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AUCTION CHOICE*

by

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One of the most fruitful areas of research in non-cooperative game theory has been the analysis of auctions and related trading mechanisms. An important strand of this research compares the two most common forms of auctions — the sealed high-bid and open ascending-bid auctions. Beginning with the striking revenue equivalence theorem (Vickrey (1961), Myerson (1981), Riley-Samuelson (1981)) theorists have sought to understand how the revenue implications of the two auctions differ as the assumptions of risk neutrality, independence and symmetry are weakened.

Risk aversion raises expected revenue in the sealed high-bid auction but does not affect bidding in the open auction. Since the two auctions are revenue equivalent under risk neutrality, it follows that the sealed high bid auction generates higher expected revenue when buyers are risk averse. Moreover, as Matthews (1987) shows, under constant absolute risk aversion the two auctions remain equivalent for all buyers. Thus the sealed high-bid auction is Pareto dominant.

Milgrom-Weber (1982), in their path-breaking paper, argue that bidders' private information is more likely to be positively correlated rather than independent. This will be the case, for example, in construction bidding when each bidder first estimates the cost of the project. Their central conclusion is that if private signals are affiliated (pairwise positively correlated) expected revenue is higher in the open auction.

All the above noted papers assume symmetry across buyers. This assumption is relaxed by Maskin-Riley (1999a) who consider the implications for expected revenue if bidders have information about how they all differ. It is shown that while different types of asymmetry will produce different rankings of expected revenue, the "typical" case leads to greater expected revenue in the sealed high-bid auction. In this paper we begin by relaxing the Maskin-Riley assumption of two bidders and derive some new theoretical results. We derive quite strong results about the equilibrium bid distributions of "strong" and "weak" bidders. In addition, starting from symmetry (and hence revenue equivalence) we are able to compare revenue when bidders' distributions are "rotated."

Since relaxing different assumptions, one-by-one, leads to different rankings, the net effect of these analytical results remains ambiguous. What one would really like to know is the economic importance of risk aversion, affiliation and asymmetry and the conditions under which one or other is likely to be the dominant factor.[1]

The paper is organized as follows. The independent private values model is summarized in section 2. Equilibrium bid distributions under asymmetry are characterized in section 3. Revenue comparison theorems are derived in section 4. The model is generalized in section 5 to include affiliated types and related preferences (or "common values.") Numerical issues are also addressed. Sections 5-9 summarize the numerical results.

[1] Our central results are summarized in Table 9.1.

1. THE BASIC MODEL

There are *n* bidders. Bidder *i*, if his type is v_i, has a gain in utility of $u(b,v_i)$ if he wins the auction and pays *b* for the item. We assume that utilty is a strictly increasing function of the bidder's type and is strictly decreasing in the bid. It will also be convenient to define a bidder's "loss aversion" $L(b,v_i)$ Suppose that bidder *i* has won the item and paid a price *b*. Let δb be the increase in the bid that bidder *i* would make in order to avoid fully insuring against a small probability δp of losing the item, that is

$$u(b+\delta b, v_i) = (1-\delta p)u(v_i, b) .$$

Taking the first order approximation and rearranging,

$$\frac{\delta b}{\delta p} \approx \frac{u(b,v_i)}{-\frac{\partial u}{\partial b}}$$

Thus loss aversion is the bidder's willingness to pay to avoid a small probability of losing the item.

Assumption 1: Increasing Loss Aversion[2]

A bidder's loss aversion is an increasing function of the bidder's type.[3]

Given Assumption 1, it follows that a higher type is more willing to bid higher. To see this, consider the slope of the indifference curve of a type v_i bidder. Let p be his probability of winning. Then his expected utility is

$$e_i(b,p) \equiv pu(b,v_i)$$

Around an indifference curve,

$$\left.\frac{db}{dp}\right|_{e_i=const} = -\frac{\frac{\partial e}{\partial p}}{\frac{\partial e_i}{\partial b}} = -\frac{u_i}{p\,\frac{\partial u_i}{\partial b}} .$$

Thus the higher is a bidder's type, the more willing he is to bid higher, given the same beliefs about the bidding behavior of his opponents.

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[2] In technical terms, this is the assumption that utility is log supermodular.

[3] Suppose $u_i = \int U(v_i + e - b)dF(e)$, and $E\{e\} = 0$, so that v_i is bidder *i*'s expected valuation. Then Assumption 1 is satisfied if $U(\cdot)$ is strictly increasing and concave.

Consider Fig. 1.1 The indifference curves of types v and $\hat{v} > v$ through (b_1, p_1) are depicted. Lemma 1 follows directly.

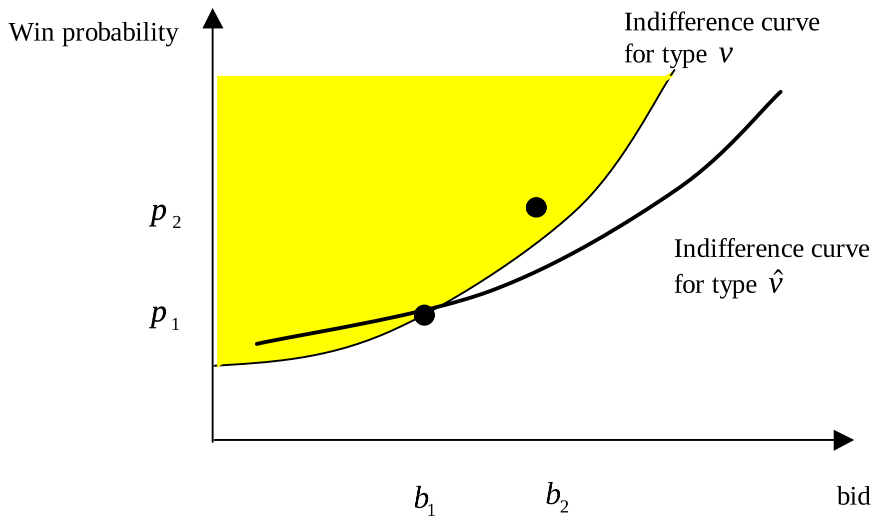

Figure 1.1: Single crossing property

Lemma 1: Monotonicity

Suppose $(b_1, p_1) < (b_2, p_2)$. If Assumption 1 holds, then for all *v*, and $\hat{v} > v$,

$$(b_2, p_2) \underset{v}{\succsim} (b_1, p_1) \Rightarrow (b_2, p_2) \underset{\hat{v}}{\succ} (b_1, p_1).$$

Since it will be helpful later, we define $L_i(b, v_i) \equiv \dfrac{-\dfrac{\partial u_i}{\partial b}}{u_i(b, v_i)}$. Given increasing loss aversion,

$$\frac{\partial}{\partial v_i} L_i(b, v_i) < 0 \ . \qquad (1.1)$$

Differentiating by *b*,

$$\frac{\partial}{\partial b} L_i(b, f_j) = A_i(b, f_j) L_i(b, f_j) + L_i(b, f_j)^2 \qquad (1.2)$$

where $A_i(b, v_i) \equiv \dfrac{\partial^2 u_i}{\partial b^2} / \dfrac{\partial u_i}{\partial b}$ is the bidder's aversion to income risk.

Thus with risk neutrality or risk aversion, $L_i(b, v_i)$ is increasing in b.

Assumption 2: Non-increasing absolute risk aversion

A bidder's absolute aversion to risk, $A_i(b,v_i) \equiv \dfrac{\frac{\partial^2 u_i}{\partial b^2}}{-\frac{\partial u_i}{\partial b}}$,is a non-decreasing function of his type.[4]

If Assumptions 1 and 2 both hold, it follows directly that

$$\frac{\partial^2}{\partial v_i \partial b} L_i(b,v_i) < 0 \qquad (1.3)$$

We assume throughout that bidder *i*'s type $\tilde{v}_i$ is continuously distributed with c.d.f $F_i(\cdot)$ and support $[a, b_i]$ $i = 1,\ldots,n$, where $b_1 \geq \ldots \geq b_n$. That is, higher indexed bidders are "weaker" in the sense that their maximum reservation values are weakly lower. We assume that $u(0,a) \geq 0$ so that, with probability 1, all bidders have a strictly positive payoff from receiving the item. Since it will simplify the analysis, we assume that the seller sets a reserve price *r* sufficiently high that the lowest types would be strictly worse off paying *r* for the item.[5] That is, there is some type $v_r \in (a, b_i)$ such that $u(r,v_r) = 0$. Since buyers of lower types cannot gain from bidding, we assume that they do not bid. Given the reserve price r, there is a positive probability that all the opponents of bidder *i* do not bid. Then it can never be an equilibrium strategy to make a bid which would lead to a negative payoff for the winner. But all bidders with types exceeding v_r have a strictly positive expected payoff if they bid *r*. Thus, in equilibrium, a bidder bids if and only if his type is at least v_r and the minimum bid is the seller's reserve price.

Since this is a special case of the model examined by Maskin and Riley (1999b), we know that there exists a monotonic equilibrium.

2. Characterizing equilibrium bid distributions in the sealed high-bid auction

Since it will be helpful in the analysis to follow, we define $p_i(b)$ to be the c.d.f for $\tilde{b}_i$, bidder *i*'s bid distribution and $f_i(b)$ to be the bidder *i* type who bids *b*. If there is no type who bids *b*, we define $f_i(b)$ to be the largest type that bids less than *b*.

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[4] Note that in the special case in which $u_i = U(w + v_i - b)$, $A_i = -\dfrac{U''(w+v_i-b)}{U'(w+v_i-b)}$. Then Assumption 2 holds if absolute risk aversion is non-increasing in wealth.

[5] Given continuity, our results also hold in the limiting case where $v_r = a$.

Lemma 2: Ordering maximum bids

If Assumption 1 holds then maximum bids satisfy $b_1^* = b_2^* \geq \ldots \geq b_n^*$. Moreover, if $b_i^* > b_{i+1}^*$, $b_i > b_{i+1}$

Proof: (see appendix)

In this paper we will establish directly that there is a vector of equilibrium bid probabilities $p(b)$ which is continuously differentiable and strictly increasing. In fact, as Maskin-Riley (1996) show, this is the unique equilibrium.

It is useful to define the equilibrium inverse bid functions

$$f_i(\cdot) = b_i^{-1}(\cdot), \ \ i = 1,\ldots,n \tag{2.1}$$

Then $p_i(b) = F_i(f_i(b))$. Finally, we define $H_i(\cdot) \equiv F_i^{-1}(\cdot)$. Then, from (2.1),

$$f_i(b) = H_i(p_i(b)) \tag{2.2}$$

Proposition 3: Necessary conditions for equilibrium

Suppose $p(b)$ is continuously differentiable on an interval $[\underline{b}, \bar{b}]$ over which the *m* lowest indexed buyers bid. Then the c.d.f.'s for the equilibrium bid distributions $p_1(b),\ldots,p_m(b)$ must satisfy the differential equation system

$$\begin{bmatrix} 0 & 1 & . & 1 \\ 1 & 0 & . & 1 \\ . & . & . & . \\ 1 & 1 & . & 0 \end{bmatrix} \begin{bmatrix} \dfrac{d}{db} \ln p_1 \\ . \\ . \\ \dfrac{d}{db} \ln p_m \end{bmatrix} = \begin{bmatrix} L_1(b, H_1(p_1)) \\ . \\ . \\ L_m(b, H_m(p_m)) \end{bmatrix}. \tag{2.3}$$

Proof: To see that this must be the case, suppose that all bidders other than *i* bid according to their equilibrium bid distributions. Then if buyer *i* bids *b*, his expected payoff is

$$e_i(b, v_i) = \prod_{\substack{j=1 \\ j \neq i}}^{n} p_i(b) u(b, v_i)$$

Taking the logarithm of this expression and differentiating by *b* we obtain

$$\frac{\partial}{\partial b}\ln e_i = \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{d}{db}\ln p_j(b) - L(b,v_i)\,. \tag{2.4}$$

By hypothesis, $b_i(v_i)$ is buyer i's equilibrium best reply. Thus $e_i(b,v_i)$ must take on its maximum at $b = b_i(v_i)$, that is, if $v_i = f_i(b)$. Appealing to (2.4), the first order condition for an equilibrium best reply is therefore,

$$\left.\frac{\partial}{\partial b}\ln e_i\right|_{v_i = f_i(b)} = \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{d}{db}\ln p_j(b) - L(b, f_i(b)) = 0\,. \tag{2.5}$$

Since $p_i(b) = F_i(f_i(b)$, this can be rewritten as

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{d}{db}\ln p_j(b) = L(b, H_i(p_i(b)))\,. \tag{2.6}$$

To see that the necessary conditions are also sufficient, we differentiate (2.4) by v_i to obtain:

$$\frac{\partial}{\partial v_i}\frac{\partial}{\partial b}\ln e_i = \frac{\partial}{\partial v_i} L(b,v_i) > 0\,.$$

Therefore

$$v_i \underset{(<)}{>} f_i(b) \Rightarrow \frac{\partial}{\partial b}\ln e_i(b,v_i) \underset{(<)}{>} \frac{\partial}{\partial b}\ln e_i(b, f_i(b)) = 0.$$

Thus over the interval $[\underline{b},\bar{b}]$,buyer i's expected payoff $e_i(b,v_i)$ takes on its maximum at $b = f_i^{-1}(v_i)$

Q.E.D.

We next show that there can be no "gaps" in the equilibrium distribution of bids.

Lemma 4: Given Assumption 1, the equilibrium bid distribution of bidder i , i=1,…,n is an interval $[r,b_i^*]$.

Proof: (see appendix)

We next characterize a solution to (2.3) satisfying the right end-point condition $p_j(b^o) = 1,\;\; j = 1,\ldots,n$. For an auction, at least two bidders must be willing to bid. Hence b^o

must satisfy the condition $u(b^o, b_2) > 0$. For any $b^o > r$ define $b_1^o = b_2^o = b^o$ and $p(b) = (p_1(b), \ldots, p_n(b))$ according to (2.3) and $b_3^o, \ldots, b_n^o$ iteratively as follows.

$$b_{m+1}^o = Max\{b | b \le b_m^o, e_{m+1}(b, b_{m+1}) \text{ has a local max at } b\} \qquad (2.7)$$

where $e_{m+1}(b, b_{m+1}) \equiv \underset{\substack{j=1 \\ j \neq m+1}}{\overset{n}{\times}} p_j(b)(b_{m+1} - b)$

Lemma 5: If $p(b)$ defined above also satisfies the left end-point condition $p_j(r) = F_j(v_r),\ \ j = 1, \ldots, n$, then for each *j*, $p_j(b)$ is the c.d.f. for bidder *j*'s equilibrium bid distribution $\tilde{b}_j,\ \ j = 1, \ldots, n$.

<u>Proof</u>: Suppose that the statement of the Lemma holds for buyer's 1,…,*m*-1. From the definition, bidder *m*, of type b_m has a local best reply at $b = b_m^o$. If it is not a global best reply, there is some $\hat{b} < b_m^o$ such that

$$e_m(\hat{b}, b_m) = e_m(b_m^o, b_m) \text{ and } e_m(b, b_m) \le e_m(b_m^o, b_m),\ \ b \in (\hat{b}, b_m^o).$$

But then we can argue exactly as in the proof of Lemma 4 to establish a contradiction.

Q.E.D.

The next result establishes that the solution to (2.7) is strictly monotonic.

Proposition 6: Suppose Assumptions 1 and 2 hold.. Then, for any $b^o > r$ such that $u(b^o, b_2) > 0$, $p(b)$ defined by (2.3) and (2.7) satisfies

$$p'(b) \ge 0,\ \ b < b^o \quad \text{and} \quad p_1'(b), \ldots, p_m'(b) > 0,\ \ b \in [b_{m+1}^o, b_m^o].$$

For some of our characterization theorems we will make further assumptions about the nature of the asymmetry across bidders.

Assumption 3: Lower indexed bidders are "stronger" in the sense of stochastic dominance

$$F_i(v) < F_{i+1}(v),\ \ v \in (a, b_{i+1}) \ \ i=1,\ldots,n\text{-}1$$

Since a higher type implies a higher reservation value, stochastic dominance carries over to reservation values. Thus a "stronger" bidder is one with a higher probability of a high reservation value.

Since we have defined $H_i(\cdot) = F_i^{-1}(\cdot),\ i = 1,\ldots,n$, an immediate implication of Assumption 3 is that

$$H_i(p) > H_{i+1}(p),\ 0 < p < 1,\ i=1,\ldots,n\text{-}1 \tag{2.8}$$

To compare equilibrium bid functions, we will need the following stronger assumption.

Assumption 4: Lower indexed bidders are stronger in the sense of conditional stochastic dominance.

$$\frac{F_i'(v)}{F_i(v)} > \frac{F_{i+1}'(v)}{F_{i+1}(v)},\ v \in (a, b_{i+1}],\ i = 1,\ldots,n-1\,.$$

Note that

$$\Pr\{\tilde{v}_i < x | \tilde{v}_i < y\} = \frac{F_i(x)}{F_i(y)} = e^{\ln\frac{F_i(x)}{F_i(y)}} = e^{-[\ln F_i(y) - \ln F_i(x)]} = e^{-\int_x^y \frac{F_i'(v)}{F_i(v)}dv}$$

Thus Assumption 4 requires that the stochastic dominance property hold across buyers, conditional upon types being no greater than *y*, for all $y \in (a, b]$.

Lemma 7: Monotonicity Property

Consider two solutions $p(b)$ and $p(b)$ to the differential equation system (2.3) in the interior of $B = \{(b, p_1,\ldots,p_n) |\ b \ge r,\ 0 < p_i \le 1,\ \text{and}\ u(b, H_i(p_i)) > 0,\ i = 1,\ldots,n\}$. If $p(b_o) \ge p(b_o)$, then $p(b) > p(b),\ b < b_o$ and

$$\frac{\mathop{\times}\limits_{i=1}^{n} p_i(b)}{\mathop{\times}\limits_{i=1}^{n} p_i(b)} > \frac{\mathop{\times}\limits_{i=1}^{n} p_i(b_o)}{\mathop{\times}\limits_{i=1}^{n} p_i(b_o)},\ b < b_o\,. \tag{2.9}$$

<u>Proof</u>: see Appendix

Consider solutions to the system of differential equations satisfying the right end-point condition $p(b^o) = 1$. Consider the right end-point $b^o = \underline{b}$, where $\underline{b}$ is sufficiently small. In this case the left end-point is a point like *M* in the figure, where $p_i(r) > F_i(v_r)$. Next consider the right end-point $b^o = \bar{b}$, where $\bar{b}$ is sufficiently large. In this case the left end-point is a point like *N* in the figure, where $u(b, H_i(p_i)) = 0$. An immediate implication of Lemma 7 is that there is a unique right end-point $b^o = b_i^*$ such that the left end-point condition $p_i(r) = F_i(v_r)$ is also satisfied. This is depicted in Figure 2.1 as the heavy curve.

One further point should be noted. For any right end-point $\bar{b} > b_i^*$, there is some minimum bid $\bar{b}_o$ such that $u(\bar{b}_o, H_i(p_i(\bar{b}_o))) = 0$. That is, $\bar{b}_o$ is the reservation price of type $f_i(\bar{b}_o)$. This is the equilibrium bid of buyer *i* if the seller sets a reserve price of $\bar{b}_o$. That is, if we choose a sufficiently large maximum bid b^o the resulting probability functions are equilibrium c.d.f.s for an auction for <u>some</u> reserve price set by the seller.

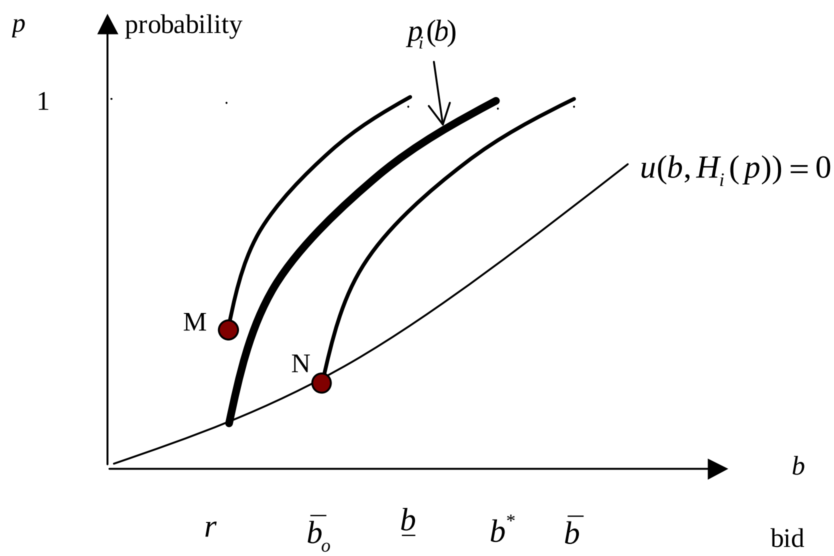

Fig. 2.1 Satisfying the End-point Conditions

Lemma 7 then yields the following implications for equilibrium bidding as the seller changes the reserve price.

Proposition 8: **Uniqueness**

Suppose Assumptions 1 and 2 hold. Then, for any reserve price *r*, there is a unique $(p_1(b),\ldots, p_n(b))$ satisfying the differential equation system (2.3) and the end-point conditions. Moreover, if the seller increases the reserve price, the equilibrium bid distributions all shift to the right.

The expected revenue of the seller from all those who bid thus rises with the reserve price. Of course the higher this price, the higher the probability that no one will bid so the seller must weigh the trade-off between these two effects.

As we shall see in section 4, this Proposition is the key to providing a robust algorithm for numerical solutions.

We now show that if the different bidders can be ranked according to one of the "strongness" criteria, there is much that can be said about the differences in the way that they bid. One comparison that we make is to two symmetric equilibria. Suppose Assumption 3 holds so that bidder 1 is the strongest and bidder n the weakest. Let $b_1^s(v)$ and $b_n^s(v)$ be the symmetric equilibria and let $p_1^s(b)$ and $p_n^s(b)$ be the corresponding c.d.f.s for the equilibrium bid distributions. The following result extends those of Maskin and Riley (1999a) to the case of n bidders.

Proposition 9: Comparison of Equilibrium Bid Distributions

If Assumptions 1-3 hold, then

(i) $p_1(b) < \ldots < p_n(b),\ b \in [r, b^*]$

(ii) there exists $d > 0$ such that, if $b_i > b_1 - d,\ b_i(\cdot) > b_{i-1}(\cdot)$ on the interval $[b_1 - d, b_1)$.

(iii) $p_1^s(b) < p_1(b)$ and $p_n^s(b) > p_n(b),\ b \in (r, b^*]$.

<u>Proof</u>:

We provide a formal proof for the case $b_1 = \ldots = b_n$. The proof is readily generalized to cover the case of different supports. To prove (i), note that, from (2.6)

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{d}{db} \ln p_j(b) = L(b, H_i(p_i(b))),\ i = 1, \ldots, n \ .$$

Setting $i = m$ and $i = k < m$ and subtracting the two expressions, it follows that

$$\begin{aligned} \frac{d}{db} \ln p_k(b) - \frac{d}{db} \ln p_m(b) &= L(b, H_m(p_m(b))) - L(b, H_k(p_k(b))) \\ &= L(b, f_m(b)) - L(b, f_k(b)) \end{aligned} \quad (2.10)$$

Suppose $p_k(b) \geq p_m(b)$. Then, since $p_i(b) = F_i(f_i(b))$, $f_k(b) \geq f_m(b)$ and the inequality is strict unless $b = b^*$ so that $p_k(b^*) = p_m(b^*) = 1$. It follows that

$$\frac{d}{db} \ln p_k(b) - \frac{d}{db} \ln p_m(b) \geq 0, \text{ with strict inequality if } b < b^* \ .$$

Since $p_k(b^*) = p_m(b^*) = 1,\ \ p_k(b) < p_m(b),\ b \in (r, b^*)$.

To prove (ii), suppose that it is false. Then over any sufficiently small interval $(\hat{b}, b^*)$, there must be some $k < m$ such that $f_k(b) < f_m(b)$. Then, from (2.10),

$$\frac{d}{db}\ln p_k(b) - \frac{d}{db}\ln p_m(b) = \frac{p_k'}{p_k} - \frac{p_m'}{p_m} < 0, \ \ b \in (\hat{b}, b^*).$$

Since $p_k(b^*) = p_m(b^*) = 1$, it follows that $p_k(b) > p_m(b), \ b \in (\hat{b}, b^*)$, contradicting (i).
To prove (iii), we first show that b^*, the maximum bid in the asymmetric sealed high bid auction lies between the maximum bids $\bar{b}_n^s$ and $\bar{b}_1^s$ in the two symmetric auctions. Suppose that $b^* < \bar{b}_n^s$. Since $f_n^s(r) = f_n(r) = v_r$, there must be some $\hat{b}$ such that $f_n(b) > f_n^s(b)$ for all $b > \hat{b}$, and $f_n(\hat{b}) = f_n^s(\hat{b})$, hence $p_n(\hat{b}) = F_n(f_n(\hat{b}) = p_n^s(\hat{b})$.
From (2.3)

$$\sum_{j=1}^{n-1} \frac{d}{db}\ln p_j(b) = L(b, H_n(p_n(b))) = L(b, f_n(b))$$

$$< L(b, f_n^s(b)) = \sum_{j=1}^{n-1} \frac{d}{db}\ln p_n^s(b) \qquad \text{for all } b \in (\hat{b}, b^*].$$

Integrating over the interval $[\hat{b}, b^*]$,

$$\mathop{\times}_{j=1}^{n-1} p_j(\hat{b}) > p_n^s(\hat{b})^{n-1}.$$

But, from (i) $p_n(\hat{b}) > p_j(\hat{b}), \ \ j < n$. Hence $p_n(\hat{b})^{n-1} > p_n^s(\hat{b})^{n-1}$ and so $p_n(\hat{b}) > p_n^s(\hat{b})$, contradicting our original hypothesis. Then there can be no such intersection and so $\bar{b}_n^s < b^*$.
We know that $p_n(r) = p_n^s(r)$, hence either $p_n(b) \le p_n^s(b),$ for all $b \in (r, \bar{b}_n^s]$ and we are done, or there exist $b_1 > r$ and $b_2 > b_1$ such that $p_n(b_1) = p_n^s(b_1)$, $p_n(b_2) = p_n^s(b_2)$ and $p_n(b) > p_n^s(b)$ over (b_1, b_2). But then $f_n(b) > f_n^s(b)$ over this interval. From (2.3), over this interval

$$\sum_{j=1}^{n-1} \frac{d}{db}\ln p_j(b) = L(b, H_n(p_n(b))) = L(b, f_n(b))$$

$$< L(b, f_n^s(b)) = \sum_{j=1}^{n-1} \frac{d}{db}\ln p_n^s(b)$$

Then, arguing as above,

$$\underset{j=1}{\overset{n-1}{\times}} p_j(b_1) > p_n^s(b_1)^{n-1} \text{ and since } p_n(b_1) \geq p_j(b_1),\ \ j<n,\ \ p_n(b_1) > p_n^s(b_1). \text{ But this}$$

contradicts our earlier hypothesis.

A symmetric argument establishes the second half of the (iii).

Q.E.D.

Proposition 9 (i) yields a central testable implication of the theory. Stochastic dominance across types implies that equilibrium bid distributions can also be ranked according to stochastic dominance, with stronger bidders bidding more aggressively. Proposition 7 (iii) provides an answer to the question of how bidders respond if, starting from symmetry, some subset of the bidders' distributions become stronger. That is, the distribution of valuations of these bidders shifts to the right. The proposition establishes that all those with unchanged bid distributions (the "weak" bidders) respond by bidding more aggressively ($p_n(b) < p_n^s(b),\ b \in (r, \bar{b}_n^s)$). And, facing this more aggressive bid distribution, the bid distributions of the stronger bidders also shift to the right (part (i).)

Intuitively, the stronger bidders, since they face bidders who are likely to have lower valuations, will shade their bids further below their valuations. However as Maskin-Riley (1999a) establish, this may not be the case. However, we now show that under the stronger Assumption 3, the intuition holds.

Proposition 10: Comparison of equilibrium bids

If Assumptions 1and 3 hold, stronger bidders shade their bids more, that is,

$$b_i(v) < b_{i+1}(v),\ \ v \in (v_r, b_i) \text{, where the inequality is strict if } b_2 > ... > b_n$$

Proof:

We provide a formal proof for the case of equal maximum reservation values. Since Assumption 3 implies Assumption 2, we may appeal to Proposition 9 (ii). Thus we know that for all sufficiently high valuations, stronger bidders shade their bids more. Suppose that $\hat{b}$ is the smallest bid for which this holds and that $\hat{b} < b_n^*$, the smallest of the maximum bids.. Then there is some k and $i > k$, such that $f_i(\hat{b}) = f_k(\hat{b})$. From (2.6)

$$\frac{d}{db}\ln p_k(\hat{b}) - \frac{d}{db}\ln p_i(\hat{b}) = L(\hat{b}, f_i(\hat{b})) - L(\hat{b}, f_k(\hat{b})) = 0$$

Since $p_j(b) = F_j(f_j(b))$ it follows that

$$\frac{F_k'(f_k(\hat{b}))}{F_k(f_k(\hat{b}))} f_k'(\hat{b}) = \frac{F_i'(f_i(\hat{b}))}{F_i(f_i(\hat{b}))} f_i'(\hat{b}).$$

Since $f_k(\hat{b}) = f_i(\hat{b})$, it follows from Assumption 4 that $f_k'(\hat{b}) < f_i'(\hat{b})$. But this is impossible since $f_k(b) > f_i(b)$ for all $b \in (\hat{b}, b^*)$. Then if there is some such intersection it must be the case that, for some $m \le n$, $b_m^* < \hat{b} \le b_{m-1}^*$. Arguing as above, it follows that, for all $b \in [b_m^*, b^*)$, $f_1(b) >...> f_{m-1}(b)$. It remains to show that $f_m(b_m^*) = b_m \le f_{m-1}(b_m^*)$.

Q.E.D.

The final question that we address here is what happens to bid distributions if one bidder, (bidder s) becomes "stronger" in the sense of conditional stochastic dominance, that is, the c.d.f of his new distribution of types, $\overline{F}_s(\cdot)$ satisfies

$$\frac{\overline{F}_s'(v)}{\overline{F}_s(v)} \ge \frac{F_s'(v)}{F_s(v)}, \quad v \in [a, b_i].$$

To simplify the discussion, we focus on the case in which the supports of the distributions are unchanged. Let $\overline{f}(b)$ be the new vector of equilibrium inverse bid functions and let $\overline{b}_o$ be the new maximum bid.

Proposition 11: If one bidder becomes "stronger" (in the sense of conditional stochastic dominance) the new maximum bid is strictly larger.

<u>Proof</u>**:** Let $\overline{f}(b)$ be the new equilibrium inverse bid functions**.** We first show that

$$\overline{f}(\hat{b}) \ge f(\hat{b}) \Rightarrow \overline{f}(b) > f(b), \ b < \hat{b} \qquad (2.11)$$

From (2.10),

$$.\frac{p_j'}{p_j} == \frac{F_j'(f_j)}{F_j(f_j)} f_j' = \frac{1}{n-1}\left(\sum_{\substack{i=1 \\ i \ne j}}^{n} \frac{1}{L(,b, f_i(b))} - \frac{n-2}{L(,b, f_j(b))}\right).$$

Then

$$\frac{F_j'(f_j)}{F_j(f_j)}(\overline{f}_j' - f_j') < \frac{\overline{F}_j'(\overline{f}_j)}{\overline{F}_j(\overline{f}_j)}\overline{f}_j' - \frac{F_j'(f_j)}{F_j(f_j)} f_j' =$$
$$\frac{1}{n-1}\left(\sum_{\substack{i=1 \\ i \ne j}}^{n} L(b, \overline{f}_i(b)) - L(b, f_i(b))\right) + (n-2)(L(b, f_j(b)) - L(b, \overline{f}_j(b))) \qquad (2.12)$$

Suppose that condition (2.11) false. Then, for some bidder j, and $\hat{b}$, $\bar{f}_j(\hat{b}) = f_j(\hat{b})$ and $\bar{f}(b) \geq f(b),\ b > \hat{b}$. But, from (2.12), it follows immediately that $\bar{f}_j{}'(\hat{b}) < f_j{}'(\hat{b})$, a contradiction of the previous statement. Suppose the new maximum bid $\bar{b}_o \leq b_o$. Then, $b = \bar{f}(\bar{b}_0) \geq f(\bar{b}_o)$ and so, from (2.11), $\bar{f}(b) > f(b),\ b < \bar{b}_o$. But, at the lower endpoint, $f_i(r) = v_r,\ i = 1,\ldots,n$, a contradiction of the previous statement. Then $\bar{b}_o > b_o$ and so $b = f(b_0) > f(b_o)$.

Q.E.D.

3. REVENUE COMPARISONS

We now compare revenue in the two most common auctions, the open ascending bid and sealed high-bid auctions. We consider only the case of risk-neutral bidders, that is, $u_i(b,v_i) = v_i - b$, so that a bidder's type is also his reservation value (or "valuation"). One simple result that carries over from the two bidder case is that if there is one bidder who is "sufficiently strong" relative to all his opponents, revenue is higher in the sealed high bid auction. Consider the extreme case in which buyer 1 always has the highest valuation. We will argue if buyer 1's minimum valuation is sufficiently high, his equilibrium strategy is to bid buyer 2's maximum valuation $\boldsymbol{b}_2$. To see this, suppose that buyer 2 bids his full valuation and all other buyers drop out. If buyer 1 bids *b*, his expected gain is then

$$e_1(b,v_1) = F_2(b)(v_1 - b)\,.$$

Taking the logarithmic derivative, we obtain

$$\frac{\partial}{\partial b}\ln e_1(b,v_1) = \frac{F_2^{'}(b)}{F_2(b)} - \frac{1}{v_1 - b} > 0,\ \ b < \boldsymbol{b}_2\,,$$

if v_1 is sufficiently large. Thus bidder 1's best response is to bid $\boldsymbol{b}_2$ as claimed. And with bidder 1 bidding in this way, bidder 2 cannot profit so his proposed strategy is also a best response.[6] Intuitively, if we consider overlapping supports and consider the effect of allowing $F_1(\boldsymbol{b}_2) \to 0$, this same result must continue to hold.

The two main cases that we consider here are (a) asymmetries induced by rotations around the upper endpoint of the distribution function, and (b) asymmetries induced by rotations around the lower end-point. In case (a), the c.d.f. for bidder i's valaution F_i satisfies

$$1 - F_i(v) = q_i(1 - F(v)),\ \ q_1 \geq \ldots \geq q_n\,. \tag{3.1}$$

[6] This is the unique equilibrium bid for buyer 1. In this extreme case there is a continuum of equilibrium strategies for the other buyers but in each case they win nothing. Thus equilibrium payoffs are unique.

Proposition 12: Revenue comparison (rotations about upper endpoint)

If the c.d.f.'s of the buyers' distributions of valuations satisfy (3.1), expected seller revenue is higher in the open ascending bid auction than in the sealed high-bid auction.

Proof: From Myerson (1981), we know that expected revenue from any auction-like mechanism can be expressed as

$$R = E\{\sum_{i=1}^{n} J_i(v_i)p_i(v) | p_i(v) \geq 0,\ i = 1,\ldots,n,\ \sum_{i=1}^{n} p_i(v) \leq 1\} \qquad (3.2)$$

where $p_i(v)$ is the probability that the item is allocated to buyer *i* if the valuation vector is *v*. As Bulow (1990) has emphasized, we can interpret

$$J_i(v_i) = v_i - \frac{1 - F_i(v_i)}{F_i^{'}(v_i)}$$

as the marginal revenue associated with increasing the probability that buyer 1 wins the item. Note that in the special case in which the asymmetry is given by (3.1),

$$J_i(v_i) = v_i - \frac{1 - F(v_i)}{F'(v_i)} \equiv J(v_i)$$

and so (3.2) becomes

$$R = E\{\sum_{i=1}^{n} J(v_i)p_i(v) | p_i(v) \geq 0,\ i = 1,\ldots,n,\ \sum_{i=1}^{n} p_i(v) \leq 1\} \qquad (3.3)$$

In the case of the two common auctions, the item remains unallocated if all valuations are less than the reserve price *r* but is otherwise allocated to one of the bidders. That is

$$\sum_{i=1}^{n} p_i(v) = 1 \text{ if, for some } i,\ v_i \geq r \text{, otherwise } \sum_{i=1}^{n} p_i(v) = 0 \qquad (3.4)$$

We now seek to characterize the mechanism which maximizes expected revenue, that is (3.3) subject to the constraint (3.4). We consider the regular case in which marginal revenue is an increasing function of a buyer's valuation. Thus the buyer with the highest valuation also has the highest marginal revenue.

From (3.3), it is clear that expected revenue is maximized by assigning the item to the buyer who contributes the highest marginal revenue. But, in the special case in which $J_i(v) = J(v),\ i = 1,\ldots,n$, this is also the buyer with the highest valuation. Then, expected revenue is maximized by allocating to the highest valued bidder subject to the constraint (3.4), that is, subject to the constraint that the highest valuation is at least *r*. But this is precisely the allocation achieved in the open ascending bid auction. Thus revenue in this auction must be at least as large as that from the sealed high-bid auction. As we have seen, the sealed high-bid auction is not *ex-post* efficient, so revenue is strictly lower.

Q.E.D.

We now consider a family of distributions for which the revenue rqanking is reversed,, that is expected revenue is higher in the sealed high bid auction. We establish that for the open ascending bid auction introducing asymmetry lowers expected revenue, while in the sealed high-bid auction revenue rises. We begin with the open auction. Consider any asymmetric case with c.d.f.s $F_1,...,F_n$ and supports $[a,b_i]$ where $b_1 \geq ... \geq b_n$.

Proposition 13: Revenue Reducing effect of Asymmetry in the open auction

Expected revenue in the open auction with asymmetric bidders is smaller than in a symmetric auction in which each bidder draws from a distribution with c.d.f equal to the logarithmic mean of the c.d.f.s of the n bidders' valuations, that is

$$\overline{F}(v)^n \equiv \underset{i=1}{\overset{n}{\times}} F_i(v),\ v \in [a, b_2]. \tag{3.5}$$

The c.d.f. for the second highest valuation is

$$G_S(v) = \sum_{i=1}^{n} (1-F_i) \underset{\substack{j=1 \\ j\neq i}}{\overset{n}{\times}} F_j + \underset{i=1}{\overset{n}{\times}} F_i = \sum_{i=1}^{n} \underset{\substack{j=1 \\ j\neq i}}{\overset{n}{\times}} F_j - (n-1) \underset{i=1}{\overset{n}{\times}} F \ . \tag{3.6}$$

Similarly, for the symmetric reference auction

$$\overline{G}_S(v) = n\overline{F}^{n-1} - (n-1)\overline{F}^n \ . \tag{3.7}$$

By construction, the second terms on the right-hand side of equations (3.6) and (3.7) are the same. Define

$$q_i \equiv \underset{\substack{j=1 \\ j\neq i}}{\overset{n}{\times}} F_i,\ i = 1,...,n .$$

Then $G_S - \overline{G}_S = \sum_{i=1}^{n} q_i - (n-1)\overline{F}^n$.

Next note that

$$\sum_{i=1}^{n} \ln q_i = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \ln F_j = (n-1)\sum_{i=1}^{n} F_i = (n-1)\ln \overline{F} .$$

Then

$$G_S - \overline{G}_S \geq \underset{q}{Min}\{\sum_{i=1}^{n} q_i - (n-1)\overline{F}^n \mid \sum_{i=1}^{n} \ln q_i \geq (n-1)n \ln \overline{F}, q \geq 0\} .$$

By inspection, the solution to the optimization problem is symmetric with $q_i = \overline{F}^{n-1}$. Hence,

$$G_s - \overline{G}_s > 0 \text{, for any } (F_1,\ldots,F_n) \neq (\overline{F},\ldots,\overline{F}).$$

In the open ascending bid auction, each bidder's dominant strategy is to bid until the asking price equals his valuation. Given such bidding, expected revenue the expectation of the second highest valuation. Thus.

$$R_S = \int_a^{b_2} v dG_S(v) = b - \int_a^{b_2} G_S(v)dv$$
$$\leq b - \int_a^{b_2} \overline{G}_S(v)dv = \overline{R}_S.$$

Q.E.D.

The next corollary is an immediate implication of this result.

Corollary 14: If the difference between the maximum bids of the second through the n-th bidders is sufficiently small, revenue is in the open auction is smaller than revenue in a symmetric reference auction in which

$$\overline{F}(v)^n \equiv \mathop{\times}_{i=1}^{n} F_i(v),\ v \in [a, b_n]. \tag{3.8}$$

We now turn to the sealed high-bid auction. The following Lemma is proved in the appendix.

Lemma 15: If $\dfrac{F_i''(v_i)}{F_i'(v_i)} \geq -\dfrac{2}{v_i}$, $L_i(b, H_i(p_i))^{-1}$ is convex in p_i.

Assumption 5: Linear Rotations about the lower support

Let $\overline{F}(v)$ be a twice continuously differentiable function with $\overline{F}(a) = 0$ and $\overline{F}(\infty) = \infty$. For $b_1 \geq \ldots \geq b_n > 0$ define g_i to satisfy $g_i \overline{F}(b_i) = 1$. Then let $F_i = g_i \overline{F}$, be the c.d.f. for $\tilde{v}_i$, $i = 1,\ldots,n$.

Proposition 16: Revenue Increasing effect of Asymmetry in the sealed high-bid auction

Suppose that $\dfrac{\overline{F}''(v)}{\overline{F}'(v)} \geq -\dfrac{2}{v}$, $a \leq v \leq b_1$, Assumptions 1 and 5 hold, and the difference between the second and n-th valuations is sufficiently small. Then if $\underset{i=1}{\overset{n}{\times}} g_i = 1$,expected revenue in the sealed high-bid auction is greater than in a symmetric auction where each bidder's valuation has support $[a, b]$ and c.d.f. $\overline{F}$.

Proof: As long as the difference between the second and n-th maximum valuations is sufficently small, we know that the equilibrium bid distribution for each bidder is the same interval $[r, b^*]$.

Define $G_H(b)$ to be the c.d.f. for the distribution of winning bids in the sealed high-bid auction. Let $\overline{G}_H(b)$ be the corresponding c.d.f. for the symmetric auction and let $\overline{b}$ be the maximum bid in this case.

In the asymmetric case, it follows from (2.6) that

$$(n-1)\sum_{j=1}^{n} \frac{d}{db} \ln p_j(b) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{d}{db} \ln p_j(b) = \sum_{i=1}^{n} L(b, H_i(p_i(b))) .$$

Hence

$$\frac{d}{db} \ln G_H(b) = \sum_{i=1}^{n} \frac{d}{db} \ln p_i(b) = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{L(b, H_i(p_i(b)))} .$$

Define $p_i = g_i \overline{F}(v_i)$. Then $v_i = H(\frac{p_i}{g_i})$, where $H(\cdot) \equiv F^{-1}(\cdot)$ and $L_i = H(\frac{p_i}{g_i}) - b$. Substituting into the previous expression we obtain,

$$\frac{d}{db} \ln G_H(b) = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{L(b, H(\frac{p_i(b)}{g_i}))} . \tag{3.9}$$

Similarly, in the symmetric case,

$$\frac{d}{db} \ln \overline{G}_H(b) = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{L(b, H(\overline{p}(b)))} . \tag{3.10}$$

Suppose that $G_H(b) \geq \overline{G}_H(b)$, that is

$$\sum_{i=1}^{n} \ln p_i(b) \geq \sum_{i=1}^{n} \ln \overline{p}(b) = n \ln \overline{p}(b)$$

From (3.9) and (3.10),

$$\frac{d}{db}\ln G_H(b) \le \frac{1}{n-1}\underset{p_1,\dots,p_n}{Max}\{\sum_{i=1}^{n} \frac{1}{H(\frac{p_i}{g_i})-b} \mid \sum_{i=1}^{n} \ln p_i \ge n\ln \bar{p}\}$$

$$= \frac{1}{n-1}\underset{y_1,\dots,y_n}{Max}\{\sum_{i=1}^{n} \frac{1}{H(y_i)-b} \mid \sum_{i=1}^{n} \ln g_i y_i \ge n\ln \bar{p}\}\,.$$

By hypothesis $\overset{n}{\underset{i=1}{\times}} g_i = 1$ and so $\sum_{i=1}^{n} \ln g_i = 0$.

Then

$$\frac{G_H^{'}}{G_H} \le \frac{1}{n-1}\underset{y_1,\dots,y_n}{Max}\{\sum_{i=1}^{n} \frac{1}{H(y_i)-b} \mid \sum_{i=1}^{n} \ln y_i \ge n\ln \bar{p}\}\,.$$

But $\sum_{i=1}^{n} \frac{1}{H(y_i)-b}$ is a decreasing and, by Lemma 14, is also convex. By inspection it follows that the maximum is achieved at the symmetric point $(y_1,\dots,y_n)=(\bar{p},\dots,\bar{p})$. Thus,

$$G_H \ge \bar{G}_H \Rightarrow \frac{G_H^{'}}{G_H} \le \frac{n}{n-1}\frac{1}{H(\bar{p}(b))-b} = \frac{\bar{G}_H^{'}}{\bar{G}_H}\,.$$

But $G_H(r) = \overset{n}{\underset{i=1}{\times}} F_i(r) = \bar{F}^n(r) = \bar{G}_H(r)$. It follows immediately that $b^* > \bar{b}$. Hence expected revenue in the asymmetric sealed high-bid auction,

$$R_H = \int_r^{b^*} bdG_H > \int_r^{\bar{b}} bdG_H = \bar{b} - \int_r^{\bar{b}} G_H(b)db > \bar{b} - \int_r^{\bar{b}} \bar{G}_H(b)db = \bar{R}_H$$

Q.E.D.

Appealing to Corollary 14 and Proposition 16 then yields our second main revenue comparison.

Proposition 17: Revenue Comparison (rotations about the left end-point)

Suppose that $\dfrac{\bar{F}''(v)}{\bar{F}'(v)} \geq -\dfrac{2}{v}$, $a \leq v \leq b_1$, Assumptions 1,2 and 5 hold, and the difference between the second and *n*-th maximum valuations is sufficiently small. Then expected revenue in the sealed high-bid auction is greater than in the open ascending bid auction.

4. NUMERICAL ISSUES

The differential equation system (2.3) has a singularity at the left end-point and the slope of the equilibrium bid distributions typically increases without bound as this singularity point is reached.[7] However, in view of our theoretical results, we know that we can solve backwards from the right end-point. Indeed for any maximum bid $\bar{b}$ which is sufficiently large, and induced left end-point $\underline{b}$, the solution to (2.3) is an equilibrium for an auction in which the seller sets a reserve price of $\underline{b}$. Moreover, since $\underline{b}$ declines monotonically with $\bar{b}$, it is, in principle, an easy matter to find the solution associated with some target reserve price *r*, where

$$G(r) = \times_{i=1}^{n} F_i(r) \equiv F_o(r) \tag{4.1}$$

However, there is a further issue. Consider right end-points b^* and $\bar{b} > b^*$, and the corresponding c.d.f.'s for the distribution of winning bids $G^*(b)$ and $\bar{G}(b)$. From Lemma 5,

$$\frac{d}{db}(\ln G^*(b) - \ln \bar{G}(b)) < 0$$

and so the gap between the log win probabilities increases to the left. This problem is extreme in the absence of a reserve price, or, equivalently, if the reserve price $r = a$. For in this case the target is $G(a) = 0$ and so $\ln G(a) = -\infty$. Thus, in the absence of a reserve price, it is a very delicate matter solving numerically for the equilibrium bid functions and win probabilities if the goal is a numerical solution which is accurate at both endpoints.

In Figure 4.1, we depict a "high" and "low" numerical estimate of $G(b)$, the c.d.f of the distribution of winning bids. Expected revenue in the sealed high-bid auction is,

$$R_H = \int_r^{\infty} b\,dG(b) = \bar{b} - \int_r^{\bar{b}} G(b)db < \bar{b} - \int_r^{\bar{b}} Max\{F_o(b), \bar{G}(b)\}db.$$

[7] In the limiting case in which the reserve price is equal to the lowest reservation value, there is a family of distribution functions for which slopes are well-defined at the endpoint. For example, in the symmetric *n* bidder case with uniform distributions, the equilibrium bid distribution has a linear c.d.f. Marshall et al exploit this property in their numerical analysis.

Also

$$R_H = \int_r^{\infty} b dG(b) = \bar{b} - \int_r^{\bar{b}} G(b) db > \bar{b} - \int_r^{\bar{b}} G^*(b) db \,.$$

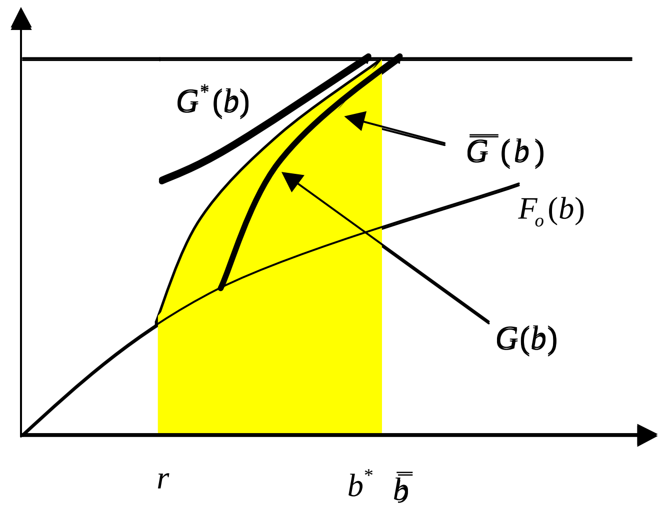

Figure 4.1: High and Low estimates of $G(b)$

Thus if the "high" and "low" estimates are close, we have a close approximation to the equilibrium bid function.

We have just argued that, with $r \leq a$, so that the reserve price is not binding, the estimates of the equilibrium win probabilities will be imprecise in the neighborhood of b=r. But, as the numbers of buyers increases, $\overline{G}(b) = \underset{i=1}{\overset{n}{\times}} \overline{p}_i(b)$ becomes small in the neighborhood of $b = a$, as long as some of the win probabilities converge to zero. In the numerical analysis, we exploit this result to obtain a very fast algorithm for computing the equilibrium probability distribution for the winning bid. While we do not focus on the issue here, a similar argument holds for the bid distribution that each buyer faces. Thus, despite the singularity at the lower endpoint, our methods should provide a useful algorithm for estimating underlying parameter values generated from sample bid distributions .

In our numerical analysis, we also consider the effects of affiliation across types and values that are related.[8] For simplicity, we consider only the 2-bidder case. Then the bidders types s_1 and s_2 are drawn from a distribution with c.d.f. $F(s_1, s_2)$ and density $f(s_1, s_2)$. Let

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[8] The traditional "common value" model assumes symmetry of preferences and beliefs. We use the term "related values" to refer to the broader class of preferences which includes private values as one limiting case ($l_1 = l_2 = 1$) and symmetric common values as a second limiting case ($l_1 = l_2 = \frac{1}{2}$).

$f_i(s_{-i}|s_i),\ i=1,2.$ be buyer *i*'s conditional density function for his opponent's type. We assume that bidder i's valuation is related to that of his opponent as follows.

$$v_i(s) = l_i s_i + (1- l_i)s_j,\ \ s_j \neq s_i,\ 0< l_i \leq 1.$$

We then define $\bar{v}_i(s) \equiv E\{v_i(s_i, \tilde{s}_{-i})|\tilde{s}_{-i} \leq s_{-i}\},\ i=1,2$. As in section 2, we define the vector of equilibrium inverse bid functions $f(b)$

If buyer i bid *b* he wins as long as his opponent bids less than b, that is, as long as $s_{-i} < f_{-i}(b)$. Buyer i's expected gain is then

$$e_i(b,s_i) = \int_{-\infty}^{f_{-i}(b)} v_i(s_i,x) f_i(x|s_i)dx - bF_i(f_{-i}(b)|s_i)$$

Arguing almost exactly as in section 2, in the sealed high-bid auction, the equilibrium inverse bid functions must satisfy the first order conditions,

$$f_i'(b) = \frac{F_i(f_{-i}|f_i)}{f_i(f_{-i}|f_i)}\frac{1}{v_j(f)-b},\ i=1,2. \qquad (4.2)$$

In addition, there are types $\underline{s}_1$ and $\underline{s}_2$ who are just indifferent to paying the reserve price *r* and staying out of the auction, that is:

$$\bar{v}_i(\underline{s}_i, \underline{s}_{-i}) = r,\ \ i=1,2.$$

These modifications of the basic model are readily incorporated into the numerical analysis.

5. NUMERICAL ANALYSIS OF THE EFFECTS OF ASYMMETRY

For our first numerical example we consider uniform distributions. As a base-line case, suppose both buyers' valuations are uniformly distributed with mean 100 and that the support of each distribution is [80,120]. From the revenue equivalence theorem we know that expected revenue is the same in the two auctions. As our first variation, suppose that we consider the effect of a rightward shift in the mean of buyer 1's distribution so that the support becomes $[m_1 - 20, m_1 + 20]$. That is, we increase the first moment of buyer 1's distribution of valuations keeping higher moments constant. The results are shown below. In this and all following tables, ΔR is the net gain in expected revenue from adopting the high-bid auction rather than the open auction.

TABLE 5.1: Shift in the Mean (Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [m_1 - 20, m_1 + 20]$	$m_2 = 100$									
$\text{supp}\{\tilde{v}_2\} = [80,120]$	$s_1 = s_2 = 11.5$									
$m_2 - m_1$	0	10	20	30	40	50	60	70	80	80+
ΔR	0.0	1.1	3.8	7.4	11.2	14.7	17.6	19.3	20.0	20.0

If the difference in means exceeds 80, the equilibrium bidding strategy of buyer 1 is always to bid the weak buyer's maximum valuation. Thus revenue is 120. In the sealed second price auction buyer 1 always wins and pays buyer 2's expected valuation so expected revenue is $m_2 = 100$. The revenue difference is thus 20. Note also that ΔR rises at an increasing rate when $0 < m_1 - m_2 < 40$.

As our second variation, suppose that F_1, buyer 1's c.d.f., is rotated to the right about the lower support. That is, F_1 is a uniform distribution with support $[80, b_1]$, where

TABLE 5.2: Rotation about Lower Support (Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [80, b_1]$							
$\text{supp}\{\tilde{v}_2\} = [80,120]$							
b	120	140	160	180	200	220	240
$m_1 - m_2$	0	10	20	30	40	50	60
ΔR	0.0	1.08	1.5	2.9	4.0	4.6	5.5

$b_1 > 120$. Thus again buyer 1 is drawing from a more favorable distribution. From Proposition 14 we know that revenue must again be higher in the high-bid auction. The table provides an indication of how big the difference can be.

Similarly, for our third variation we consider a rotation of buyer 2's c.d.f. about its upper support. That is buyer 2's valuation is uniformly distributed on $[a_2, 120]$ so that buyer 2 has a distribution with a longer left tail than that of buyer 1. In this case, since the upper supports of the two distributions are the same, it is not all clear that the "fear of loss" is central. However, once again revenue is higher in the sealed high bid auction, although quantitatively, the effects are much smaller.

TABLE 5.3: Rotation about Upper Support
(Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [80,120]$	$\text{supp}\{\tilde{v}_2\} = [a_2, 120]$					
a	80	70	60	50	40	20	0
$m_2 - m_1$	0	5	10	15	20	30	40
ΔR	0.0	0.4	1.2	2.3	3.5	6.4	9.4

Finally, suppose that the two distribution have the same mean but that the variance of buyer 1's valuation is higher. Since this can be thought of as combining the two previous variations, it is a reasonable conjecture that revenue will again be higher in the high-bid auction. This is illustrated in Table 5.4.

TABLE 5.4: **Increasing Uncertainty (Uniform Distributions)**

$\text{supp}\{\tilde{v}_1\} = [100 - 2g, 100 + 2g]$ $\text{supp}\{\tilde{v}_2\} = [80,120]$	$m_1 = m_2 = 100$				
$\text{supp}\{\tilde{v}_1\}$	[80,120]	[60,140]	[40,160]	[20,180]	[0,200]
s_2 / s_1	1	2	3	4	5
ΔR	0.0	0.5	1.0	1.4	1.9

Clearly one must be careful about drawing strong inferences from a numerical analysis of one distribution. However, at least for symmetric distributions, we find broadly similar results, regardless of the form the distribution takes. We conclude this section by returning to our first variation, this time under the assumption that valuations have truncated normal distributions.[9] We take as our base-line case two (truncated) normal distributions, with means of 100 and standard deviations of 10.

TABLE 5.5: Shift in the Mean
(Independent Normal Distributions)

$m_2 = 100$, $s_1 = s_2 = 10$	$r = 0$				
$m_1 - m_2$	0	10	20	30	40
ΔR	0.0	1.1	3.7	6.6	8.9

6. AFFILIATED VALUATIONS

By themselves, the results of the previous section strongly suggest that it is distinctly to the seller's advantage to employ the sealed high-bid auction. However, we have thus far

[9] Unless otherwise noted, we truncate at three standard deviations. Adding longer tails has very little effect on equilibrium bid distributions and hence on expected revenue.

assumed that buyers' valuations are independent. We know from Milgrom and Weber (1982) that, in the symmetric case, whenever valuations are affiliated (pairwise positively correlated), expected revenue is lower in the high-bid auction. The loose intuition is as follows. In the open auction, each bidder acts as if he is up against an opponent with the same valuation. However, in the high-bid auction a bidder must outbid those with lower valuations. In the case of independent valuations, we have revenue equivalence. But if valuations are affiliated, a buyer with a lower valuation has a more conservative estimate of his opponent's valuation and therefore bids more conservatively. Then the optimal response is to bid more conservatively as well.

The case of affiliation is very naturally analyzed under the assumption that valuations have a joint normal distribution. We will write this as $N(m_1, m_2, s_1, s_2, r)$, where r is the correlation coefficient.[10]

Buyers' valuations are affiliated, if for any $x' > x$ and $y' > y$, the joint density $f(v_1, v_2)$ satisfies:

$$\frac{f(x',y')\,f(x,y)}{f(x',y)\,f(x,y')} \geq 1.$$

Then a natural measure of affiliation is

$$A(x,y) \equiv \lim_{x'\downarrow x,\, y'\downarrow y} \frac{\ln \dfrac{f(x',y')\,f(x,y)}{f(x',y)\,f(x,y')}}{(x'-x)(y'-y)} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\ln f(x,y).$$

In the normal case it is readily confirmed that $A(v_1, v_2) = \dfrac{r}{1-r^2}\dfrac{1}{s_1 s_2}$. Thus the second moments of the joint distribution completely determine the measure of affiliation. For the first variation we therefore consider the effect of increasing the correlation coefficient. At $r = 0$, we have independence and thus the revenue equivalence theorem holds. Moreover, at $r = 1$ valuations are perfectly correlated thus each buyer knows that his opponent has the same valuation. In this full information case the buyers bid away all the surplus in both auctions so again we have revenue equivalence. Results for intermediate cases are summarized in Table 6.1.

Two things are striking about these results. First the revenue difference continues to increase with the correlation coefficient until correlation is very high. Second, the revenue differences are small over the entire range, and are never more than one quarter of one

[10] That is, the joint density

$$f(v_1, v_2) = K \exp\left(\frac{-\left(\dfrac{v_1 - m_1}{s_1}\right)^2 + 2r\left(\dfrac{v_1 - m_1}{s_1}\right)\left(\dfrac{v_2 - m_2}{s_2}\right) - \left(\dfrac{v_2 - m_2}{s_2}\right)^2}{2(1-r^2)}\right).$$

TABLE 6.1: Increasing Correlation (Normal Distributions)

$N(m_1, m_2, s_1, s_2, r) = N(100{,}100{,}10{,}10, r)$ Private or common values								
ρ	0	0.1	0.3	0.5	0.7	0.8	0.9	1.0
ΔR	0.0	-0.4	-1.1	-1.7	-2.6	-2.0	-1.7	0

standard deviation. Thus the revenue difference due to affiliation is very small unless the correlation coefficient is very high and the underlying correlation between valuations is very large.

Given this conclusion, it is tempting to also conclude that whenever heterogeneity in valuations is significant, this will very likely be the dominant factor. Table 6.2 below provides some strong confirming support for this conjecture. Note that even for a correlation as high as 0.3, a difference in mean of one standard deviation is enough to reverse the sign of the revenue difference so that again the high-bid auction generates greater revenue.

TABLE 6.2: Shift in the Mean with Positive Correlation (Normal Distributions)

$N(m_1, m_2, s_1, s_2, r) = N(m_1, 100{,}10{,}10, r)$ Private values			
	$m_1 - m_2$		
r	0	10	20
0	0	1.1	3.7
0.1	-0.4	0.7	3.0
0.3	-1.1	0.1	2.8
0.5	-1.7	-0.3	2.5

The above analysis assumes private values. With common values the impact of introducing asymmetry is further reinforced. To illustrate, consider Table 6.3.

TABLE 6.3: Shift in the Mean with Common Values (Independent Normal Distributions)

$m_2 = 100$, $s_1 = s_2 = 10$, $r = 0$						
		$m_1 - m_2$				
	(e_1, e_2)	0	10	20	30	40
Private values	(1,1)	0.0	1.1	3.7	6.6	8.9
Common values	$(\frac{3}{4}, \frac{3}{4})$	0.0	1.4	4.8	9.0	13.4
Symmetric common values	$(\frac{1}{2}, \frac{1}{2})$	0.0	1.9	6.6	12.3	18.2

The first row of results assumes private values. That is, while types are affiliated, a buyer's value depends only on his own type. In the second row each buyer's valuation is a function of both types but each puts more weight on his own valuation. In the third row we have the case of symmetric common values. Each bidder puts equal weight on his own and his opponent's type in valuing the item for sale. While there is no theoretical analysis of the reinforcing effect of common values, the loose intuition is clear. With common values, the higher is a buyer's type, the higher is his opponent's valuation as well as his own. Being up against a stronger opponent, the "fear of loss" effect is reinforced.

7. OPTIMAL RESERVE PRICES

Thus far we have focused on auctions in which the bidders adapt to known heterogeneity but the auctioneer remains passive. This is a natural assumption in an environment in which the bidders are better informed about their opponents than the auctioneer (or seller) is about the bidders' valuations. For example in bidding for construction contracts, the bidders will often have much tighter beliefs about their opponents' costs than the agent putting the contract out to bid. This seems especially likely in the case of government procurement.

However, suppose that the seller has the same beliefs about bidders as they have about each other. In this case the seller can potentially exploit this information by announcing a minimum or "reserve" price. It is easy to see that in extreme cases, this will dramatically affect the conclusion of the previous section. Suppose, for example that the upper support of buyer 2's distribution of valuations is less than the lower support of buyer 1's distribution. If the seller knows this and can commit to a reserve price, one option is to set the reserve price equal to buyer 1's lowest valuation. Buyer 2 will not bid and buyer 1 knows this. The auction then reduces to a simple take-it-or-leave-it offer. Exactly the same holds for any other higher reserve price. Thus with an optimal reserve price the two auctions collapse to a take-it-or-leave-it offer and hence revenue equivalence again obtains. Of course if the supports of the distributions overlap, this simple argument breaks down. However the basic insight remains. When the seller can set an optimal reserve price, the "fear of loss" effect on revenue will be small as the bidder heterogeneity grows large.

To understand the effect of seller reserve prices in less extreme cases, we return to our numerical analysis. In each case, we calculate the optimal seller reserve for the open auction and then compare revenue in the two auctions at this reserve price. However there is no reason to think that the optimal reserve for the high-bid auction will be the same as for the open auction. Thus our approach yields an underestimate of the revenue advantage to utilizing the high-bid auction.

We first consider the uniform case and examine the effect of shifting up the mean of buyer 1's distribution of valuations. To facilitate comparison we first present some results with no reserve price (see Table 7.1). he cost of introducing a reserve price is that the item may not sell. Just how costly this is to the seller depends upon how much he values continuing to hold the item. Thus the optimal reserve price for the seller depends critically on his own valuation.

TABLE 7.1: Shift in the Mean with No Reserve Price (Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [m_1 - 20, m_1 + 20]$ $\text{supp}\{\tilde{v}_2\} = [80,120]$	$s_1 = s_2 = 11.5$				
$m_1 - m_2$	0	10	20	30	40	50
ΔR	0.0	1.1	3.8	7.4	11.2	14.7

We therefore consider two fairly extreme cases. In the first, the seller's valuation is far below the minimum valuations of the buyers. In the second, the seller's valuation is equal to the minimum valuation of the buyers.

TABLE 7.2: Shift in the Mean with Optimal Reserve Price (Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [m_1 - 20, m_1 + 20]$ $\text{supp}\{\tilde{v}_2\} = [80,120]$			$s_1 = s_2 = 11.5$ Seller’s valuation = 0			
$m_1 - m_2$	0	10	20	30	40	50
Reserve price	90	94	100	110	120	
ΔR	0.0	0.4	1.0	1.1	0	0

TABLE 7.3: Shift in the Mean with Optimal Reserve Price (Uniform Distributions)

$\text{supp}\{\tilde{v}_1\} = [m_1 - 20, m_1 + 20]$ $\text{supp}\{\tilde{v}_2\} = [80,120]$			$s_1 = s_2 = 11.5$ Seller’s valuation = 80			
$m_1 - m_2$	0	10	20	30	40	40+
Reserve price	100	103	108	114	120	
ΔR	0.0	0.2	0.6	0.7	0	0

The striking feature of both Tables is that revenue differences are small in all cases. This suggests that with correlation, expected revenue might be actually higher in the open auction. We therefore turn once more to the case of normally distributed valuations. Table 7.4 summarizes results from the previous section with no reserve price and a correlation coefficient of 0.3. In Table 4.5 the seller's reserve price is set to maximize revenue in the open auction.

TABLE 7.4: Shift in the Mean with No Reserve Price (Normal Distributions)

$N(m_1, m_2, s_1, s_2, r) = N(m_1, 100, 10, 10, 0.3)$					
$m_1 - m_2$	0	10	20	30	40
ΔR	-1.1	0.1	2.8	5.3	5.7

TABLE 7.5: Shift in the Mean with Optimal Reserve Price (Normal Distributions)

$N(m_1, m_2, s_1, s_2, r) = N(m_1, 100, 10, 10, 0.3)$ Seller's valuation = 0					
$m_1 - m_2$	0	10	20	30	40
Reserve price	77.4	87.5	100.3	111.8	121.9
ΔR	-1.1	0.1	0.11	0.03	0.00

Note that the maximum difference in expected revenue is not much more than 10% of one standard deviation and is typically much less than this. From this we conclude that when the seller has the same information as the buyers and can credibly commit to a reserve price, revenue differences between the two auctions are typically going to be insignificant. That is, an approximate revenue equivalence result re-emerges from the numerical analysis.

8. MORE BIDDERS

We now consider the effects of adding additional bidders. First suppose that there is one "strong" bidder, with a higher mean. Instead of being faced by a single weaker opponent, he must compete with several such opponents. Results are shown in Table 8.1.

TABLE 8.1: Single Strong Bidder (Independent Normal Distributions)

$N_A(m_A, s_A) = N(100, 10)$ $N_B(m_B, s_B) = N(m_B, 10)$	One strong type A bidder				
	$m_A - m_B$				
# of type B bidders	0	10	20	30	40
1	0	1.1	3.7	6.6	8.9
2	0	0.5	2.4	5.0	7.7
3	0	0.3	1.8	4.2	6.8
4	0	0.2	1.4	3.7	6.2

The first row of results assumes private values. That is, while types are affiliated, a buyer's value depends only on his own type. In the second row each buyer's valuation is a function of both types but each puts more weight on his own valuation. In the third row we have the case of symmetric common values. Each bidder puts equal weight on his own and his opponent's type in valuing the item for sale.

Next suppose that there are two strong bidders competing. There is also a "competitive fringe" of other bidders with lower expected valuations. The central issue now is how much do the weak ("fringe") bidders influence the behavior of the two strong bidders.
As Table 8.2 shows, even when the strong bidders' means are only one standard deviation higher, the effect of adding a third bidder is quite small (one third of a standard deviation). Adding further weak bidders has a successively smaller impact on expected revenue.

TABLE 8.2: Competitive Fringe (Independent Normal Distributions)

$N_A(m_A, s_A) = N(m_A, 10)$ $N_B(m_B, s_B) = N(100, 10)$				$m_B = 2$		
	$m_B - m_A$					
	10			20		
# of Type A bidders	R_H	R_S	ΔR	R_H	R_S	ΔR
0	104.4	104.4	0.0	114.4	114.4	0.0
1	107.8	106.9	0.2	115.3	115.1	0.2
2	108.7	108.5	0.2	116.0.	115.7	0.2
3	109.8	109.6	0.2	116.5	116.2	0.3
4	110.7	110.5	0.2	116.9	116.6	0.3

Most strikingly, the difference in expected revenue from the two auctions is very small and is almost independent of the number of weak bidders. We therefore conclude that the effect of asymmetry is strong if and only if there is a single strong bidder.

Finally, we consider the effect of introducing risk aversion with different numbers of bidders. Since the risk aversion parameter a is the bidders' degree of absolute risk aversion, even a parameter value of $a = 0.1$ is very large since it implies a relative risk aversion of a times a bidder's wealth.

The first three columns of results in Table 8.3 show the effect of risk aversion on revenue differences in the high-bid auction when all bidders have the same underlying distribution of types. Since risk aversion has no effect on equilibrium bidding in the open auction, these revenue differences are the total effect of risk aversion on expected revenue. Looking at column 3, even when the risk aversion parameter is absurdly high, the effect on revenue is small.

TABLE 8.3: Risk Aversion

$\text{supp}\{\tilde{v}_A\} = [100,140]$ $\text{supp}\{\tilde{v}_B\} = [80,120]$,						
	# of Type A bidders					
	0			1		
	Risk aversion parameter *a*			Risk aversion parameter *a*		
# of Type B bidders	0	0.1	1.0	0	0.1	1.0
1				3.7	3.7	4.2
2	0	0.0	0.3	2.4	2.4	2.7
3	0	0.0	0.3	1.7	1.7	1.9
4	0	0.0	0.2	1.3	1.3	1.5
5	0	0.0	0.1	1.1	1.1	1.2

The final three columns show the impact of risk aversion when there is a single "strong" bidder. Again the impact of risk aversion on expected revenue is small. We conclude from this that there is little to be lost by assuming risk neutrality when analyzing bidding in the sealed high-bid auction.[11]

9. CONCLUDING REMARKS

In this paper we have focused on the role of heterogeneity in auctions with affiliated private valuations. Our main results are summarized in Table 9.1 below.

Clearly the results obtained to date have led us to make some strong claims. Possibly there are other important classes of distributions for which quite different conclusions may emerge. We therefore strongly encourage any suggestions along these lines. Moreover, because conclusions drawn from a numerical analysis are never going to be as convincing as those from a theorem with weak assumptions, we have taken pains to develop a FORTRAN program with the power and flexibility to handle a wide variety of cases.[12] Because the differential equations which define the equilibrium bid functions are very poorly behaved at the lower end-point, there are some quite complex technical issues which had to be dealt with before we could develop such a program. These are summarized in the Appendix.

[11] The impact of risk aversion on revenue will be far greater if there is uncertainty about the value of the item for sale. However such uncertainty after the auction is a factor in both auctions. We conjecture that the small differential effect holds very generally.

[12] The *.EXE files are available at http://www.econ.ucla.edu/riley. If you wish to modify the program, please let us know and we will forward a copy of the fortran program.

TABLE 9.1: THE CENTRAL RESULTS

Independent Private Values	With a single "strong" bidder, whose type is drawn from a more favorable distribution, there is an economically important net advantage to adopting the sealed high-bid auction.
Correlated Values (affiliation)	With symmetric bidders there is only a small revenue loss if the sealed high-bid auction is utilized rather then the open (or Vickrey) auction. Introducing asymmetry in the form of a single strong bidder quickly reverses the revenue ranking.
Common Values	The asymmetry effect resulting in higher revenue from the sealed high-bid auction is further strengthened if values are common.
Reserve Prices	If the seller/procurer knows as much about the underlying distributions as the bidders and sets an optimal reserve price, revenue differences are small.
Competitive Fringe	If there are two equally strong bidders, the effect of adding a competitive fringe of weaker bidders has a very similar effect on revenue from the two auctions.
Risk Aversion	Even a very high degrees of risk aversion has only a small positive effect on bids in the sealed high-bid auction (and none in the open auction.)

We have argued that to understand revenue differences in the two common auctions, it is critical to separate out cases in which the seller has the information and credibility to set a reserve price from those in which he does not. In the latter case, even a small degree of buyer heterogeneity is likely to imply that revenue will be greater in the high-bid auction. This is so even though revenue is lower in the high-bid auction with correlated valuations and homogeneous bidders. The reasons are two-fold. First, even with high correlation there is only a small difference in expected revenue from the two auctions. Second, the "fear of loss" effect which comes into play when one buyer has a higher mean valuation is typically quite strong, even when the difference in means is as small as one standard deviation.

If, however, the seller knows the distributions of buyers' valuations and is able to set a reserve price which incorporates this knowledge, the "fear of loss" effect is exploited via the seller's reserve price and thus pushes up revenue in both auctions rather than in the high-bid auction alone. While in our examples the sealed high-bid auction again generates higher expected revenue with only modest heterogeneity, the revenue differences are typically small. Thus for this case an approximate revenue equivalence theorem once again emerges. We believe these results shed striking new light on the puzzle as to why bidding for contracts is essentially always by sealed bid while the famous auction houses use open auctions to trade antiquities.[13] In the former case the procurer typically has much less information about the cost of completing the contract than the bidders. In the case of government contracts, there are additional incentive problems associated with setting reserve prices. In particular it is very hard to imagine that the agency putting the contract out to bid is able to establish a meaningful opportunity cost of not completing the project. Instead there will be some overall budget ceiling established further up the political hierarchy.

A very different picture emerges if the auctioneer (and seller) together are in a position to "evaluate the market" as effectively as any potential bidder can evaluate his likely competitors. For then a seller's reserve can be established which takes into account any observable heterogeneity and approximate revenue equivalence ensues. This is precisely the situation in the sale of important paintings in the famous auction houses. The seller is himself often an expert and the auction house operates not just as a middle-man but also as a market analyst. The fact that these auction houses charge high commissions suggests strongly that they do provide valuable information services.

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[13] We refer here to the sale of valuable items to "serious" and experienced bidders. Clearly the open auction has one great advantage over other schemes. It creates excitement. While auction houses' advertising of this factor is self-serving, the recent Kennedy auction suggests that the "buzz" of an open auction can play a powerful role.

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