*Auctions Workshop (Prof. J. Riley), Spring, 99*
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Study note on the paper "Multi-unit auctions with uniform prices" [*Economic Theory* 12 (1998): 227-258] by Richard Engelbrecht-Wiggans & Charles M. Kahn (EW-K).

**What is a uniform price auction?**
A multi-unit auction is the simultaneous auction of multiple identical objects. There are two common forms of multi-unit auctions: the uniform price auction and the pay-your-bid auction. In the uniform price auction, the highest bid wins the units, but all units are sold for the same price. In the pay-your-bid auction, each unit is sold at its corresponding bid. In the paper, EW-K address the question of characterization of Nash equilibria in undominated for the uniform price auction in the independent private values case.

**Model and Assumptions**
The model assumes there are more than two units of a good (or object) to be auctioned (*M ³ 2*) and that there are more bidders than half the number of objects (*N+1* bidders with *2(N+1) > M*). This implies that not all bidders end up satisfied. Bidders are willing to bid for at most two units. After observing his valuations $(v_1^n, v_2^n)$, the bidder *n* submits a pair of bids $(c_1^n, c_2^n)$. This is of course a simplification and EW-K provide some insights to extend it (section 6.1). It is assumed, without loss of generality, that $v_1^n \geq v_2^n > 0$ and $c_1^n \geq c_2^n$. The *M* highest bidders get the goods, and for each good received, the winner pays an amount equal to the highest losing bid. Then bidders have pairs of valuations and announce pairs of bids. The reserve price is zero. The individual valuation pairs are *IID* with distribution function $G(v_1, v_2)$. Focusing on the strategies of player 1, let $b_1$ and $b_2$ be the *(M-1)*th and *M*th highest of other players' bids, respectively. Let $F(b_1, b_2)$ be their joint distribution and $F_1(b_1)$ and $F_2(b_2)$ their marginal distributions. It is implied by the description above that $Pr(b_1 \geq b_2) = 1$. The following assumptions are widely used in the paper:

*Assumption (Pr):* $Pr(c_1 = b_2) = Pr(c_2 = b_1) = 0$.
*Assumption (À -): M ³ 3 and N < M.*
*Assumption (À +): M ³ 3 and N ³ M.*

Assumption *(Pr)* rules out the possibility of ties, *(À -)* assumes there are "many" units (more than three) and few bidders, and *(À +)* says that there are many units but more bidders than units. The number of buyers compared with the number of units to be auctioned plays a crucial role in the model.

**The S-strategies**
An *S-strategy* is one such that the higher of the two bids equals the higher of the two valuations and the lower of the two bids is nonnegative and no greater than the lower of the two valuations, that is: $b_1 = v_1$ and $0 \leq b_2 \leq v_2$. Let *S* be the set of S-strategies. An *S-equilibrium* is one in which players use S-strategies. EW-K shows that S is the set of weakly dominating strategies [lemma 2.2]. Moreover, every (trembling-hand) perfect equilibrium is an S-equilibrium [theorem 2.3].

**How is the expected utility?**
Assume that *(Pr)* holds. Then the expected payoff *U* for the player given others' strategies, his own valuations and his own bids is:

$$U = \iint_{\{(b_1,b_2)\,/\,b_2 < c_1\}} (v_1 - \max\{c_2, b_2\})dF(b_1, b_2) + \iint_{\{(b_1,b_2)\,/\,b_1 < c_2\}} (v_1 + c_2 - 2b_1\})dF(b_1, b_2)$$

If in addition we restrict $U$ to S-strategies, then $U$ takes the simpler form:

$$U(c_2, v_1, v_2) = \int_{c_2}^{v_1} F_2(b_2)db_2 + 2\int_0^{c_2} F_1(b_1)db_1 + (v_2 - c_2)F_1(c_2)$$

**Symmetric S-equilibria are monotonic. Well, low values are pooled at zero**

EW-K characterize symmetric equilibria in the following way: If there are many units and less bidders than units, i.e., assumption *(À -)*, then, in any symmetric S-equilibrium, bids on the second unit are strictly increasing in the valuation of the second object [theorem 3.1(1)]. If there are many units and more bidders than units, i.e., assumption *(À +)*, then, in any symmetric S-equilibrium, bids are strictly increasing in the valuation of the second object, except that a range of low values may be pooled at zero [theorem 3.1(2)]. The consequences of these statements are important. For instance, it may be the case that in equilibrium the lower of a player's bid is shaded below the corresponding valuation. Under standard assumptions on the distributions of valuations, symmetric S-equilibria are characterized by the fact that the second bid is strictly below the second valuation [theorem 3.2]. Bidders reveal their valuations for the first object by bidding the true valuation, but bid less than their valuation for the second object. The question of when players bid zero for the second object arises then. EW-K also provide some necessary and/or sufficient conditions under which in symmetric S-equilibria bidders indeed bid zero for the second object. If there are many units and more bidders than units, i.e., assumption ($\aleph$+), and the hazard function for the high valuation for any individual is sufficiently low [i.e., if $(M-1)(v^*-c)h_1(c)$£ $1$, where $h_1(c) = G_1'(c)/(1 - G_1(c))$ is the hazard function and $v^*$ is the highest valuation in the support of $G_1$], then it is optimal to bid zero for the second object [theorem 4.2].

**The two-units case**

When there are exactly two units, strict monotonicity of the second bid in the S-equilibria can not be guaranteed. EW-K provide necessary and sufficient conditions for equilibrium. By assuming that all opponents follow a strategy in which the bid for the second unit is weakly increasing in the valuation of the second unit, the bid distribution takes a simpler form. Then the bidder's first order condition takes also a special form. From this EW-K define the function:

$$\Gamma(c, v) = \int_0^c NG_1^N(x)[G_1(x) - G_2(x) + G_1'(x)(v - x)]dx$$

The necessary and sufficient conditions for an S-strategy to yield a symmetric S-equilibrium is related to the requirement that the strategy is a local maximizer of to $\Gamma(\cdot, v)$ [theorems 5.2 and 5.3].

**Summary of results**

(1) The S-strategies weakly dominate the other strategies. This dominance is strict under trembles, and hence every perfect equilibrium is an S-strategy. (2) If there are many units and few bidders, bids on the second unit are strictly increasing in the valuation of the second unit. If there are many units and many more bidders, the same holds, but low valuations are pooled at zero. (3) Bids on the second unit are actually zero provided there are many more bidders than units and the hazard function for the high valuation for any bidder is sufficiently low. (4) The two-units case deserves special attention, because none of the $\aleph$ conditions are satisfied. (5) EW-K provide an example in which there is a continuum of equilibria, none of them strictly monotonic [example 4], and an example in which the only equilibria are discontinuous [example 3]. (6) EW-K provide a three-unit example in which the bid for the third unit is pooled at zero. The condition in this case is that the number of bidders is less than the number of units (and more than a half).