ASYMMETRIC AUCTIONS*

by

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The revenue-equivalence theorem[1] for auctions predicts that expected seller revenue is independent of the bidding rules, as long as equilibrium has the properties that the buyer with the highest reservation price wins and any buyer with the lowest possible reservation price has zero expected surplus. Thus, in particular, the two most common auction institutions - the open "English" auction and the sealed high-bid auction - are equivalent despite their rather different strategic properties.

This strong prediction of equivalence seems at odds, however, with the empirical observation that rarely is any given kind of commodity sold through more than one sort of auction. Thus, for example, art is nearly always auctioned off according to the English rules, whereas job contracts are normally awarded through sealed bids.

Admittedly, in the public sector, there have been a few attempts to use both methods (lumber contracts in the Pacific Northwest) or to switch from one to the other (Treasury Bills). But changes have typically met great resistance. This is also in conflict with theory, since a corollary of the revenue equivalence theorem is that the expected surplus for any buyer is the same in the two auctions.

These discrepancies suggest that the hypotheses of the revenue-equivalence theorem may be too strong. The principal assumptions are (i) risk neutrality, (ii) independence of different buyers' private signals about the item's value, (iii) lack of collusion among buyers, and (iv) symmetry of buyers' beliefs. Over the last fifteen years, a number of papers have explored the implications of relaxing these assumptions. Typically a clear-cut prediction has emerged in each of these papers in favor of either the high-bid or open auction.

Thus under increasingly general assumptions, Holt (1980), Riley and Samuelson (1981), and Maskin and Riley (1984) show that, when buyers are risk averse, the high-bid auction should be favored by a seller even if he also exhibits risk aversion.[2]

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[1]This was independently derived by Myerson (1981) and Riley and Samuelson (1981). Twenty years earlier Vickrey (1961) established the revenue equivalence of the two most common auctions (open and sealed high-bid) under the assumption that reservation prices were independent draws from a uniform distribution.

[2]Matthews (1987) extends this result to show that if all buyers exhibit constant absolute risk aversion, the auctions are equivalent from their perspective. The sealed high bid auction thus Pareto-dominates the open auction.

In turn, the assumption of independence of private signals of the items value is relaxed by Milgrom and Weber (1982). If reservation prices are "affiliated" (technically, pair-wise positively correlated), they show that the English auction generates higher expected revenue than the high-bid auction.

Finally Graham and Marshall (1989) and McAfee and McMillan (1992) allow for the possibility that buyers collude. In particular, Graham and Marshall argue that such collusion is facilitated in an open auction, where buyers can directly inspect one another's behavior. Hence expected revenue will tend to be higher in high-bid auctions.

In all these papers, buyers are symmetric *ex ante* in the sense that their preference parameters (i.e., their "types") are drawn from a symmetric joint probability distribution. Thus if two buyers are of the same type they will have the same beliefs about the remaining buyers' preferences. Given this symmetry, there will exist a symmetric equilibrium (if an equilibrium exists at all[3].) In such an equilibrium, all buyers use the same equilibrium strategy as a function of their type. (Most authors have confined attention to symmetric equilibria.[4])

In this paper we drop the symmetry assumption.[5] Asymmetries are often important in contract bidding. Each potential contractor has essentially the same information about the nature of the project but a different opportunity cost of completing it. Whenever some aspect of these differences is common knowledge, beliefs are asymmetric. In major art auctions as well, there are obvious *ex ante* asymmetries associated with differing budget constraints.

In the following section we describe the basic model and then present three illustrative examples. These show that, with asymmetry, revenue equivalence no longer

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[3]See Maskin and Riley (1995) for conditions under which an equilibrium exists in the high-bid auction.

[4]Maskin and Riley (1996) provide conditions under which the high-bid auction has no asymmetric equilibria.

[5]Although previous literature on asymmetric auctions is not large, it does date back many years. Vickrey (1961) analyzed equilibrium under the extreme assumption that one buyer's reservation value is public information while the other buyer's valuation is drawn from a uniform distribution. Griesmer, Levitan and Shubik (1967) extend Vickrey's analysis to the case of two uniform distributions.

holds and that, under different assumptions about the nature of the heterogeneity, expected revenue in the high-bid auction may be higher or lower than in the open auction.[6] We lay stress on the informal logic behind these results.

In Section 2 we begin the formal analysis by characterizing buyers' equilibrium bidding strategies in the high-bid auction (Propositions 2.2 and 2.4). We also show quite generally that "strong" buyers prefer the open auction, whereas "weak" buyers prefer the high-bid auction (Proposition 2.5). Then, in section 3, we extend the examples of section 1 to obtain general comparisons of the revenue yielded by the high-bid and open auctions (Propositions 3.3-3.5). Finally, in order to illustrate the economic significance of our analytical results, we present some numerical solutions in section 4.

1. THREE EXAMPLES

Consider a two-buyer auction for a single item. In this paper, we assume private values, i.e., that one buyer's information does not affect the other's preferences and that buyers are risk neutral. Let v_i be buyer *i*'s reservation price or "valuation." Then we can express buyer *i*'s surplus if he wins the item and pays *b* as

(1.1) $$u_i(b, v_i) = v_i - b$$

We assume that buyer *i*'s valuation is private information. From the other buyer's perspective, it is a random variable $\tilde{v}_i$ with c.d.f. $F_i(\cdot)$. We assume that the support of v_i is an interval $[b_i, a_i]$. Heterogeneity is thus captured by the assumption that the buyers' valuations are drawn from different distributions. Throughout we will describe one buyer as "strong" (*s*) and the other as "weak" (*w*). (Roughly speaking, these terms mean that $F_s(\cdot)$ first-order stochastically dominates $F_w(\cdot)$.) Hence the index *i* ranges in the set $\{s,w\}$. Finally we suppose that the random variables for the two buyers' valuations, denoted, $\tilde{v}_s$ and $\tilde{v}_w$, are independent.[7]

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[6] Graham and Marshall (1995), Bulow, Huang and Klemperer (1997) and Klemperer (1997) also consider asymmetries. Each paper points to cases in which expected revenue is higher in the high-bid auction.

[7] As Li and Riley (1998) show, essentially equivalent results hold if the true valuation is a convex combination of the private signals. Thus buyer *i*'s payoff is
$u_i(b, v_i, v_j) = a v_i + (1-a) v_j - b, \quad j \neq i,$

In the sealed high-bid auction, buyers submit bids simultaneously. The winner is the high bidder (in our formal analysis we work with continuous distributions and so the probability of a tie is zero), and he pays his bid. Suppose that the strong buyer's equilibrium bidding strategy is to bid $b_s = b_s(v_s)$ as a function of his valuation v_s. Under weak assumptions, one can show that both $b_s(\cdot)$ and $b_w(\cdot)$ (the equilibrium bidding strategies of the strong and weak buyers) are strictly increasing functions (see Maskin and Riley (1996)). Hence for each v_s, $b_s(v_s)$ solves

(1.2) $$\underset{b}{Max}\, F_w(b_w^{-1}(b))(v_s - b),$$

where $b_w^{-1}(b)$ is the inverse of $b_w(\cdot)$. Similarly, $b_w(v_w)$ solves

(1.3) $$\underset{b}{Max}\, F_s(b_s^{-1}(b))(v_w - b).$$

In the open (or "English") auction, buyers call out successively higher bids. The last buyer to bid is the winner, and he pays his bid. Under our assumptions, a buyer will clearly be willing to top his opponent's current bid as long as that bid is less than his own valuation. Hence, bidding will proceed until the lower of the two buyers' valuations is reached. Thus, in equilibrium the winner will be the high-valuation buyer, but he will pay a bid equal to the other buyer's valuation. The English auction is, therefore, equivalent to a sealed-bid auction in which the high bidder wins but pays only the second-highest bid (a second-price or "Vickrey" auction), since, as Vickrey (1961) showed, it is a dominant strategy in such an auction to bid one's valuation. Thus although we are interested in comparing the high-bid and open auctions, we will analyze the latter as a second-price auction.

We begin by examining several leading examples of deviations from symmetry.

Example 1: The strong buyer 's distribution is "shifted" to the right.

Suppose that the weak buyer valuation is distributed uniformly on the interval [0,1] and that the strong buyer's valuation is distributed on the interval [2,3]. That is, the strong buyer's distribution is shifted to the right.

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where $0 < a \leq 1$. The critical assumption is that the signals be independent.

Let us first consider the high-bid auction. Assume, for the moment, that the weak buyer bids his valuation, i.e., he sets $b_w(v_w) = v_w$. What is the strong buyer's best response? If she bids $b \in [0,1]$, she wins with probability $F_w(b_w^{-1}(b)) = b$. Hence, her maximization problem is $\underset{b\in[0,1]}{Max}\, b(v_s - b)$. But for $v_s \geq 2$, the solution is $b = 1$. That is, the strong buyer's best-response is $b_s(v_s) = 1$ for v_s in the interval [2,3]. Notice, moreover, that if she behaves in this way, it is indeed optimal for the weak buyer to set $b_w(v_w) = v_w$, since he cannot profitably win the auction anyway.

We have therefore exhibited an equilibrium for the high-bid auction. Moreover, as shown in Maskin-Riley (1995), this equilibrium is (essentially) unique. The salient feature is that the strong buyer stands to gain so much from winning that it pays her to be sure that she out-bids the weak buyer. Clearly, the expected revenue from the auction is 1.

Now let us turn to the open auction. Notice that (as in the high-bid auction) the strong buyer is always the winner, since she always has the higher valuation. However, because she pays only the second-highest valuation, her expected payment is $E\{\tilde{v}_w\} = \frac{1}{2}$.

We have thus exhibited an example in which an asymmetry between the buyers favors the high-bid over the open auction. More generally, whenever the strong buyer's distribution is such that, with high probability, her valuation is a great deal higher than that of the weak buyer, the high-bid auction will tend to generate more revenue; to guarantee winning, the strong buyer will be inclined to enter a bid equal to the maximum valuation in the weak buyer's support, whereas, under the open auction, she will pay only the expected value of the weak buyer's valuation.

This principle might termed the "Getty effect," after the wealthy art museum known for out-bidding its competition. And clearly, as the art world illustrates, it is an important principle in practice. But as our generalization of Example 1 (Proposition 3.3) demonstrates, the high-bid auction also emerges as superior even for horizontal shifts of the strong buyer's distribution that are not so extreme as to induce her to preempt her opponent.

Example 2: The strong buyer 2's distribution is "stretched".

For our second example, suppose that the weak buyer's valuation is distributed uniformly on $[0,\dfrac{1}{1+z}]$ and the strong buyer's valuation is distributed uniformly on $[0,\dfrac{1}{1-z}]$, for $z>0$. That is, the strong buyer has the same distribution as the weak buyer, only "stretched out" over a wider interval.

When $z=0$ (i.e., the distributions are both uniform on [0,1]), it is easy to verify that

(1.4) $\quad b_i^{-1}(b)=2b$

is buyer *i*'s equilibrium inverse bid function in the high-bid auction.[8]

Because a buyer with valuation 2*b* wins with probability $F_j(2b)=2b,\;\; j\neq i,$ his expected payment is $F_j(2b)b=2b^2$. In the open auction, a 2*b*-buyer also wins with probability 2*b*, and his payment (conditional upon winning) is the mean of the other buyer's valuation, conditional upon it being less than 2*b*. Because of our uniformity assumption, this conditional mean is just b. Thus the unconditional expected payment is $2b^2$, the same as in the high-bid auction (this is an illustration of the revenue equivalence theorem).

Now let us see what happens as z becomes positive. In the open auction, the weak buyer with valuation 2b wins with probability $2b(1-z)$, and his unconditional expected payment is $2b^2(1-z)$. In the high-bid auction, if the strong buyer were to continue to use the strategy given by (1.4), the weak buyer's best response would also be given by (1.4). Thus the weak buyer's expected payment would be $F_j(2b)b=(1-z)2b^2$, exactly as in the open auction. Similarly, the payment by the strong buyer would be $(1+z)2b^2$ in either auction. Thus revenue would be the same in the two auctions.

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8 When $z=0$ and one buyer bids according to (1.4), the probability that the other buyer, with valuation v, wins with a bid of *b* is $2b$. Then the strong buyer has an expected profit of $2b(v-b)$. This takes on its maximum at $b=\frac{1}{2}v$.

In the high-bid auction, however, buyers do not continue to use (1.4). If they did, a $\frac{1}{1-z}$ – valuation strong buyer would outbid a $\frac{1}{1+z}$ – valuation weak buyer by $\frac{1}{2(1-z)}$ to $\frac{1}{2(1+z)}$, and so can reduce her bid to the latter while still winning with probability 1. Thus, for equilibrium, the strong buyer must reduce her bid as a function of her valuation, relative to (1.4). But such a reduction will induce the weak buyer to bid more aggressively than he would if the strong buyer used (1.4). This is because the strong buyer's bids are now distributed more densely than before. Hence the marginal benefit to the weak buyer of bidding slightly higher rises: a small increase in his bid leads to a greater increase in his probability of winning than under (1.4). In equilibrium,[9] the weak and strong bidder's inverse bid functions are

$$b_w^{-1}(b) = \frac{2b}{1+z(2b)^2} \quad \text{and} \quad b_s^{-1}(b) = \frac{2b}{1-z(2b)^2}.$$

What effect does this change in bidding strategies have on revenue? For the high-bid auction, the c.d.f. $G_H(b)$ for the winning bid satisfies

$$G_H(b) = \Pr(b_s(\tilde{v}_s) \le b) \times \Pr(b_w(\tilde{v}_w) \le b) = F_s(b_s^{-1}(b))F_w(b_w^{-1}(b))$$
$$= (1-z)b_s^{-1}(b)(1+z)b_w^{-1}(b) = \frac{(1-z^2)(2b)^2}{1-z^2(2b)^4}.$$

It is readily confirmed that this is a decreasing function of z. Thus expected revenue in the high-bid auction rises with z. For the open auction the second valuation is less than b if and only if it is not the case that both valuations are higher. Thus

$$G_O(b) = 1-(1-F_s(b))(1-F_w(b)) = F_s + F_w - F_sF_w$$
$$= (1-z)b + (1+z)b - (1-z^2)b^2 = 2b - (1-z)b^2.$$

[9] It is readily confirmed that these inverse bid functions satisfy the equilibrium differential equations and endpoint conditions that we give below (see (2.12) and (2.13)).

The c.d.f. for the open auction is therefore increasing in z. Since the two distributions yield the same expected revenue in the symmetric case (z=0), expected revenue is strictly greater for the high-bid auction than for the open auction when z>0.

We have been discussing the case of the uniform distribution. But, as Proposition 3.4 below makes clear, the same conclusion applies to a large class of other distributions.

Let us turn to our last example.

Example 3: Probability is reallocated to the lower end point of buyer 1's distribution.

The idea of the example is easiest to present for two-point distributions - ones in which all probability mass is confined to the points 0 and 2. Suppose first that both buyers have degenerate distributions in which all probability is concentrated on 2. Suppose that we now shift half the probability mass for the weak buyer to the point 0. Thus the probabilities that the weak buyer has valuation 0 or valuation 2 are $\frac{1}{2}$ each.

In the open auction, expected revenue is just the probability that the weak buyer's valuation is high (i.e., $\frac{1}{2}$) times the payment made in that case (i.e., 2). Thus expected revenue is 1. In the high-bid auction, the strong buyer can win with probability (at least) $\frac{1}{2}$ if she bids (just above) zero. Her expected payoff from doing so is (at least) 1. This implies that she will never bid more than 1 in equilibrium (since her payoff would then be strictly less than 1). Hence the weak buyer can win for certain by bidding $1+e$, and so his <u>ex</u> <u>ante</u> expected payoff (i.e., his payoff before his type is realized) exceeds

$$\tfrac{1}{2}(2-(1+e)) = \tfrac{1}{2}(1-e) \text{ for all } e > 0.$$

That is, his expected payoff is at least $\frac{1}{2}$. We conclude that the sum of the equilibrium expected payoffs to the two buyers is at least $1\frac{1}{2}$. Because the winner always has a valuation of 2, the social surplus from the auction is 2. It follows that expected revenue - the difference between surplus and buyers' expected payoffs - is at most $\frac{1}{2}$. Hence, the open auction is superior.

The important feature in this example is that in the high-bid auction the strong buyer does not get a positive payoff from bidding so high that she is assured of winning (given that the weak buyer has valuation 2 with positive probability, the strong buyer would have to bid 2 to be assured of winning.) However, she does obtain a positive expected payoff from bidding very low. It is this incentive to "low ball" that works against the high-bid auction here. Our general result along these lines can be found in Proposition 3.5.

2. EQUILIBRIUM BIDDING WITH ASYMMETRIC BELIEFS

We now turn to the detailed analysis. For $i \in \{s, w\}$, buyer i's valuation $\tilde{v}_i$ has support $[b_i, a_i]$, $0 \le b_i < a_i$, with c.d.f $F_i(\cdot)$ on $[b_i, a_i]$ that is twice continuously differentiable on $(b_i, a_i]$. We assume also that the density $F_i'(\cdot)$ is strictly positive on $[b_i, a_i]$.

Remark 1: We allow for the possibility of a mass point at the lower end of the distribution (i.e. $F_i(b_i) > 0$). One way that this could come about (even if intrinsically buyers' valuations are distributed continuously, is if the seller sets a reserve price b that is higher than the minimum valuation of at least one buyer. In equilibrium, anyone with a valuation greater than the seller's reserve bids strictly greater than b. Thus by bidding b, the strong buyer wins with probability $F_w(b)$ and the weak buyer wins with probability $F_s(b)$.

We shall assume that the distribution of the strong buyer's valuation first order stochastically dominates that of the weak buyer:

(2.1) $\quad F_w(v) > F_s(v),\;$ for all $v \in (b_w, a_s)$.

Notice that (2.1) implies that

(2.2) $\quad b_w \le b_s$ and $a_w \le a_s$.

Actually, we will require a condition somewhat stronger that (2.1), viz., conditional stochastic dominance. Specifically, suppose that, for all $x < y$ in (b_s, a_w)

(2.3) $$\Pr\{\tilde{v}_s < x | \tilde{v}_s < y\} = \frac{F_s(x)}{F_s(y)} < \frac{F_w(x)}{F_w(y)} = \Pr\{\tilde{v}_w < x | \tilde{v}_w < y\}$$

Rearranging (2.3) we obtain:

$$\frac{F_s(x)}{F_w(x)} < \frac{F_s(y)}{F_w(y)} \quad \text{for all } x < y \text{ in } (b_s, a_s).$$

Thus for (2.3) to hold, $\frac{F_s(v)}{F_w(v)}$ must be strictly increasing on the interval $[b_s, a_s]$. The definition of Conditional Stochastic Dominance that we shall use is a slightly weaker condition than this.

Conditional Stochastic Dominance (CSD)

Suppose that (2.2) holds. There exists $l \in (0,1)$ and $g \in [b_s, a_w]$ (with $g = b_s$ if $b_s > b_w$) such that,

(i) $F_s(v) = lF_w(v)$ for all $v \in [b_s, g]$

and

(ii) $\frac{d}{dv}\frac{F_s(v)}{F_w(v)} > 0$[10] for all $v \in [g, a_s]$,

Note that CSD implies that

(2.4) $\frac{F_s'(v)}{F_s(v)} > \frac{F_w'(v)}{F_w(v)}$, for all $v \in (g, a_w]$.

In addition, we have the following result.

Lemma 2.1: CSD implies first order stochastic dominance, that is, (2.1). In addition, it implies that either

$$F_w(b_s) > F_s(b_s) \text{ or } F_w(b_s) = F_s(b_s) = 0.$$

Proof: If $b_w < b_s$, then for $v \in (b_w, b_s)$, $F_w(v) > 0 = F_s(v)$ and for $v \in [b_s, a_s)$, CSD (ii) implies

(2.5) $$\frac{F_s(v)}{F_w(v)} < \frac{F_s(a_s)}{F_w(a_s)} = 1.$$

If $b_w = b_s$, then, for $v \in (b_s, g)$

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[10] If $v = g > b_s$, this should be interpreted as the right derivative of $\frac{F_s}{F_w}$.

$$F_w(v) > lF_w(v) = F_s(v)$$

and for $v \in (g, a_s)$, (ii) implies that (2.5) holds. Hence CSD implies (2.1). But if

(2.6) $\quad b_s = b_w = b$

and $F_s(b) = F_w(b) > 0$, then (2.4) implies that $F_s(v) > F_w(v),$ for all v near b, a contradiction of (2.1). We conclude that if (2.6) holds, and $F_s(b) = F_w(b)$, then

(2.7) $\quad F_s(b) = F_w(b) = 0$.

Q.E.D.

Before presenting the main results, we first consider what insights can be gained from a mechanism-design perspective. From the revenue-equivalence theorem, revenue can be computed immediately for any mechanism, once the probability of winning for each buyer-type is ascertained. With two bidders having valuations v_s and v_w, let $p_i(v_s, v_w),\ i \in \{s, w\}$, be the equilibrium probability that buyer i wins in a given mechanism. We consider the case in which the minimum valuations are zero $(b_w = b_s = 0)$. Then as Myerson (1981) shows, expected revenue can be expressed as

(2.8) $\quad R = \int_0^{a_s} \int_0^{a_w} [J_w(v_w)p_w(v_w, v_s) + J_s(v_s)p_s(v_w, v_s)]dF_w dF_s$

where

(2.9) $\quad J_i(v) = v - \dfrac{1 - F_i(v)}{F_i'(v)}$.

We are interested in understanding the difference in expected revenue generated by the high-bid and open auctions. These auctions belong to the class of selling mechanisms for which the item is always sold in equilibrium. That is, for all possible pairs of valuations (v_s, v_w),

$$p_w(v_s, v_w) + p_s(v_s, v_w) = 1$$

Thus (2.8) can be rewritten as

(2.10) $\quad R = \int_0^{a_s} \int_0^{a_w} [J_s(v_s) - J_w(v_w)]p_s(v_s, v_w)dF_w dF_s + \int_0^{a_s} \int_0^{a_w} J_w(v_w)dF_w dF_s$

As Bulow and Roberts (1989) point out, $J_i(v_i)$ is the expected marginal revenue generated if the item is assigned to buyer i of type v_i. Thus, under the assumption that J_s and J_w are strictly increasing, and given the constraint that the item must be sold, expected total revenue is maximized by selling to the strong buyer (setting $p_s = 1$) if and only if J_s exceeds J_w.

Suppose, to begin with, that, for each valuation, the weak buyer's marginal revenue is no greater than that of the strong buyer $(J_w(v) \le J_s(v).)$ Figure 1 depicts the implicit mapping $v_s = T(v_w) \equiv J_s^{-1}(J_w(v_w))$. That is, the pairs of valuations (v_s, v_w) where $J_w(v_w) = J_s(v_s)$. If $J_w(v) \le J_s(v)$, this curve lies on or below the $45°$ line. In the open auction the high-valuation buyer is the winner. Thus to maximize revenue,

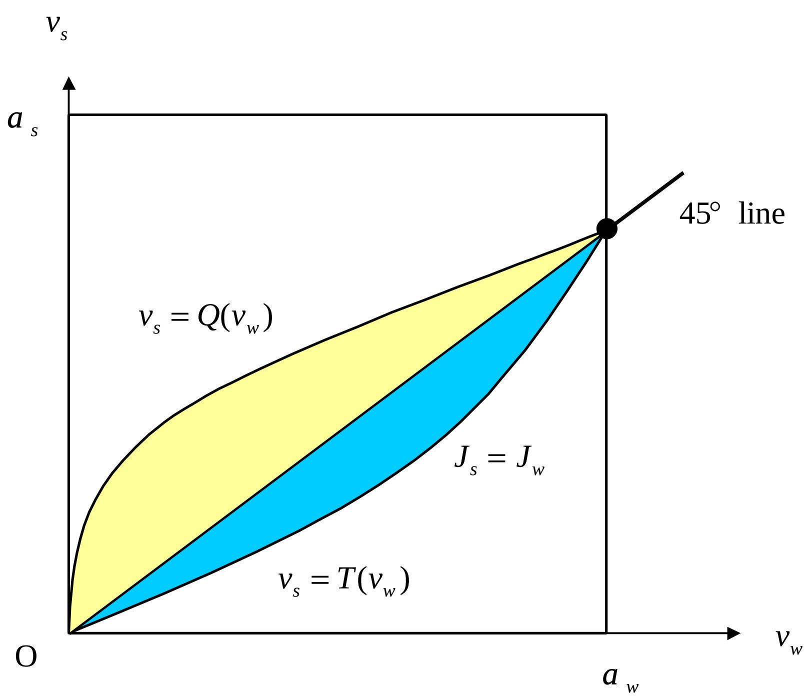

Figure 1: Open auction superior

the playing field needs to be tilted in favor of the strong bidder.[11] [12]

In this paper, however, we are concerned with comparing the revenue from the two common auctions which both employ symmetric rules (a level playing field.) In the open auction, the high-valuation bidder wins. Thus, from the revenue perspective, the item is incorrectly assigned in the open auction whenever the valuations lie in the heavily shaded region bounded by the 45° line and the curve $v_s = T(v_w)$.

As we shall show, it is typically the case in the high-bid auction that the strong buyer shades his bid more than the weak buyer, that is

$$b_s(v) < b_w(v), \; v \in (0, a_w) \; . \tag{2.11}$$

Then, for any valuation v_w of the weak buyer, he wins if $v_s < b_s^{-1}(b_w(v_w) \equiv Q(v_w)$. Given (2.11) the implicit mapping $Q(\cdot)$ lies above the $45°$ line. Thus misallocation occurs in the high-bid auction whenever the valuations lie either in the heavily or lightly shaded regions. It follows immediately that, under these conditions, expected revenue is higher in the open auction.

The assumption that $J_w(v) \le J_s(v)$ is, however, a strong one. Suppose, for example, that the weak buyer's valuation is distributed according to $F_w = v, \; v \in [0,1]$, while $F_s = v - a, \; v \in [a, 1+a]$. That is, the strong bidder's distribution is shifted to the right by a. It is readily confirmed that marginal revenue is respectively $J_w(v) = 2v - 1$, and $J_s(v) = 2v - 1 - a$. Thus the weak bidder has a higher marginal revenue.

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[11] In the special case $F_w(v) = wn + (1-w)F_s(v), \; v \in [0, a], \quad 1 - F_w(v) = (1-w)(1 - F_s(v))$ and so $J_w(v) = J_s(v)$. Thus the critical curve $v_s = T(v_w)$ is the $45°$ line. In this case the open auction is optimal from the revenue perspective.

[12] One way to do so is to use a modified Vickrey auction in which the weak buyer is given an "effective bid" of $T(v_w)$ if he submits a bid of v_w. See also McAfee and McMillan (1989).

Figure 2 depicts this case. In the open auction the misallocation is again the heavily shaded region. For the high-bid auction the misallocation occurs when the valuations lie between the $45°$ line and the curve $Q(v)$. Note that for high valuations the

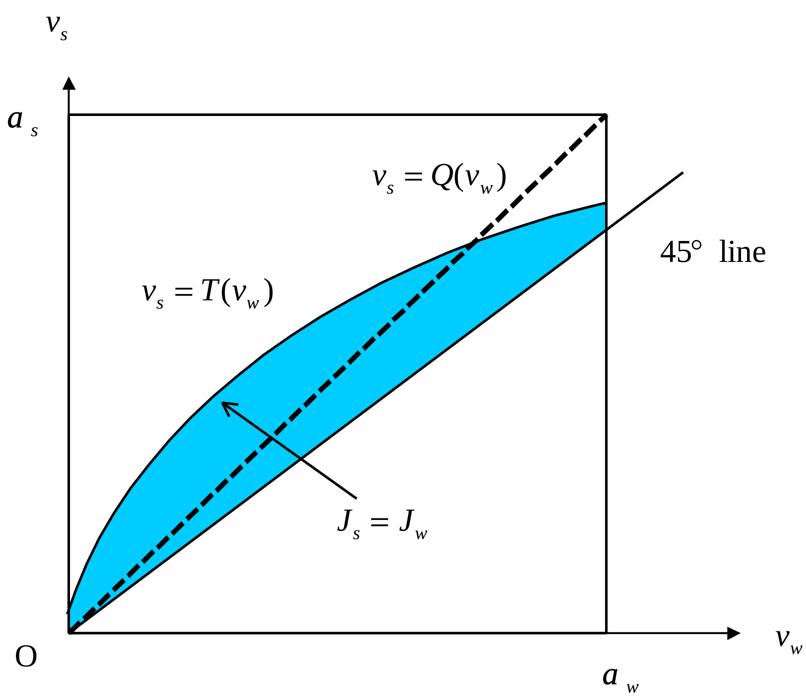

Figure 2: Shift in the mean

high bid auction allocates the item to the weak buyer too often while the reverse is true for low valuations. It follows that, from geometry alone, we cannot immediately rank the two auctions (although, in fact the high-bid auction turns out to be better, as we shall see in the next section.) Thus in the case in which $J_w(v) \leq J_s(v)$ need not hold, mechanism design considerations do not settle the matter of which auction generates more revenue.

We now characterize equilibrium in the high-bid auction. Rather than attempting to solve directly for the equilibrium bid functions, it is convenient to work with inverse bid functions. From Maskin and Riley (1995 and 1996), we know that, provided that

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b_s is not too much bigger than a_w[13], there are unique minimum and maximum winning bids b_* and b^* for which there exists a solution to the following pair of differential equations

(2.12)
$$\begin{cases} \dfrac{F_w'(f_w)}{F_w(f_w)} f_w'(b) = \dfrac{1}{f_s - b} \\ \\ \dfrac{F_s'(f_s)}{F_s(f_s)} f_s'(b) = \dfrac{1}{f_w - b} \end{cases}$$
(all functions f_i and f_i' are evaluated at b for all $b \in [b_*, b^*]$.)

satisfying the boundary conditions

(2.13)
$$F_i(f_i(b^*)) = 1,\ \ i \in \{s, w\},$$
$$b_w = b_s \ \Rightarrow \ b_* = f_s(b_s) = f_w(b_s) = b_s$$
$$b_w < b_s \ \Rightarrow \ b_* = Max \arg \underset{b}{Max}\{(b_s - b)F_w(b)\},\ \ f_w(b_*) = b_*$$

Moreover, this solution is unique and constitutes the (unique) equilibrium pair of <u>inverse bid</u> functions. That is,

(2.14) $\qquad f_i(b) = b_i^{-1}(b),\ \ i \in \{s, w\},$

where $b_i(\cdot)$ is buyer i's equilibrium bid as a function of his valuation.

To see that (2.14) holds, note that if the strong buyer bids according to f_s and the weak buyer submits a bid of *b*, then he wins if and only if $v_s < f_s(b)$. (It can be shown that the solution functions to (2.12) and (2.13) are strictly increasing and twice differentiable.) The weak buyer's expected surplus is therefore

(2.15) $\qquad (v_w - b)\Pr\{v_s < f_s(b)\} = F_s(f_s(b))(v_w - b)$

Taking logarithms and then differentiating by *b*, we obtain the first-order condition

$$\frac{F_s'(f_s)}{F_s(f_s)} f_s' = \frac{1}{v_w - b}$$, with boundary condition $y_i(b_i) = b_i$.

at $v_w = f_w(b)$, which is the second equation in (2.12).

---

[13] As we have seen, in Example 1, the strong bidder will bid a_w regardless of his valuation, if b_s is sufficiently bigger than a_w.

We characterize the equilibrium bid functions by comparing bidding in the asymmetric auction with that when buyers are symmetric, i.e. either both strong or both weak. Let $y_i(b)$ be the symmetric equilibrium inverse bid function. From (2.12)

(2.16) $$\frac{F_i'(y_i)}{F_i(y_i)}y_i' = \frac{1}{y_i - b}, \; i \in \{s, w\}$$, with boundary condition $y_i(b_i) = b_i$.

Rearranging we obtain

$$bF_i'(y_i)\frac{dy_i}{db} + F_i(y_i) = y_i F_i'(y_i)\frac{dy_i}{db}.$$

Let $\bar{b}_i(v)$ be the corresponding equilibrium bid function (i.e., the inverse of $y_i(b)$). Integrating the last equation, we have

(2.17) $$\bar{b}_i(v)F_i(v) = \int_{b_i}^{v} yF_i'(y)dy .$$

It follows immediately that in the equilibrium where both buyers are of type i $(= s \text{ or } w)$, a buyer's maximum possible bid is equal to the mean valuation

$$\bar{b}_i(a_i) = \int_{b_i}^{a_i} yF'(y)dy = E\{\tilde{v}_i\} \equiv m_i .$$

By Lemma 2.1 the CSD assumption implies (2.1), in which case

(2.18) $$m_w < m_s .$$

Note that we can rewrite (2.17) as

(2.19) $$\bar{b}_i(v) = v - \int_{b_i}^{v} \frac{F_i(x)}{F_i(v)}dx .$$

From (2.19) and CSD, we have,

(2.20) $$\bar{b}_w(v) \le \bar{b}_s(v) \text{ for all } v \in (b_s, a_w) .$$

Our first general results concern the buyers' equilibrium bid distributions. Define

(2.21) $$p_i(b) = F_i(f_i(b)), \; i \in \{s, w\} .$$

Also define

(2.22) $$H_i(\cdot) \equiv F_i^{-1}(\cdot), \; i \in \{s, w\} .$$

Given stochastic dominance, $H_s(p) > H_w(p)$ for all $p \in (0,1)$.

Substituting (2.21) and (2.22) into (2.12) we obtain

$$(2.23)\qquad \begin{cases} \dfrac{p_w'}{p_w} = \dfrac{1}{H_s(p_s) - b} \\ \dfrac{p_s'}{p_s} = \dfrac{1}{H_w(p_w) - b} \end{cases} .$$

Similarly, for the symmetric equilibria, define

$$(2.24)\qquad p_i(b) = F_i(y_i(b)),\ i \in \{s, w\} .$$

Then from (2.16)

$$(2.25)\qquad \begin{cases} \dfrac{p_w'}{p_w} = \dfrac{1}{H_w(p_w) - b} \\ \dfrac{p_s'}{p_s} = \dfrac{1}{H_s(p_s) - b} \end{cases} .$$

The results that follow are proved in the appendix. Proposition 2.3 part (ii) tells us that in the high-bid auction the equilibrium bid distribution of the strong buyer stochastically dominates that of the weak buyer. Part (iii) indicates that if a weak buyer faces a strong buyer rather than another weak buyer, he responds with a more aggressive bid distribution (in the sense of stochastic dominance). And symmetrically, Proposition 2.3 part (iv) establishes that if a strong buyer faces a weak buyer rather than another strong buyer, he will respond with a less aggressive bid distribution.

As for Proposition 2.5, part (ii) indicates that, in the asymmetric equilibrium, the strong bidder shades his bid further below his valuation than the weak bidder. Part (iii) tells us that if a weak bidder faces a strong bidder rather than a weak bidder he will bid more aggressively (closer to his valuation.[14]) Arguing symmetrically, part (iv) indicates

[14] It may seem as though first-order stochastic dominance would be enough to imply this result. Consider, however, the example in which $F_w(v) = 3v - v^2$ and $F_s(v) = 3v - 2v^2$. It follows from equations (A.4)-(A.7) in the appendix that $f_w(b) > y_w(b)$ for b sufficiently small.

that if a strong bidder faces a weak bidder rather than a strong bidder she will bid less aggressively.

As we shall see in Proposition 2.6, these results allow us to rank the high-bid and open auctions from perspective of each buyer.

Lemma 2.2: If (2.1) holds, $b_s = b_w \equiv b$, $F_s(b) = F_w(b) = 0$ and $\left.\frac{d}{dv}\frac{F_s(v)}{F_w(v)}\right|_{v=b} > 0$, then there exists $d > b$ such that for all $b \in [b, d]$

(i) $p_w(b) > p_s(b)$ (ii) $p_w(b) > p_s(b)$

(iii) $p_w(b) > p_w(b)$ (iv) $p_s(b) > p_s(b)$

Proposition 2.3: Comparison of equilibrium bid distributions

If CSD holds,[15] then

(i) $p_w(b) > p_s(b)$, for all $b \in (b_s, m_s)$ (ii) $p_w(b) > p_s(b)$, for all $b \in (b_*, b^*)$

(iii) $p_w(b) > p_s(b)$, for all $b \in (b_*, b^*)$ (iv) $p_w(b) > p_s(b)$, for all $b \in (b_*, m_s)$

Corollary 2.4: Given CSD, we have

$$m_w \le b^* \le m_s$$

with at least one strict inequality.

Proof: If $b^* < m_w$, then $p_s(b^*) > p_w(b^*)$ and so $p_s(b) > p_w(b)$ for b near b^*, a contradiction of part (iii) of Proposition 2.3. A similar contradiction follows from $m_s \le b^*$. The result then follows from (2.14).

Q.E.D.

15 Actually, Proposition 2.3 goes through provided that (2.1) holds and that, if $b_s = b_w = b$, the hypotheses of Lemma 2.2 hold.

Proposition 2.5: Characterization of equilibrium inverse bid functions

If CSD holds, then

(i) $y_w(b) \geq y_s(b)$, for all $b \in (b_s, m_s)$ and (ii) $f_s(b) > f_w(b)$, for all $b \in (b_*, b^*)$

(iii) $y_w(b) \geq f_w(b)$, for all $b \in (b_*, b^*)$ and (iv) $f_s(b) > y_s(b)$, for all $b \in (b_s, b^*)$

Using Proposition 2.5 we can derive the following comparative result.

Proposition 2.6: Ranking of the two auctions by the buyers

If CSD holds, the strong buyer strictly prefers the open auction while the weak buyer prefers the high-bid auction (where the preference is strict for all valuations exceeding the minimum bid b_* in the high-bid auction.[16]

Proof: For $i \in \{s, w\}$, let $U_i^H(v, F_s, F_w)$ be buyer *i*'s expected equilibrium surplus from the high-bid auction when his reservation price is *v* and the two buyers' reservation prices are distributed according to F_s and F_w respectively. Similarly, let $U_i^O(v, F_s, F_w)$ be buyer *i*'s expected surplus from the open auction. From part (iv) of Proposition 2.5 $F_s(f_s(b)) \equiv p_s(b) > p_s(b) \equiv F_s(y_s(b))$ for all $b \in (b_*, b^*)$. Thus for all $v \in (b_*, a_w]$

$$
\begin{aligned}
U_w^H(v, F_s, F_w) &= \underset{b}{Max}\, p_s(b)(v-b) \\
&\geq p_s(\bar{b}_s(v))(v - \bar{b}_s(v)) \\
&> p_s(\bar{b}_s(v))(v - \bar{b}_s(v)) \\
&= U_s^H(v, F_s, F_s) \\
&= U_s^O(v, F_s, F_s) \qquad \text{from symmetry and revenue equivalence} \\
&= \int_b^v (v-x)\, dF_s(x) \\
&= U_w^O(v, F_s, F_w)
\end{aligned}
$$

[16]Under the weaker assumption of first order stochastic dominance, it can be shown that the ranking by buyers continues to hold for all those buyers with sufficiently high valuations.

As for $v \in [b_w, b_*]$, $U_w^O(v, F_s, F_w) = 0$ because $v < b_s$ and so the weak buyer weakly prefers the high-bid auction.

From part (iii) of Proposition 2.5 $p_w(b) > p_w(b)$ for all $b \in (b_w, b^*)$. Hence for all $v \in (b_s, a_s]$,

$$
\begin{aligned}
U_s^O(v, F_s, F_w) = U_w^O(v, F_w, F_w) & \\
&= U_w^H(v, F_w, F_w) \quad \text{by the Revenue Equivalence Theorem} \\
&= \underset{b}{Max}\, p_w(b)(v-b) \\
&\geq p_w(b_s(v))(v - b_s(v)) \\
&> p_w(b_s(v))(v - b_s(v)) \\
&= U_s^H(v, F_s, F_w),
\end{aligned}
$$

where $b_s(\cdot)$ is the strong buyer's equilibrium bid function in the high-bid auction when the distributions are (F_s, F_w).

Q.E.D.

This last result seems to have been understood by buyers who perceived themselves to be "strong" before the recent spectrum auctions held by the F.C.C. There was a clear preference for some form of open auction rather than a sealed high-bid auction. Similarly, in the lumber tract auctions in the Pacific Northwest, the local "insiders" with neighboring tracts have forcefully (and successfully) lobbied for open auctions and the elimination of sealed high-bid auctions.

Our result also provides some insight into the logic behind Proposition 3.3 below. If the strong buyer is a much stronger bidder, then the weak buyer wins with only a small probability, so that his expected payoff is small in either auction. It follows that the difference in expected payoffs from the two auctions is small for the weak buyer. Total surplus is lower in the high-bid auction since the high-valuation buyer wins with probability less than 1. But again, if the strong buyer almost always wins, this loss in surplus is small. Then, roughly speaking, the lower expected payoff for the strong buyer in the high-bid auction is offset by an increase in payoff to the third party -- the seller. That is, expected revenue is higher in the high-bid auction.

3. REVENUE COMPARISONS -- GENERAL RESULTS

We now derive our general revenue comparisons. Throughout this section we shall invoke Proposition 2.5 (ii). Hence, the function $Q(v)$, implicitly defined by the equation

(5.1) $\quad f_s(b) \equiv Q(f_w(b))$,

is a mapping from $[b_*, a_w]$ onto $[b_s, a_s]$ with $Q(v) > v$ for all $v \in (b_*, a_w)$. Let us adopt the convention that $Q(v) = v$ for all $v \in [b_w, b_*)$.

For each valuation, the strong buyer bids lower than the weak buyer (i.e., $f_s(b) > f_w(b),\ b \in (b_*, b^*)$) in equilibrium. However, from Proposition 2.3 (ii), the distribution of his bids first-order stochastically dominates that of his opponent, that is, for all $b \in (b_*, b^*)$,

$$\Pr\{\tilde{b}_s \le b\} \equiv p_s(b) = F_s(f_s(b)) < F_w(f_w(b)) = p_w(b) \equiv \Pr\{\tilde{b}_w \le b\}$$

Thus, from the definition of $Q(v)$, it follows that

(5.2) $\quad F_w(v) > F_s(Q(v))$, for all $v \in (b_*, a_w)$.

We have the following general expressions for expected seller revenue in the two auctions. Proofs of these and later propositions can be found in the Appendix.

Lemma 3.1: Expected seller revenue from bidder i $(i = s, w)$ in the sealed high bid auction is R_i^H, where

(5.3) $$R_w^H = \int_{b_*}^{a_w} (1 - F_s(Q(v))) \frac{d}{dv}(v(1 - F_w(v)))dv + b_*(1 - F_w(b_*))$$

and

(5.4) $$R_s^H = \int_{b_*}^{a_w} (1 - F_s(Q(v)))Q(v)F_w'(v)dv + b_* F_w(b_*)(1 - F_s(b_s))$$

Lemma 3.2: Expected seller revenue from buyer i in the open auction can be expressed as R_i^O, where

(5.5) $$R_w^O = \int_{b_s}^{a_w} (1-F_s(v))\frac{d}{dv}(v(1-F_w(v))dv + b_s(1-F_w(b_s))$$

and

(5.6) $$R_s^O = \int_{b_w}^{a_w} (1-F_s(v))vdF_w(v) + b_w F_w(b_w)(1-F_s(b_w)).$$

We now turn to the revenue comparison. If $b_w = b_s = b$, the minimum bid, $b_* = b$. Then, from the above Lemmas, the difference in expected revenue from the two auctions is

$$D = R_s^H + R_w^H - R_s^O - R_w^O$$

$$= \int_{b_*}^{a_w} (1-F_s(Q(v)))[(1-F_w(v)) + (Q(v)-v)F_w{}'(v)]dv \; - \; \int_{b_*}^{a_w} (1-F_s(v))(1-F_w(v))dv$$

Rearranging we obtain

(5.7) $$D = \int_{b_*}^{a_w} [1-F_s(Q(v))(Q-v)F_w{}'(v) - \int_{b_*}^{a_w} (1-F_w(v))(F_s(Q)-F_s(v))]dv$$

$$= \int_{b_*}^{a_w} (Q-v)(1-F_s(Q))(1-F_w(v))C(v,Q)dv,$$

where

(5.8) $$C(v,Q) \equiv \frac{F_w{}'(v)}{1-F_w(v)} - \frac{F_s(Q)-F_s(v)}{(1-F_s(Q))(Q-v)}.$$

If $b_w < b_s$, then, from (2.8),

$$R_w^O < \int_{b_*}^{a_w} (1-F_s(v))\frac{d}{dv}(v(1-F_w(v)))dv + b_*(1-F_w(b_*)),$$

provided that $-vF_w{}'(v)+1-F_w(v) \geq 0$ for all $v \in (b_*, b_s)$. Thus we have an upper bound for the expected revenue from the weak buyer in the open auction. And, therefore, a lower bound for the difference between the high-bid and open auction's revenue is

$$\int_{b_*}^{a_w} (Q-v)(1-F_s(Q))(1-F_w(v))C(v,Q)dv - \int_{b_w}^{b_*} (1-F_s(v))vdF_w(v)$$

$$+b_* F_w(b_*)(1-F_s(b_s)) - b_w F_w(b_w)(1-F_s(b_w)),$$

which is no less than the right-hand side of (1.4).

The following revenue rankings are obtained by finding conditions sufficient to sign the function $C(Q,v)$.

Proposition 3.3: High-bid auction superior for distribution shifts.

Suppose that

$$\text{(5.9)} \qquad \text{(a) } F_w''(v) \geq 0, \text{ and (b) } \frac{d}{dv}\frac{F_w'(v)}{F_w(v)} < 0 \text{ on } [b_w, a_w].$$

Given $a < a_w - b_w$, suppose that, for all $v \in [b_w, a_w + a]$

$$\text{(5.10)} \qquad F_s(v) = \begin{cases} 0, & v < a + b_w \\ F_w(v-a), & v \geq a + b_w \end{cases}.$$

and that $-vF_w'(v) + F_w(v) \geq 0$ for all $v \in [b_w, b_w + a]$. Then the high-bid auction generates higher expected revenue than the open auction.

Notice that F_s in Proposition 3.3 is just a shift to the right (by a) of the distribution F_w. Thus the Proposition is a generalization of Example 1 in section 1. (In that example a=2.) We next turn to a generalization of Example 2.

Imagine "stretching out" distribution F_w by multiplying it by a scalar $l < 1$. Since $lF_w(a_w) < 1$, we have to say what happens for values $v > a_w$, in order to obtain a new c.d.f. F_s. Let $G(\cdot)$ be the "extension" of F_s to this range of values. (In Example 2, $G = \frac{v}{1+\Delta x}$.) We have

Proposition 3.4: High-bid auction superior for distribution "stretches"

Suppose that $F_w(v)$ satisfies $F_w(b_w) = 0$ and

$$\text{(5.11)} \qquad \frac{d}{dv}\frac{F_w'(v)}{F_w(v)} < 0 \text{ on } [b_w, a_w].$$

For $l \in (0,1)$, let the strong buyer have distribution $F_s(v)$, where $v \in [b_w, a_s]$ $(a_w < a_s)$, such that

(5.12) $$F_s(v) = \begin{cases} lF_w(v), & v \in [b_w, a_w] \\ G(v), & v \in (a_w, a_s] \end{cases} .$$

where $G(a_w) = l$, $G(a_s) = 1$, and

(3.13) $$F_w{}'(v) \geq G'(w) > 0, \text{ for all } v \in [b_w, a_w] \text{ and } w \in [a_w, a_s].$$

Then the high-bid auction generates more expected revenue than the open auction.

Recall that Example 3 was obtained by taking a one-point distribution and shifting probability mass to the zero point. We conclude the section by generalizing this example so that, at each point of a distribution $F_s(v)$, a fraction $1-q(v)$ of the density is shifted to the lower end- point of the distribution,

Proposition 3.5: **Open auction superior for shifts of probability mass to the lower end point.**

Suppose that the strong buyer's valuation $\tilde{v}_s$ is distributed according to $F_s(v),\ v \in [b, a_s]$ where $F_s(b) = 0$ and

(3.14) $$\frac{F_s{}'(v)}{1-F_s(v)} \text{ is increasing.}$$

Buyer 1's valuation $\tilde{v}_w$ is distributed so that, for all $v \in [b, a_s]$, its density at v is a fraction $q(v) \in (0,1)$ (with $q'(v) \geq 0$) of $F_s{}'(v)$ where the remaining density is reassigned to b. That is,

(3.15) $$F_w(v) = \int_b^v q(t)dF_s(t) + g$$

where

(3.16) $$g = \int_b^{a_s} (1-q(t))dF_s(t)$$

Then, the open auction generates higher revenue than the high-bid auction.

Finally, we note that, in combination, our three revenue comparisons, Propositions 3.3-3.5 embrace all first-order deviations from symmetry. Specifically, given $F_w(v)$, such deviations can be obtained by taking:

$$F_s(v) = lF_w(wv + a) + d$$

and considering $l \neq 0$, $w \neq 1$, $a \neq 0$, and $d \neq 0$. Decreases in a from 0 correspond to Proposition 3.3; variations in l (or similarly w), where $d = 0$, to Proposition 3.4; and decreases in l, where d increases correspondingly, to Proposition 3.5. Hence, by combining the effects described by the three propositions, one can examine any first-order asymmetry.

4. Numerical Results

Because the results in section 3 are purely qualitative, it seems worthwhile to complement them with some examples[17] to see how large the quantitative effects can be. The uniqueness results of Maskin and Riley (1995) provide a natural way to solve numerically for the unique equilibrium.

First we consider the case of uniform distributions and consider the effect of a simple shift of one distribution as in Proposition 3.3.

TABLE 4.1: Percentage gain in revenue under the high-bid auction over the open auction under simple shifts in the uniform distribution $F_i(v,a) = v - a_i$, $v \in [a_i, 1 + a_i]$

c.d.f. for weak buyer	$a_w = 0$	$a_w = 0$	$a_w = 0$	$a_w = 0$

[17]See Riley and Li (1996) for a much more complete numerical analysis, and also Marshall et al (1994). The program "BIDCOMP²" which computes inverse equilibrium bid functions and compares expected revenue is available for use by any interested reader. The FORTRAN source files are also available to researchers wishing to compile modified versions of the program.

c.d.f. for strong buyer	$a_s = 0$	$a_s = \frac{1}{4}$	$a_s = \frac{1}{2}$	$a_s = \frac{3}{4}$
Revenue in high-bid auction	.33	.456	.573	.682
Revenue in open auction	.33	.430	.479	.497
Percentage difference	0	6.1	19.6	37.2

We next consider the effect of a distributional stretch, as in Proposition 3.4. Suppose that F_w is uniform on [0,1] and that F_s is uniform on $[0, a_s]$.

TABLE 4.2: Percentage increase in expected revenue under the high-bid auction.

$F_i(v) = v / a_i, v \in [0, a_i]$

c.d.f. for weak buyer c.d.f. for strong buyer	$a_w = 1$ $a_s = 1$	$a_w = 1$ $a_s = 2$	$a_w = 1$ $a_s = 3$	$a_w = 1$ $a_s = 4$
Revenue in high-bid auction	.33	.459	.538	.514
Revenue in open auction	.33	.417	.444	.459
Percentage difference	0	10.1	21.2	29.7

Finally we turn to Proposition 3.6. Suppose that

(4.1) $$F_i(v) = \frac{v + h_i}{1 + h_i}, \; v \in [0,1]$$

Then, with probability $\frac{h_i}{1+h_i}$ buyer i has a valuation of zero and with probability $\frac{1}{1+h_i}$ his valuation is a draw from a uniform distribution with support [0,1].

Table 4.3 summarizes differences in expected revenue for different parameter values. Note that the high-bid auction does worse as the weak buyer's probability of not bidding increases. This is because the strong buyer's incentive to "lowball" also rises.

TABLE 4.3: Percentage increase in expected revenue under the high-bid auction

$$F_i(v) = \frac{v + h_i}{1 + h_i}, \; v \in [0,1]$$

c.d.f. for weak buyer c.d.f. for strong buyer	$h_w = 0$ $h_s = 0$	$h_w = 0$ $h_s = 1$	$h_w = 0$ $h_s = 2$	$h_w = 0$ $h_s = 3$
Revenue in high-bid auction	.33	.150	.086	.056
Revenue in open auction	.33	.167	.111	.083
Percentage difference	0	-9.7	-28.9	-48.5

5. Concluding Remarks

We noted in the introduction that art auctions are nearly always conducted openly whereas job-contract bidding normally is sealed. It is tempting to try to explain these regularities using our results on asymmetries. (Of course focusing on other violations of the revenue equivalence theorem's hypotheses might give rise to alternative explanations.)

One of the well-known peculiarities about people's tastes for art is that these are idiosyncratic. Idiosyncratic tastes mean that the market for any given item may be extremely thin. Suppose, for example, that a given buyer happens to be enthusiastic about a particular painting. He might reasonably conjecture that he is alone in his enthusiasm. But, if so, low-balling in a sealed high-bid auction becomes a good strategy. As we have seen, an open format helps safeguard the seller in such a situation, i.e., Proposition 3.5 applies.

If we take the case of defense contracting, by contrast, we find at least two bidders in serious competition on almost every occasion (see Alexander (1992)). Thus low-balling tends not to be a viable strategy. Rather, Propositions 3.3 and 3.4 are the relevant findings. We therefore expect the sealed high-bid auctions to work better than open auctions for defense procurement. Indeed, the evidence appears to bear this out. The U.S. defense Department has almost always used either "prototype" or "paper" competitions for awarding contracts. In the prototype mode, a competitor must produce an actual working airplane, or whatever, as its "bid." Since such prototypes are normally constructed in secret and are difficult to modify *ex post*, this competition resembles a high-bid auction. By contrast, in a paper auction, a would-be contractor need only produce the blueprints for the airplane. These are comparatively easy to modify after the contractor learns what its competitors have done, and so the competition more closely resembles an English auction. Alexander has found that the Defense Department has fared considerably better with the prototype than with paper competitions.

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APPENDIX

Suppose that $b_s = b_w = b$. Define

$$e_i(v) = \frac{(v-b)F_i'(v)}{F_i(v)}, \; i \in \{s, w\}.$$

We can therefore rewrite (2.12) as follows:

$$\text{(F.1)} \quad \begin{aligned} e_s(f_s)\, f_s'(b) &= \frac{f_s - b}{f_w - b} \\ e_w(f_w)\, f_w'(b) &= \frac{f_w - b}{f_s - b} \end{aligned} \quad .$$

Applying l'Hôpital's Rule we infer from the definition of $e_i(v)$ that

$$\text{(F.2)} \quad F_i(b) = 0 \Rightarrow \begin{cases} e_i(b) = 1 \\ e_i'(b) = \dfrac{F_i''(b)}{2F_i'(b)} \end{cases} \quad .$$

Differentiating $F_s(v)/F_w(v)$, we obtain

$$\frac{d}{dv}\frac{F_s(v)}{F_w(v)} = \frac{F_s'F_w - F_w'F_s}{F_w^2}, \text{ for } F_s > 0 \text{ and } F_w > 0.$$

Again applying l'Hôpital's Rule, we obtain

$$\text{(F.3)} \quad F_s(b) = F_w(b) = 0 \Rightarrow \left.\frac{d}{dv}\frac{F_s(v)}{F_w(v)}\right|_{v=b} = \frac{1}{2}\left[\frac{F_s''(b)}{F_s'(b)} - \frac{F_w''(b)}{F_w'(b)}\right]\frac{F_s'(b)}{F_w'(b)}.$$

If $\left.\dfrac{d}{dv}\dfrac{F_s(v)}{F_w(v)}\right|_{v=b} > 0$, the bracketed expression in (1.4) is strictly positive.

Lemma 2.2: If (2.1) holds, $b_s = b_w \equiv b$, $F_s(b) = F_w(b) = 0$ and $\left.\dfrac{d}{dv}\dfrac{F_s(v)}{F_w(v)}\right|_{v=b} > 0$, then

there exists $d > b$ such that for all $b \in [b, d]$

(i) $p_w(b) > p_s(b)$ (ii) $p_w(b) > p_s(b)$

(iii) $p_w(b) > p_w(b)$ (iv) $p_s(b) > p_s(b)$

<u>Proof</u>:

Since $F_s(b) = F_w(b) = 0$, $p_i(b) = p_i(b) = 0$, $i \in \{s,w\}$. Applying l'Hôpital's Rule to (2.3), we obtain

$$e_w(b)\, f_w{}'(b) = \frac{f_w{}'(b)}{f_s{}'(b) - 1} \quad \text{and} \quad e_s(b)\, f_s{}'(b) = \frac{f_s{}'(b)}{f_w{}'(b) - 1}.$$

It follows from (2.2) that

(F.4) $\qquad f_i{}'(b) = 2,\ i \in \{s,w\}$,

and, in symmetric equilibrium,

(F.5) $\qquad y_i{}'(b) = 2,\ i \in \{s,w\}$.

Next, taking the logarithm of (2.3) and differentiating, we find

$$\frac{e_w{}'}{e_w} f_w{}' + \frac{f_w{}''}{f_w{}'} = \frac{f_w{}'}{f_w - b} - \frac{f_s{}' - 1}{f_s - b}.$$

Substituting for $f_w{}'$ on the right hand side, using (2.3) yields

$$\frac{e_w{}'}{e_w} f_w{}' + \frac{f_w{}''}{f_w{}'} = \frac{1 + \frac{1}{e_w} - f_s{}'}{f_s - b}.$$

Applying l'Hôpital's Rule to this last expression when $b = b$ and making use of (2.8) and (2.17) gives us

$$\frac{F_w{}''(b)}{F_w{}'(b)} + \frac{1}{2} f_w{}''(b) = -\frac{F_w{}''(b)}{2F_w{}'(b)} - f_s{}''(b).$$

Rearranging, we obtain

$$\frac{1}{2} f_w{}''(b) + f_s{}''(b) = -\frac{3F_w{}''(b)}{2F_w{}'(b)}.$$

A symmetric argument implies that

$$\frac{1}{2} f_s{}''(b) + f_w{}''(b) = -\frac{3F_s{}''(b)}{2F_s{}'(b)}.$$

Solving these equations yields:

(F.6) $$f_i''(b) = [\frac{F_i''(b)}{F_i'(b)} - 2\frac{F_j''(b)}{F_j'(b)}], \quad j \neq i\,.$$

and, in symmetric equilibrium,

(F.7) $$y_i''(b) = -\frac{F_i''(b)}{F_i'(b)}\,.$$

By definition of $p_i(b)$ and $p_{i}(b)$ we have

(F.8) $$p_i'(b) = F_i'(f_i)\,f_i'(b) \quad \text{and} \quad p_{i}'(b) = F_i'(y_i)\,y_i'(b)\,.$$

It follows immediately from (2.21) and (2.22) that

(F.9) $$p_i'(b) = 2F_i'(b) = p_{i}'(b)\,.$$

From (2.12)

(F.10)
$$p_i''(b) = F_i''(b)(f_i'(b))^2 + F_i'(b)\,f_i''(b)$$
and
$$p_{i}''(b) = F_i''(b)(y_i'(b))^2 + F_i'(b)\,y_i''(b) \quad .$$

Substituting using (2.16) - (F.7), we obtain

(F.11) $$p_i''(b) = 4F_i''(b) + F_i'(b)[\frac{F_i''(b)}{F_i'(b)} - 2\frac{F_j''(b)}{F_j'(b)}], \quad j \neq i$$

and

(F.12) $$p_{i}''(b) = 4F_i''(b) + F_i'(b)[\frac{F_i''(b)}{F_i'(b)} - 2\frac{F_i''(b)}{F_i'(b)}] \quad .$$

By (F.9) $p_w'(b) = p_{w}'(b)$. From (F.3) and the hypotheses of the Proposition,

$\frac{F_s''(b)}{F_s'(b)} > \frac{F_w''(b)}{F_w'(b)}$, and so, from (F.3), (F.11), and (F.12), $p_{w}''(b) > p_w''(b)$. Hence (iii) holds. A symmetric argument establishes (iv) holds also. From (2.1), $F_w'(b) \geq F_s'(b)$. If the inequality is strict, then (F.8) and (F.9) imply that (i) and (ii) hold. If

$F_w'(b) = F_s'(b)$ then, since $\left.\dfrac{d}{dv}\dfrac{F_s(v)}{F_w(v)}\right|_{v=b} > 0$, it follows from (A.3) that

$F_s''(b) > F_w''(b)$, but this contradicts (2.1). Q.E.D.

Proposition 2.3: Comparison of equilibrium bid distributions

Given CSD,

(i) $p_w(b) > p_s(b)$, for all $b \in (b_s, m_s)$ (ii) $p_w(b) > p_s(b)$, for all $b \in (b_*, b^*)$

(iv) $p_w(b) > p_s(b)$, for all $b \in (b_*, b^*)$ (iv) $p_w(b) > p_s(b)$, for all $b \in (b_*, m_s)$

Proof:

To establish (i), note first that, from (2.18), $1 = p_w(b) > p_s(b)$ for all $b \in [m_w, m_s)$.

Contrary to (i), suppose that there is some $\hat{b} \in (b_s, m_w)$ such that $\dfrac{p_w(\hat{b})}{p_s(\hat{b})} = 1$. We shall

argue that $\dfrac{p_w(b)}{p_s(b)}$ is increasing at $\hat{b}$. Since $H_s(p) > H_w(p)$ for $p \in (0,1)$, $p_s \geq p_w$

implies that $H_s(p_s) \geq H_s(p_w) > H_w(p_w)$. Then from (2.23)

$$\frac{p_w'}{p_w} = \frac{1}{H_w(p_w) - b} > \frac{1}{H_s(p_s) - b} = \frac{p_s'}{p_s} \quad \text{at } b = \hat{b}.$$

Hence

$$\frac{d}{db}\frac{p_w}{p_s} = [\frac{p_w'}{p_w} - \frac{p_s'}{p_s}]\frac{p_w}{p_s} > 0, \text{ at } b = \hat{b}.$$

It follows that, for some $d > 0$,

(F.13) $p_s(b) > p_w(b)$ for all $b \in (\hat{b} - d, \hat{b})$.

Let d be the biggest value for which (F.13) holds. If $\hat{b} - d > b_s$, then

(F.14) $p_s(\hat{b} - d) = p_w(\hat{b} - d)$,

and from the above argument, $p_w(b) > p_s(b)$ for b $(> \hat{b} - d)$ near $\hat{b} - d$, a contradiction

of (F.13). Assume, therefore, $\hat{b} - d = b_s$. In the symmetric auction with two strong buyers, both buyers bid above b_s if and only if they have valuations exceeding b_s. Hence $p_w(b_s) \geq F_w(b_s) \geq F_s(b_s) = p_s(b_s)$ and so, from (2.6), (2.14) holds. We conclude that $F_w(b_s) = F_s(b_s)$ and so $b_s = b_w = b$. Thus, from Lemma 2.1, we must have $F_w(b) = F_s(b) = 0$. If $g > b$, then $F_s(v) = lF_w(v)$, $v \in [b, g]$, and so, from (2.16) $y_s(b) = y_w(b)$ for b in some neighborhood of b, and so $p_w(b) > p_s(b)$ in that neighborhood, a contradiction of (2.11). Hence $g = b$. Then, from part (i) of Lemma 2.2, $p_w(b) > p_s(b)$ for all b in a neighborhood of b, a contradiction of (F.13). We conclude that $\hat{b} \in (b, m_w)$ satisfying $\dfrac{p_w(\hat{b})}{p_s(\hat{b})} = 1$ does not exist, and so (i) is established.

To prove (ii), suppose that there exists $\hat{b} \in (b_*, b^*)$ such that $\dfrac{p_s(\hat{b})}{p_w(\hat{b})} = 1$. Since $H_s(p) > H_w(p)$, for $p \in (0,1)$, it follows from (2.23) that

$$\frac{p_w{}'}{p_w} = \frac{1}{H_s(p_s) - b} < \frac{1}{H_w(p_w) - b} = \frac{p_s{}'}{p_s} \quad \text{at } b = \hat{b}.$$

Hence $\dfrac{p_s(b)}{p_w(b)}$ is increasing at $\hat{b}$. Because the same argument applies to any $b^o > \hat{b}$ for which $\dfrac{p_s(b^o)}{p_w(b^o)} = 1$, $p_w(b) < p_s(b)$ for all $b \in (\hat{b}, b^*)$. But from (2.13) $p_w(b^*) = p_s(b^*)$, and so $\hat{b}$ cannot exist. Hence (ii) holds unless, for all $b \in (b_*, b^*)$, $p_w(b) < p_s(b)$, which would conflict with Lemma 2.2 (ii).

To prove (iii), suppose that for some $\hat{b} \in (b_*, b^*)$, $\dfrac{p_w(\hat{b})}{p_s(\hat{b})} = 1$. If $m_w \leq \hat{b}$, then $1 = p_w(\hat{b}) = p_s(\hat{b})$ and $1 > p_s(\hat{b})$, a contradiction. Hence, assume that $\hat{b} < m_w$. Since (ii) holds, $p_w(\hat{b}) = p_s(\hat{b}) < p_w(\hat{b})$. Thus

$$\text{(F.15)} \qquad \frac{p_w{}'}{p_w} = \frac{1}{H_w(p_w(b)) - b} > \frac{1}{H_w(p_w(b)) - b} = \frac{p_s{}'}{p_s} \quad \text{at } b = \hat{b}.$$

The rest of the proof parallels that of (i) but uses part (iii) of Lemma 2.2 instead of part (i). A symmetrical argument establishes (iv).

Q.E.D.

Proposition 2.5: Characterization of equilibrium inverse bid functions

If CSD holds, then

(i) $y_w(b) \geq y_s(b)$, for all $b \in (b_s, m_s)$ and (ii) $f_s(b) > f_w(b)$, for all $b \in (b_*, b^*)$

(iii) $y_w(b) \geq f_w(b)$, for all $b \in (b_*, b^*)$ and (iv) $f_s(b) > y_s(b)$, for all $b \in (b_s, b^*)$

Proof: For $b \in [m_w, m_s]$, $1 = y_w(b) > y_s(b)$. For, $b \in (b_s, m_w)$, CSD implies that (i) follows immediately from (2.20). To demonstrate (ii), we first argue that (ii) holds in a punctured neighborhood of b^*. If $a_w < a_s$, this is immediate because then $a_s = f_s(b^*) > f_w(b^*) = a_w$. If $a_w = a_s$, then $f_s(b^*) = f_w(b^*)$, and so, from (2.12),

(F.16) $$\frac{F_w'(f_w)}{F_w(f_w)} f_w' = \frac{1}{f_s - b} = \frac{1}{f_w - b} = \frac{F_s'(f_s)}{F_s(f_s)} f_s', \text{ at } b = b^*.$$

Given CSD it follows that $f_s'(b) < f_w'(b)$ and so (ii) holds in a punctured neighborhood of b^*, as claimed.

Suppose that there exists $\hat{b} \in (b_*, b^*)$, such that $\dfrac{f_w(\hat{b})}{f_s(\hat{b})} = 1$. Then (F.16) holds at $b = \hat{b}$. Hence, Assumption CSD implies that $\dfrac{f_w(b)}{f_s(b)} \geq 1$ for all $b \in (\hat{b}, b^*)$, a contradiction of our finding above. Thus $\dfrac{f_w(b)}{f_s(b)} < 1$ for all $b \in (b_*, b^*)$.

To prove (iv), note first that, by Corollary 2.4, $b^* \leq m_s$. Hence $f_s(b^*) \geq y_s(b^*)$. From (2.13), (2.16) and part (ii) of this Proposition, for any $b \in (b_s, b^*)$ such that $f_s(b) \leq y_s(b)$

$$\frac{F_s'(f_s)}{F_s(f_s)} f_s' = \frac{1}{f_w - b} > \frac{1}{f_s - b} \geq \frac{1}{y_s - b} = \frac{F_s'(y_s)}{F_s(y_s)} y_s'.$$

Hence,

(F.17) $$f_s(b) \le y_s(b) \Rightarrow \frac{d}{db}\frac{F_s(f_s(b))}{F_s(y_s(b))} > 0.$$

For some $\hat{q} \le 1$, suppose that there exists $\hat{b} \in (b_s, b^*)$ satisfying

(F.18) $$\frac{F_s(f_s(\hat{b}))}{F_s(y_s(\hat{b}))} = \hat{q}$$

By (F.17), $\dfrac{F_s(f_s(b))}{F_s(y_s(b))}$ is strictly increasing at $b = \hat{b}$.

Hence

(F.19) $$f_s(b) < y_s(b) \text{ and } \frac{d}{db}\frac{F_s(f_s(b))}{F_s(y_s(b))} > 0, \text{ for all } b \in [b_s, \hat{b}).$$

But $y_s(b_s) = b_s$ and so $f_s(b_s) \ge y_s(b_s)$, a contradiction of (F.19). We conclude that $\hat{b}$ cannot exist, and so (iv) holds after all. A symmetric argument establishes that (iii) holds also.

Q.E.D.

Lemma 3.1: Expected seller revenue from bidder $i \in \{s, w\}$ in the sealed high bid auction is R_i^H, where

(3.3) $$R_w^H = \int_{b_*}^{a_w} (1 - F_s(Q(v)))\frac{d}{dv}(v(1 - F_w(v)))dv + b_*(1 - F_w(b_*))$$

and

(3.4) $$R_s^H = \int_{b_*}^{a_w} (1 - F_s(Q(v)))Q(v)F_w{}'(v)dv + b_* F_w(b_*)(1 - F_s(b_s))$$

<u>Proof:</u> The weak buyer's expected payment if he bids $b \ge b_*$ is $bF_s(f_s(b))$. Since his equilibrium bid distribution has c.d.f. $F_w(f_w(\cdot))$, the expectation over all bids is

$$R_w^H = \int_{b_*}^{b^*} bF_s(f_s(b))dF_w(f_w(b)) \ .$$

which, after integration by parts, can be rewritten as

$$R_w^H = b_* F_s(b_*)(1 - F_w(b_*)) + \int_{b_*}^{b^*} (1 - F_w(f_w(b))) \frac{d}{db} b F_s(b) db .$$

From (2.12) $\frac{d}{db} b F_s(b) = f_w F_s'(f_s) \frac{d f_s}{db}$.

Substituting this expression into the integral we then obtain

$$\text{(F.20)} \qquad R_w^H = b_* F_s(b_s)(1 - F_w(b_*)) + \int_{b_*}^{b_*} (1 - F_w(f_w(b))) f_w(b) F_s'(f_s(b)) \frac{d f_s}{db} db .$$

Since $f_s(b) \equiv Q(f_w(b))$, we can rewrite this expression as

$$R_w^H = b_* F_s(b_s)(1 - F_w(b_*)) + \int_{b_*}^{a_w} (1 - F_w(v)) v F_s'(Q(v)) Q'(v) dv .$$

Integrating again by parts, we obtain

$$R_w^H = b_*(1 - F_w(b_*)) + \int_{b_*}^{a_w} (1 - F_s(Q(v))) \frac{d}{dv} (v(1 - F_w(v))) dv .$$

Appealing to symmetry, we infer from (F.20) that

$$R_s^H = b_* F_w(b_*)(1 - F_s(b_s)) + \int_{b_*}^{b^*} (1 - F_s(f_s(b)) f_s(b) F_w'(f_w(b)) \frac{d f_w}{db} db .$$

Again using the fact that $f_s(b) \equiv Q(f_w(b))$, we then obtain

$$R_s^H = b_* F_w(b_*)(1 - F_s(b_s)) + \int_{b_*}^{a_w} (1 - F_s(Q(v)) Q(v) F_w'(v) dv .$$

Q.E.D.

Lemma 3.2: Expected seller revenue from buyer $i \in \{s, w\}$ in the open auction can be expressed as R_i^O, where

$$\text{(3.5)} \quad R_w^O = \int_{b_s}^{a_w} (1 - F_s(v)) \frac{d}{dv} (v(1 - F_w(v)) dv + b_s(1 - F_w(b_s))$$

and

$$\text{(3.6)} \quad R_s^O = \int_{b_w}^{a_w} (1 - F_s(v)) v dF_w(v) + b_w F_w(b_w)(1 - F_s(b_w)).$$

Proof: If the weak buyer has a valuation $v_w > b_s$ his expected payment is

$b_s F_s(b_s) + \int_{b_s}^{v_w} b dF_s(b)$. Taking the expectation over v_w, the expected revenue from the weak buyer is

$$R_w^O = \int_{b_s}^{a_w} (b_s F_s(b_s) + \int_{b_s}^{v_w} b dF_s(b)) dF_w(v_w) = b_s F_s(b_s)(1 - F_w(b_s)) + \int_{b_s}^{a_w} v F_s{}'(v)(1 - F_w(v)) dv .$$

Integrating by parts once more, we obtain

$$R_w^O = -\int_{b_s}^{a_w} F_s(v) d(v(1 - F_w(v))) ,$$

which can be rewritten as (3.5). Appealing to symmetry, we also find

$$R_s^O = b_w F_w(b_w)(1 - F_s(b_w)) + \int_{b_w}^{a_w} v F_w{}'(v)(1 - F_s(v)) dv .$$

Q.E.D.

Proposition 3.3: High-bid auction superior for distribution shifts.

Suppose that CSD holds and, in addition

(3.9) (a) $F_w{}''(v) \geq 0$, and (b) $\dfrac{d}{dv} \dfrac{F_w{}'(v)}{F_w(v)} < 0$, on $[b_w, a_w]$.

Given $a < b_w - a_w$, suppose that, for all $v \in [b_w, a_w + a]$

(3.10) $$F_s(v) = \begin{cases} 0, & v < a + b_w \\ F_w(v - a), & v \geq a + b_w \end{cases}$$

and that $-F_w{}'(v) + 1 - F_w(v) \geq 0$ for all $v \in [b_w, b_w + a]$. Then the high-bid auction generates higher expected revenue than does the open auction.

Proof: From (3.9b) and (3.10), Proposition 2.5 applies and so, from part (ii), $Q > v$ for all $v \in [b_*, a_w]$. From (3.7)-(3.9), we need only show that $C(v, Q)$ is positive. Since F_w is convex (from (3.9)), so is F_s. Therefore

$$\frac{F_s(Q) - F_s(v)}{Q - v} \leq F_s{}'(Q) \equiv F_w{}'(Q - a) .$$

Thus

$$C(v,Q) \geq \frac{F_w{}'(v)}{1-F_w(v)} - \frac{F_w{}'(Q-a)}{1-F_w(Q-a)}\,.$$

But $\frac{d}{dv}[\frac{F_w{}'(v)}{1-F_w(v)}] = [\frac{F_w{}''(v)}{F_w{}'(v)} + \frac{F_w{}'(v)}{1-F_w(v)}]\frac{F_w{}'(v)}{1-F_w(v)} > 0$, since F_w is convex.

Moreover, from (3.2), $F_w(v) > F_s(Q(v)) = F_w(Q(v)-a)$. Thus $Q-a<v$ and so $C(v,Q)$ is indeed positive.

Q.E.D.

Proposition 3.4: High-bid auction superior for distribution "stretches"

Suppose that $F_w(v)$ satisfies $F_w(b_w)=0$ and

(3.11) $$\frac{d}{dv}\frac{F_w{}'(v)}{F_w(v)} < 0 \text{ on } [b_w, a_w].$$

For $l \in (0,1)$, let the strong buyer have distribution $F_s(v)$, where $v \in [b, a_s]$ $(a_w < a_s)$, such that

(3.12) $$F_s(v) = \begin{cases} lF_w(v), & v \in [b_w, a_w] \\ G(v), & v \in (a_w, a_s] \end{cases}$$

where $G(a_w) = l$, $G(a_s) = 1$, and

(3.13) $$F_w{}'(v) \geq G'(w) > 0, \text{ for all } v \in [b_w, a_w] \text{ and } w \in [a_w, a_s].$$

Then the high-bid auction generates more expected revenue than the open auction.

Proof: From (3.12) and (3.13), CSD holds. Hence, from Proposition 2.5 (ii), $Q(v) > v$, for all $v \in (b_w, a_w)$. Now, (3.7) and the fact (from (3.2)) that $F_s(Q(v)) < F_w(v)$ implies that the difference in revenue, D, satisfies

$$D > \int_{b_w}^{a_w} (Q(v)-v)(1-F_w(v))[F_w{}'(v) - \frac{F_s(Q(v)) - F_s(v)}{Q-v}]dv\,.$$

Since $Q(v) > v$, there exists some $\hat{v} \in [v, Q(v)]$ such that

$$\frac{F_s(Q(v) - F_s(v)}{Q-v} = F_s{}'(\hat{v})\,.$$

Then D is positive if $F_w'(v) \geq F_s'(\hat{v})$. For the case in which $\hat{v} > a_w$, this follows immediately from (3.13). Thus suppose that $\hat{v} < a_w$. By (3.11) and (3.12)

(F.21) $$\frac{F_s'(\hat{v})}{F_s(\hat{v})} = \frac{F_w'(\hat{v})}{F_w(\hat{v})} \leq \frac{F_w'(v)}{F_w(v)}.$$

Also since $\hat{v} \leq Q(v)$, $F_s(\hat{v}) \leq F_s(Q) < F_w(v)$ (where the last inequality follows from (3.2)). Then, from (F.21) $F_w'(v) \geq F_s'(\hat{v})$.

Q.E.D.

Proposition 3.5: **Open auction superior for shifts to of probability mass to the lower end point.**

Suppose that the strong buyer's valuation $\tilde{v}_s$ is distributed according to $F_s(v)$, $v \in [b, a_s]$ where $F_s(b) = 0$ and

(3.14) $$\frac{F_s'(v)}{1 - F_s(v)} \text{ is increasing.}$$

Buyer 1's valuation $\tilde{v}_w$ is distributed so that, for all $v \in [b, a_s]$, its density at v is a fraction $q(v) \in (0,1)$ (with $q'(v) \geq 0$) of $F_s'(v)$ where the remaining density is reassigned to b. That is,

(3.15) $$F_w(v) = \int_b^v q(t) dF_s(v) + g$$

where

(3.16) $$g = \int_b^{a_s} (1 - q(t)) dF_s(t).$$

Then, the open auction generates higher revenue than the high-bid auction.

<u>Proof</u>: We first establish that CSD holds, i.e.,

(F.22) $$\frac{F_s'(v)}{F_s(v)} > \frac{F_w'(v)}{F_w(v)} \quad \text{for all } v \in (b, a_w).$$

But from (3.15), (F.22) can be rewritten as

$$\frac{F_s'(v)}{F_s(v)} > \frac{q(v)F_s'(v)}{\int_b^v q(t)dF_s(t) + g}$$

i.e.,

(F.23) $$\int_b^v q(t)dF_s(t) + g > q(v)F_s(v).$$

And from (3.16), the left-hand side of (F.23) can be rewritten as

$$F_s(v) + \int_v^{a_s} (1 - q(t))dF_s(t),$$

and so (F.23) indeed holds. Thus, from (3.1) $Q(v) > v$, $v \in (b, a_s)$, i.e. Q appears as depicted in Figure 1.

We next show that, under our assumptions the weak buyer has a lower marginal revenue. Given the discussion at the beginning of section 2, this will imply that we have the case depicted in Figure 1 and so revenue is higher in the open auction. From (3.16) and (3.17).

$$\frac{F_w'(v)}{1 - F_w(v)} = \frac{q(v)F_s'(v)}{\int\limits_v^{a_s} q(v)F_s'(v)dv}.$$

Then

$$\frac{F_w'(v)}{1 - F_w(v)} - \frac{F_s'(v)}{1 - F_s(v)} = \frac{q(v)F_s'(v)}{\int\limits_v^{a_s} q(v)F_s'(v)dv} - \frac{F_s'(v)}{\int\limits_v^{a_s} F_s'(v)dv}$$

$$= \frac{F_s'(v)}{\int\limits_v^{a_s} q(v)F_s'(v)dv}[q(v) - \frac{\int\limits_v^{a_s} q(v)F_s'(v)dv}{\int\limits_v^{a_s} F_s'(v)dv}]$$

≤ 0, since $q(v)$ is non-decreasing.

Since $J_i(v) = v - \frac{1-F_i}{F_i'}$, it follows immediately that $J_w \leq J_s$.