let $\hat{b} = b_i(s_i)$. If no other buyers bid in $[\hat{b} - e, \hat{b}]$ with positive probability, $G_i(\hat{b} - e) = G_i(\hat{b})$. Then buyer 1 is strictly better off bidding $\hat{b} - e$ than $\hat{b}$, contradicting the definition of $\hat{b}$. If only one other buyer bids in $[\hat{b} - e, \hat{b}]$ with positive probability, buyer i must also. For otherwise the other buyer (call him buyer j) has the same probability of winning if he bids $\hat{b} - e$ as if he bids in $[\hat{b} - e', \hat{b}]$, for $e' < e$. It follows that buyer j bids in $[\hat{b} - e', \hat{b}]$ with zero probability. But then no buyer bids in $[\hat{b} - e', \hat{b}]$ with positive probability, contradicting our earlier result.

Q.E.D.

Lemma 6: If $f_i(b)$ is strictly increasing to the right (from the left) at $b = b$, then b is a best response for $\hat{s}_i = f_i(b)$.

Proof: Since both cases are handled in the same way, we consider only the case in which $f_i(b)$ is strictly increasing from the right. If $f_i(b)$ is also strictly increasing from the left, the lemma follows immediately. Then suppose that $f_i(b) = \hat{s}_i$ if and only if $b \in [a, b]$.

That is, for some $\hat{b} \in [a, b]$, $y_i(\hat{b}) = \hat{s}_i$. Since $f_i(b)$ is strictly increasing to the right at b, there exists a decreasing sequence $\{b^1,\ldots,b^t,\ldots\}$ approaching b and a corresponding nonincreasing sequence $\{y_i(b^1),\ldots, y_i(b^t),\ldots\}$ approaching $\hat{s}_i$. Since b^t is optimal for parameter $y_i(b^t)$, we have

(A.1) $G_i(b^t)u_i(b^t, y_i(b^t)) - G_i(\hat{b})u_i(\hat{b}, y_i(b^t)) \geq 0$, for all t.

Since $G_i(\cdot)$ and u_i are continuous, we have in the limit,

(A.2) $G_i(b)u_i(b, \hat{s}_i) - G_i(\hat{b})u_i(\hat{b}, \hat{s}_i) \geq 0$

From (A.2) it follows that buyer *i*, with parameter $\hat{s}_i$, is at least as well off choosing b as $\hat{b}$.

Q.E.D.

Lemma 7: If $f_i(b)$ is strictly increasing to the right (from the left) at $b = b > b_*$, $G_i(b)$ is right (left) differentiable at b. Moreover, the right (left) derivative satisfies

(A-3) $$G_i^{'}(b)u_i(b, f_i(b)) + G_i(b)\frac{\partial}{\partial b}u_i(b, f_i(b)) = 0$$

Proof: Since the two cases are handled in the same way, we consider only the case in which $f_i(b)$ is strictly increasing to the right. We know that $f_i(b)$ is continuous. Then at b there exists a decreasing sequence

$\{b^1, b^2, \ldots\}$ approaching b such that $y_i(b^t)$ is defined for all t and approaches $\hat{s} = y_i(b)$ monotonically from above.

Since b^t is optimal for $s^t = y_i(b^t)$ we require

$$G_i(b)u_i(b, y_i(b^t)) \leq G_i(b^t)u_i(b^t, y_i(b^t))$$

Subtracting $G_i(b^t)u_i(b, y_i(b^t))$ from both sides, we obtain

$$[G_i(b) - G_i(b^t)]u_i(b, y_i(b^t)) \leq G_i(b^t)[u_i(b^t, y_i(b^t)) - u_i(b, y_i(b^t))]$$

Dividing through by $(b^t - b)u_i(b, y_i(b^t))$ we then obtain

(A.4) $$\frac{G_i(b^t)-G_i(b)}{b^t-b} \geq \frac{-G_i(b^t)}{u_i(b,y_i(b^t))}\left[\frac{u_i(b^t,y_i(b^t))-u_i(b,y_i(b^t))}{b^t-b}\right]$$

By Lemma 6 b is optimal for $\hat{s}_i = f_i(b)$. Then

$$G_i(b)u_i(b,\hat{s}) \geq G_i(b^t)u_i(b^t,\hat{s}) \text{ for all } t.$$

Subtracting $G_i(b^t)u_i(b,\hat{s})$ from both sides and then dividing by

$(b^t-b)u_i(b,\hat{s})$ we then obtain

(A.5) $$\frac{G_i(b^t)-G_i(b)}{b^t-b} \leq \frac{-G_i(b^t)}{u_i(b,\hat{s})}\left[\frac{u_i(b^t,\hat{s})-u_i(b,\hat{s})}{b^t-b}\right]$$

In the limit as $b^t \to b$ the right hand sides of (A.4) and (A.5) coincide. Then $G_i(b)$ is right differentiable at b. Moreover the right derivative satisfies (A.3).

Q.E.D.

Lemma 8: $f_i(b)$ is right (left) differentiable for all $b > b_*$ and all i.

Proof: Suppose $f_i(b),\ldots, f_k(b)$ are strictly increasing at $\hat{b}$ and that $f_{k+1}(b),\ldots, f_n(b)$ are constant at $\hat{b}$. By Lemma 5 $k \geq 2$. By Lemma 6 $G_i(b)$ is differentiable at $\hat{b}$, $i=1,..,k$.

Also, since $\hat{b} > b_o$, $f_i(\hat{b}) > \underline{s}_i$. Then $F_i(f_i(\hat{b})) > 0$ and we may take the logarithm of both sides of

(A.6) $$\ln G_i(b) = \times_{j\neq i} F_i(f_i(b)),\ b > b_o$$

to obtain

(A.7) $$\ln G_i = \sum_{\substack{j=1 \\ j\neq i}}^{k} \ln F_j(f_j(b)) + c_i$$

where

$$c_i = \sum_{j=k+1}^{n} \ln F_j(f_j(b))$$

Subtracting c_i from both sides we can express (A.7) in matrix form as follows:

$$\begin{bmatrix} \ln G_i - c_i \\ . \\ . \\ . \\ \ln G_k - c_k \end{bmatrix} = \mathbf{B} \begin{bmatrix} \ln F_1(f_1(b)) \\ . \\ . \\ . \\ \ln F_k(f_k(b)) \end{bmatrix}$$

where **B** is defined in (3-8). It follows that $\ln F_i(f_i(b))$ and hence $f_i(b)$ 6is right differentiable.

Lemma 9: Endpoint condition if no bidder has a positive probability of winning at the minimum bid.

Suppose that $F_i(\underline{s}_i) = 0$ and $u_i(b_*, \underline{s}_i) = 0, i = 1,\ldots,n$. Define

$$e_i \equiv \frac{\underline{s}_i F_i^{'}(\underline{s}_i)}{F_i(\underline{s}_i)}$$

Then if the vector of equilibrium inverse bid functions $f(b) \equiv (f_1(b),\ldots, f_n(b))$

satisfies the endpoint condition

$$f_i(b_*) = \underline{s}_i, i = 1,\ldots,n,$$

and is strictly increasing at b_*,

$$\text{(2-7)} \quad f_i^{'}(b_*) = \Big(1 + \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} e_j}\Big) \frac{\frac{¶}{¶b} u_i(b, \underline{s}_i)}{u_i(b, \underline{s}_i)}$$

Proof:

Inverting (2-6), $\left[\frac{F_i^{'}(f_i)}{F_i(f_i)} f_i^{'}\right] = \mathbf{B}\left[\frac{\frac{\partial}{\partial b}u_j(b, f_j)}{u_j(b, f_j)}\right]$

Then premultiplying the ith component by f_i

$$\frac{f_i F_i^{'}(f_i)}{F_i(f_i)} f_i^{'} = f_i \mathbf{B}_i \left[\frac{\frac{\partial}{\partial b}u_j(b, f_j)}{u_j(b, f_j)}\right]$$

where $\mathbf{B}_i$ is the *i*th row of **B**.

Applying l'Hôpital's Rule

$$e_i f_i^{'} = f_i^{'} \mathbf{B}_i \left[\frac{\frac{\partial}{\partial b}u_j}{\frac{\partial}{\partial b}u_j f_j^{'} - \frac{\partial}{\partial s}u_j}\right]$$

Then

$$\left[e_i\right] = \mathbf{B}\left[\frac{\frac{\partial}{\partial b}u_j}{\frac{\partial}{\partial b}u_j f_j^{'} + \frac{\partial}{\partial s}u_j}\right] = \mathbf{B}\left[\frac{m_j}{f_j^{'} - m_j}\right]$$

where $m_i \equiv \frac{-\frac{\partial}{\partial b}u_i(b_*, \underline{s}_i)}{\frac{\partial}{\partial s}u_i(b_*, \underline{s}_i)} > 0.$

Inverting once more and rearranging, we obtain finally,

$$f_i^{'}(\underline{s}) = m_i\left(1 + \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^{n} e_j}\right).$$

Q.E.D.

Proposition: **Uniqueness when buyers can bid more than their reservation prices**

If the two highest minimum reservation prices are either non-positive or positive and equal, there is a unique distribution of winning bids.

Proof: Consider the case in which the highest minimum reservation price is non-positive. By Lemma 3, the lower support of the distribution of winning bids, $b_* = 0$. Then if buyer i has a positive reservation price, he has a strictly positive expected payoff by bidding on the interval $(b_*, \bar{b}_i(s_i))$. Thus all such buyers are strictly worse off overbidding. And if buyer *i* has a reservation price $\bar{b}_i(s_i) < 0$, his expected payoff is negative if he submits a strictly positive bid. He is therefore strictly better off remaining out of the auction or possibly bidding 0. If two or more buyers bid zero with positive probability, they win with positive probability and thus have a negative expected payoff. Then at most one buyer can behave in this way.

Thus the only possible difference between an equilibrium with overbidding and a no-overbidding equilibrium is that one buyer may bid zero with positive probability. Then all our previous arguments hold for bids strictly greater than zero. It follows that only bidders with zero reservation prices will bid zero in equilibrium. Such bidders are indifferent between bidding zero and not bidding. If they all choose the latter strategy the overbidders bid of zero never wins. Thus this is an equilibrium. However, the new equilibrium has the same distribution of winning bids as before.

Q.E.D.