Summary of Benoit, J.P., and V. Krishma (1998): "Multiple-Object Auctions with Budget Constrained Bidders", mimeo.

This note summarizes the work of the authors in auctions of multiple objects. Their paper presents three big ideas: i) the optimal sale of the objects is a sequence of open auctions in descending order of value; ii) the sequential bidding yields more revenue than a simultaneous ascending auction does; iii) bidders choose their budget constraint endogenously.
An important field of application of these ideas is the privatization of public utilities. In the last years governments have sold public assets to private hands. Their magnitude was such that potential buyers could run up against liquidity or borrowing constraints.
The literature on auctions has shown that, when bidders face liquidity constraints, the revenue equivalence no longer holds. It has also shown that when multiple objects are sold and bidders have budget constraints, it may be convenient for a bidder to bid aggressively on one object with the aim of raising opponents' prices and reducing their budget, and hence obtain the another object at a lower price.

The model is presented as follows. Two objects are sold to a group of (n) financially constrained bidders (i: 1,2,3…). The amount to be spent by individual i is $y_i \geq 0$. The common value of the item A (B) is V^A (V^B), and the value of both objects is V^{AB} (it may be larger or smaller than $V^A + V^B$, i.e., they may be complements or substitutes). All bidders know all values. There is link between bids through bidders' strategic behavior. They may act aggressively in a bid with the aim that they can win another bid at a low price.

Sequential auctions: In this section the authors show that the order of sale matters in terms of seller's revenue. If $V^A > V^B$, the revenue derived from selling first A and then B is higher than the revenue generated by a sale in the opposite order.
Two examples are presented to show the results:
1) $y_1 = 55$, $y_2 = 30$, $y_3 = 20$, $V^A = 50$, $V^B = 39$, $V^{AB} = 89$.
If the order of sale is AB, bidder 1 is willing to bid up to 31 for A (because the alternative is to let bidder 2 to buy item A at 30 and then 1 buy item B at 20, getting a surplus of 19). Bidder 1 wins item A and pays 30, bidder 2 wins item B and pays 25 and the revenue is 55.
If the order is BA, bidder 2 is willing to bid up to 25 for B. Then 1 prefers to win the second auction. Bidder 2 wins B (paying 25) and bidder 1 wins A (paying 20). The revenue is 45.
2) $y_1 = 70$, $y_2 = 30$, $y_3 = 5$, $V^A = 60$, $V^B = 40$, $V^{AB} = 110$.
If the order is AB, bidder 1 wins both items paying 30 for each one and the revenue is 60. Instead, if the order is BA, bidder 2 wins B (paying 30), bidder 1 wins A (paying 5) and the revenue is 35.

Simultaneous ascending auction: all items are auctioned at the same time until no bidder wish to raise the bid (i.e., bidding on all objects remains open as long as there is bidding activity on any of them).[1]
There are three results: i) when the values of the two objects are substantially different the revenue generated under optimal sequential auction is higher than the revenue under the

[1] This is justified by the fact that there may be synergies between objects (for example, between radio licenses in adjoining areas) and bundles are not allowed.

simultaneous auction; ii) when the values are significantly close the opposite result holds; iii) the sequential auction also generates higher revenue when both items are strong complements. The following example shows these results:

1) $y_1 = 25$, $y_2 = 6$, $y_3 = 0$, $V^A = 20$, $V^B = 10$, $V^{AB} = V^A + V^B$.

In the sequential action (with order AB) bidder 1 wins both items paying 6 for each one. The revenue is 12. In the simultaneous auction there are many equilibria. In one equilibrium, in the first round bidder 1 offers 0.01 and 5.99 for A and B, and bidder 2 offers 6 for B. Bidder 1 wins A and bidder s wins B. The total revenue is 6.01.

If $V^A = 9$ and $V^B = 7$, the revenue under sequential AB auction is 6 (bidder 2 wins A paying 6 and bidder 1 wins B for free). Instead, the revenue under an equilibrium of the simultaneous auction is 7 (Round 1: both bidders bid 5 on A and 0.99 on B; Round 2: 1 bids 5.99 on A and 1 on B; Round 3: 2 bids 6 on A).

Endogenous Budget Constraints: There is a sequence of actions. First, the order of items is preannounced. Prior to the auction the bidders choose budgets y_1 and y_2, which are known at the beginning of the auction.

The main results are: i) the budget constraints may arise endogenously as a result of rational calculation by bidders, ii) there is at least one bidder who chooses a constrained budget.[2] The following example shows the results:

$V^A = 10$, $V^B = 6$, $V^{AB} = 16$. Bidder 1 chooses $y_1 = 11$ and bidder 2 chooses $y_2 = 5$. Also bidder 2 wins A by paying 5 and bidder 1 wins B for free.[3]

Some extensions. (1) When there are three or more bidders, there is still one whose budget is still constrained, but there may be more than one equilibrium. (2) The fact that bidders (who now have private values) choose a constrained budget may be beneficial for the seller. (3) Under some circumstances the seller may prefer to loss in efficiency (by selling an item to a bidder who does not value the most) since in this way he gains in revenue. (4) The imposition of reservation prices, even though they are not binding, increases the revenue to the seller. This result holds even if the seller's reservation price is zero or the price is below both individual values. (5) When more than two objects are sold it is not always the case that the optimal order of sale is descending.

[2] This can be understood with the following example. A bidder is a company who hires a bidding team to represent it, and this company gives instructions not to spend more than a specified amount.

[3] Note that bidder 2 chooses a budget of 5 so as to make bidder 1 indifferent between winning B for free and winning both objects (in both cases 1 gets a surplus of 6). Given the choice of 11 by 1, bidder 2 gets the highest profit by choosing a budget of 5 and win A. The same intuition holds for bidder 1's choices.