AUCTION DESIGN

The designer announces the strategy sets S_1 and S_2 and an allocation rule. Bidder i chooses $s_i \in S_i$. In the single unit auction, this is an assignment of the item and a payment by each bidder. Let $\bar{p}_i(s),\ i=1,2$ be the probability that the item is assigned to bidder *i*. Let $\bar{c}_i(s)$ be bidder *i*'s expected payment. Bidder *i*'s type t_i is a random variable with support $[a_i, b_i]$ and continuously differentiable c.d.f. F_i. Bidders are risk neutral so, without loss of generality, we may let a bidder's type be his valuation.

Let $s_i(t_i),\ i=1,2$ be a Nash equilibrium.

Suppose that bidder 2 adopts his equilibrium strategy while bidder 1, with type t_1 chooses a strategy $s_1(x)$. By hypothesis, bidder 1's best response is $s_1(t_1)$. (This is the Revelation Principle in action.)

Bidder 1's expected utility is

$$U_1(x,t_1) = \underset{t_2}{E}\{\bar{p}_1(s_1(x),s_2(t_2))t_1 - \bar{c}_1(s_1(x),s_2(t_2))\}.$$

If we define

$$\bar{P}_1(x) = \underset{t_2}{E}\{\bar{p}_1(s_1(x),s_2(t_2))\} \text{ and } \bar{C}_1(x) = \underset{t_2}{E}\{\bar{c}_1(s_1(x),s_2(t_2))\}$$

then

$$U_1(x,t_1) = \bar{P}_1(x)t_1 - \bar{C}_1(x)$$

Standard (revealed preference) arguments establish that a necessary condition for incentive compatibility is that $\bar{P}_1(\cdot)$ must be a non-decreasing function.

Next define $p_i(x,y) \equiv \bar{p}_i(s_1(x), s_2(y))$

And $c_i(x,y) \equiv \bar{c}_i(s_1(x), s_2(y))$

Then

$$U_1(x,t_1) = \underset{t_2}{E}\{p_1(x,t_2)t_1 - c_1(x,t_2)\}$$

$$= \int_{a_2}^{b_2} p_1(x,t_2)t_1 dF_2 - \int_{a_2}^{b_2} c_1(x,t_2)dF_2 . \qquad (1.1)$$

We will denote the utility of the lowest type by $\underline{U}_i \equiv U_i(a_i, a_i)$.

Equilibrium

$U_1(x,t_1)$ takes on its maximum at $x = t_1$.

Necessary condition for equilibrium:

$$\frac{\partial U_1}{\partial x}(x,t_1)\Big|_{x=t_1} = 0.$$

Now for the bit of cunning!

$$\frac{dU_1}{dt_1}(t_1,t_1) = \frac{\partial U_1}{\partial x}(x,t_1)\Big|_{x=t_1} + \frac{\partial U_1}{\partial t_1}(x,t_1)\Big|_{t_1=x}$$

$$= 0 + \int_{a_1}^{b_1} p_1(x,t_2)dF_2 \quad \text{(from (1.1))}$$

Integrating,

$$U_1(t_1,t_1) = U_1(a_1,a_1) + \int_{a_1}^{t_1} \int_{a_2}^{b_2} p_1(x,t_2)dF_2$$

We then integrate by parts to obtain an expression for the expected utility of bidder 1.

$$\overline{U}_1 = \int_{a_1}^{b_1} U_1(t_1,t_1)dF_1(t_1)$$
$$= \underline{U}_1 + \int_{a_1}^{b_1} (1-F_1(t_1)\frac{dU_1}{dt_1}dt_1$$
$$= \underline{U}_1 + \int_{a_1}^{b_1} (1-F_1(t_1))\int_{a_2}^{t_2} p_1(t_1,t_2)F_2'(t_2)dt_2$$

$$= \underline{U}_1 + \int_{a_2}^{b_2}\int_{a_1}^{t_1} p_1(t_1,t_2)\frac{(1-F_1(t_1)}{F_1'(t_1)}F_1'(t_1)F_2'(t_2)dt_1dt_2 \qquad (1.2)$$

Also, from (1.1)

$$U_1(t_1,t_1) = \int_{a_2}^{b_2} p_1(t_1,t_2)t_1dF_2 - \int_{a_2}^{b_2} c_1(t_1,t_2)dF_2 .$$

Integrating by parts,

$$\overline{U}_1 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} p_1(t)t_1dF_1dF_2 - \overline{C}_1 \qquad (1.3)$$

Next define

$$J_1(t_1) \equiv t_1 - \frac{1-F_1(t_1)}{F_1'(t_1)} \qquad (1.4)$$

Combining these last three expressions yields the expected payment by bidder 1.

$$\overline{C}_1 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} p_1(t)J_1(t_1)dF_1dF_2 - \underline{U}_1$$

A symmetrical argument holds for bidder 2.

Thus expected seller revenue

$$R_0 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} [p_1(t)J_1(t_1) + p_2(t)J_2(t_2)]dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2)$$

Let t_0 be the seller's valuation. The item remains unsold with probability

$$p_0 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} [1 - p_1(t) - p_2(t)]dF_1 dF_2 .$$

Then the expected gain of the seller is

$$U_0 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} [p_1(t)J_1(t_1) + p_2 J_2(t_2)]dF_1 dF_2 + \int_{a_1}^{b_1}\int_{a_2}^{b_2} t_0[1 - p_1(t) - p_2(t)]dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2)$$

$$= t_0 + \int_{a_1}^{b_1}\int_{a_2}^{b_2} [p_1(t)(J_1(t_1) - t_0) + p_2(J_2(t_2) - t_0)]dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2) .$$

Thus the seller's expected gain is a function only of the assignment rule and the payoffs to the lowest types.

Henceforth we will simplify a little and take the seller's valuation to be zero. Then

$$U_0 = R_0 = \int_{a_1}^{b_1}\int_{a_2}^{b_2} [p_1(t)J_1(t_1) + p_2 J_2(t_2)]dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2) .$$

SYMMETRIC AUCTIONS - - the regular case

$J_1(v) = J_2(v) \equiv J(v)$ and $J(v)$ is strictly increasing.[1]

[1] This is a mild restriction as can be seen by trying some examples.

$$U_0 = R_0 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} [p_1(t) J(t_1) + p_2 J(t_2)] dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2). \qquad (1.5)$$

Suppose that the item MUST be assigned to one of the bidders. That is

$p_1(t) + p_2(t) = 1$.

Then (1.5) becomes

$$U_0 = R_0 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} J(t_2) dF_1 dF_2 + \int_{a_1}^{b_1} \int_{a_2}^{b_2} p_1(t)[J(t_1) - J(t_2)] dF_1 dF_2 - (\underline{U}_1 + \underline{U}_2)$$

.

Thus revenue is maximized by setting $p_1(t) = 1$ if $t_1 > t_2$ and $p_1(t) = 0$ if $t_1 < t_2$.

We can choose any assignment rule if the two bidders have the same type.

The standard auctions are thus revenue maximizing among all mechanisms which always assign the item to one of the bidders.

Asymmetry

$$\frac{F_1'(f_1)}{F_1(f_1)} f_1'(b) = \frac{1}{f_2 - b} \quad \text{and} \quad \frac{F_2'(f_2)}{F_2(f_2)} f_2'(b) = \frac{1}{f_1 - b}.$$

Suppose that $f_1(\bar{b}) = f_2(\bar{b})$. Then

$$\frac{F_1'(f_1)}{F_1(f_1)} f_1'(\bar{b}) = \frac{1}{f_2(\bar{b})-\bar{b}} = \frac{1}{f_1(\bar{b})-\bar{b}} = \frac{F_2'(f_2)}{F_2(f_2)} f_2'(\bar{b})$$

Thus if bidder 1 is "stronger" in the sense of Conditional Stochastic Dominance $\frac{F_1'(t)}{F_1(t)} > \frac{F_2'(t)}{F_2(t)}$,

$$f_1(\bar{b}) = f_2(\bar{b}) \Rightarrow f_1'(\bar{b}) < f_2'(\bar{b})$$