CHARACTERIZING ASYMMETRIC AUCTIONS

by

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In a recent series of papers, Maskin and Riley have examined equilibrium bidding when buyer's valuations are drawn from different distributions. For the case of independent private valuations they provide a quite general existence result. When preferences are identical they also establish uniqueness.[1] A third paper argues that it will typically be the case that revenue is higher in the sealed high bid auction than in the open ascending bid (or sealed Vickrey auction.)

Given these conclusions, the natural next question is whether it is possible to provide some strong characterization results as well. The main question addressed in this paper is what happens to equilibrium bidding strategies of one or more buyers' bid distributions draws his valuation from a more favorable distribution.

A quick answer might go as follows. If buyer 1 draws from a more favorable distribution, but bids exactly as before, his bid distribution shifts up. Other buyers, facing a flatter bid distribution than before, will respond by making higher bids. But then buyer 1 will end up facing bid distributions which have shifted up. He too will then make an upward adjustment in the amount that he bids for an given valuation of the object. Thus in the new equilibrium, all buyers will make higher bids.

As it turns out, this simple argument is not correct. Indeed for low enough valuations, buyer 1 always bids lower. Despite this result, it is possible to draw some relatively strong conclusions about the effect on the equilibrium bid distributions.

These remarks build on the work of Lebrun (1995) which addresses the same issues under the simplifying assumption that there are just two groups of bidders. His main result is that if the distribution of one group shifts to the right, then, in the new equilibrium, the members of the other group make higher bids. Moreover, the equilibrium bid distributions of both groups of buyers shift to the right. As we shall see, significant generalizations are possible.

1. The Model

There are n buyers. Buyer i has a valuation v_i which is a draw from a distribution with continuously differentiable c.d.f. $F_i(\cdot), i = 1,\ldots,n$. For each buyer the support of the distribution is $[0,m]$. There is a single item for sale. The seller announces a reserve price $\alpha \geq 0$, and indicates that he will sell the item to the bidder who submits the highest bid, at his bid price.

Let $b_i(v_i), i = 1,\ldots,n$ be equilibrium bid functions. From Maskin and Riley (1995) we know that a buyer with a valuation equal to the seller's reserve price α bids α. Moreover each buyer's bid function is strictly increasing with the same maximum bid β. Then each buyer has an inverse equilibrium bid function $\phi_i(b) = b_i^{-1}(b), i =,\ldots,n$.

[1] Lebrun has also examined these issues under stronger assumptions.

Proposition 1: Necessary conditions for an equilibrium

The inverse equilibrium bid functions $\phi_i(b), i = 1,\dots,n$ satisfy the following system of differential equations.

$$(1) \qquad \sum_{j\neq t} \frac{F_i'(\phi_j(b))}{F_j(\phi_j(b))}\phi_j'(b) = \frac{1}{\phi_t(b) - b}, t = 1,\dots,n$$

Proof: Suppose all buyers other than t have bid functions with inverse functions satisfying (1). If buyer t bids b his expected payoff is

$$U_t(b,v) = \times_{j\neq t} F_j(\phi_j(b))(v-b).$$

Taking the logarithm and then differentiating, we obtain

$$\frac{\partial U_t / \partial b}{U_t} = \sum_{j\neq t} \frac{F_j'}{F_j}\phi_j' - \frac{1}{v-b}.$$

Substituting from (1), this can be rewritten as:

$$\frac{\partial U_t / \partial b}{U_t} = \frac{1}{\phi_t(b) - b} - \frac{1}{v-b}.$$

This is positive if and only if $b < b_t(v)$, that is $v < \phi_t(b)$. Thus buyer t's bid function is indeed a best response.

Q.E.D.

It is helpful to define $P_j(b) = F_i(\phi_j(b)), j = 1,..,n$. This is the equilibrium probability that buyer j bids b or less. We also define $p_j = \ln F_i(\phi_j(b)), j = 1,..,n.$ Given the seller's reserve, α, the minimum valuation with a chance of winning is α. Then $p_j(b):[\alpha,\beta] \to [\ln F_j(\alpha), 0]$.[2] We also define the inverse function

$$(2) \qquad H_j(p) = F_j^{-1}(e^p), j = 1,\dots,n.$$

Then the system of differential equations (1) can be rewritten as follows.

[2] A strictly positive reserve price simplifies the analysis somewhat in that it ensures that the mapping is onto a range which is bounded from below. However all the results of this paper hold as well when the reserve price is zero.

(3) $\sum_{j \neq t} p_j'(b) = \frac{1}{H_t(p_t(b)) - b}, t = 1,\dots,n, b \in [\alpha, \beta].$

From Maskin and Riley (1995b) we know that there is a unique solution to (3) satisfying the endpoint conditions. That is, here is a unique maximum bid β such that $p_j(\beta) = 0, j = 1,\dots,n$ and $p_j(\alpha) = \ln F_j(\alpha), j = 1,\dots,n.$

The central issue addressed here, is how the equilibrium bid distributions change if one buyer draws instead from a more favorable distribution. A natural assumption is that the new distributions should stochastically dominate the old one. Suppose that it is the distributions of buyers *1,...,k* which change. Let the new cumulative distribution functions be $\overline{F}_j(v), j = 1,\dots,k.$.

Assumption 1: The new distributions stochastically dominate the old distributions.

For each *j=1,...,k.*

$$\overline{F}_j(v) < F_j(v), \alpha < v < m$$

and either

$$\overline{F}_j(m) < F_j(m) = 1 \text{ or } \overline{F}_j(m) = 1 \text{ and } \overline{F}_j'(m) > F_j'(m).$$

Actually, for some of our results, we will need a somewhat stronger assumption.

Assumption 2: Conditional stochastic Dominance

$$\frac{\overline{F}_j'(v)}{\overline{F}_j(v)} > \frac{F_j'(v)}{F_j(v)}, \text{ for all } j = 1,\dots,k \text{ and } v < m.$$

As Maskin and Riley (1995b) show, Assumption 2 holds as a weak inequality if and only, for all v and all
x < v,

$$\frac{\overline{F}_j(x)}{\overline{F}_j(v)} \le \frac{F_j(x)}{F_j(v)}, j=1,\dots,k$$

Let buyer *j*'s new bid distribution of bids have support $[\alpha, \overline{\beta}]$, and c.d.f. $Q_j(b), j = 1,\dots,n,$ and define $q_j(b) = \ln Q_j(b)$. Corresponding to (2) we define

(2') $\overline{H}_j(q) = \overline{F}_j^{-1}(e^q), j = 1,\dots,n.$

Then, from (3), the new equilibrium must satisfy:

$$\text{(3')} \quad \sum_{j \neq t} q_j'(b) = \frac{1}{\overline{H}_t(q_t(b)) - b}, t = 1,\dots,n, b \in [\alpha, \overline{\beta}].$$

Writing the differential equation system (3) in matrix form, we have

$$\text{(4)} \quad \begin{bmatrix} 0 & 1 & 1 & . & . & 1 \\ 1 & 0 & 1 & . & . & 1 \\ 1. & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \\ . \\ . \\ . \\ p_n' \end{bmatrix} = \begin{bmatrix} \frac{1}{H_1(p_1) - b} \\ . \\ . \\ . \\ \frac{1}{H_n(p_n) - b} \end{bmatrix}$$

Inverting we obtain

$$\text{(5)} \quad \begin{bmatrix} p_1' \\ p_2' \\ . \\ . \\ . \\ p_n' \end{bmatrix} = \frac{1}{n-1} \begin{bmatrix} -(n-2) & 1 & 1 & . & . & 1 \\ 1 & -(n-2) & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & -(n-2) \end{bmatrix} \begin{bmatrix} \frac{1}{H_1(p_1) - b} \\ . \\ . \\ . \\ \frac{1}{H_n(p_n) - b} \end{bmatrix}$$

Differentiating by b,

$$\text{(6)} \quad \begin{bmatrix} p_1'' \\ p_2'' \\ . \\ . \\ . \\ p_n'' \end{bmatrix} = \frac{1}{n-1} \begin{bmatrix} -(n-2) & 1 & 1 & . & . & 1 \\ 1 & -(n-2) & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & -(n-2) \end{bmatrix} \begin{bmatrix} \frac{-(H_1'(p_1) - 1)}{(H_1(p_1) - b)^2} \\ . \\ . \\ . \\ \frac{-(H_n'(p_n) - 1)}{H_n(p_n) - b} \end{bmatrix}$$

We have corresponding expressions for $q'(b) = (q_1'(b),\dots,q_n'(b))$ and $q''(b)$.

2. Results

The following three Lemmas are proved in the appendix.

Lemma 1: If the stochastic dominance assumption holds,

$H_j(p) < \overline{H}_j(p)$ for all *p<0*. and if $H_j(0) = \overline{H}_j(0) = m$, then $H_j'(0) > \overline{H}_j'(0)$.

If, the conditional stochastic dominance assumption holds, then

$$H_j(p) = \overline{H}_j(q) \Rightarrow H_j'(p) > \overline{H}_j'(q)$$

Lemma 2: Suppose that for some $b = \overline{b}$ and some buyer $t > k$, a solution to the system of differential equations satisfies:

$$\text{(i)}\quad q_t(b) - p_t(b) > 0 \quad \text{and} \quad \text{(ii)}\quad \sum_{j \neq t} q_j(b) - p_j(b) > \delta$$

Then (ii) holds for all $b < \overline{b}$.

Lemma 3: Suppose that only buyer 1's distribution shifts *(k=1)*. If Assumption 2 holds and for some $b = \overline{b}$, and some buyer $t \neq 1$, a solution to the system of differential equations satisfies:

$$\text{(i)}\quad \overline{H}_1(q_1(b)) - H_1(p_1(b)) > 0 \text{ and} \quad \text{(ii)}\quad q_t(b) - p_t(b) > 0, t \neq 1$$

Then (ii) holds for all $b < \overline{b}$.

We now turn to our main results.

Proposition 2: Suppose that the distribution of valuations of buyers *j=1,...k* shifts so that the new distributions stochastically dominate the old. Then, for all sufficiently large bids, the new equilibrium bid distribution of each of the *n* buyers stochastically dominates his old equilibrium bid distribution..

Proof: Let $\overline{\beta}$ be the maximum bid under the new distribution. Under our assumptions we know that the equilibrium bid distributions must satisfy

$$(7)\quad P_j(\beta) = Q_j(\overline{\beta}) = 1, j = 1,\ldots,n, \; P_j(\alpha) = F_j(\alpha), \text{ and } Q_j(\alpha) = \overline{F}_j(\alpha)$$

From (7), the logarithm of the bid distributions must satisfy the endpoint conditions:

$$(8)\quad \begin{array}{ll} p_j(\beta) = q_j(\overline{\beta}) = 0, j = 1,\ldots,n. & p_j(\alpha) = q_j(\alpha), j \geq k, \\ \text{and} & \\ p_j(\alpha) > q_j(\alpha), j = 1,\ldots,k. & \end{array}$$

Case A: Suppose $\overline{\beta} < \beta$. Then in some left neighborhood of $\overline{\beta}, q(b) > p(b)$. Then the conditions of Lemma 2 hold and so $\sum_{j \neq t} q_j(b) - p_j(b) > \delta$, for all $b < \overline{\beta}$. But this contradicts (8).

Case B: Suppose then that $\overline{\beta} = \beta$. and $\overline{m} > m$. Then $p(\beta) = q(\beta)$. Since $\overline{m} > m$, $H_1(p_1(\beta)) = m < \overline{m} = \overline{H}_1(q_1(\beta))$. It follows from (5) and the corresponding expression for $q(b)$, that for all $t \neq 1$,

$$p_t'(\beta) - q_t'(\beta) = \frac{1}{n-1}\left(\frac{1}{H_1(p_1) - \beta} - \frac{1}{\overline{H}_1(q_1) - \beta}\right) > 0.$$

Then in some left neighborhood of β, $q_t(b) - p_t(b) > 0$. Since at $\beta, \overline{H}_1(0) > H_1(0)$, the conditions of Lemma 3 are satisfied. But then we cannot also satisfy the lower end-point condition.

Case C: Finally, suppose that $\overline{\beta} = \beta$ and $\overline{m} = m$. From (5) and the corresponding expression for $q(b)$, $p(b)$ and $q(b)$ have the same derivative at $b = \beta$. From Lemma 1 it follows that in some left neighborhood of $b, \overline{H}_1(q_1(b)) > H_1(p_1(b))$. From (6), for all $t \neq 1$,

$$p_t''(\beta) - q_t''(\beta) = \frac{1}{n-1} \frac{(\overline{H}_1'(0) - H_1'(0))}{(m-\beta)^2}$$

$$< 0, \text{ by Lemma 1.}$$

Since the first derivatives are the same at β, it follows that $p_t'(b) > q_t'(b)$ in some left neighborhood of β. Then since $p_t(\beta) = q_t(\beta) = 0$, it follows that $q_t(b) - p_t(b) > 0$ in some left neighborhood of β. The argument then proceeds exactly as in Case B.

The only remaining possibility is that the new maximum bid $\overline{\beta}$ is strictly larger. But then, $q_j(\beta) < 1 = p_j(\beta), j = 1, \ldots, n$.

Q.E.D.

Proposition 3: Suppose that the new distributions of valuations for buyers *1,...,k* stochastically dominate the old. Then the new bid distributions faced by the remaining *n-k* buyers stochastically dominate the old bid distributions.

Proof: The c.d.f. of bids initially faced by buyer t (t>k) is $\times_{j \neq t} P_j(b)$ and the new c.d.f is $\times_{j \neq t} Q_j(b)$. Taking logarithms, we need to show that

$$\sum_{j\neq t} q_j(b) - p_j(b) \leq 0.$$

From Proposition 2, this holds for b sufficiently large. Suppose then that there is some $\bar{b}$ such that $\sum_{j\neq t} q_j(b) - p_j(b)$ changes sign at $\bar{b}$. It follows that

(9) $$\sum_{j\neq t} q_j'(\bar{b}) < \sum_{j\neq t} p_j'(\bar{b}).$$

Hence, from (3) $q_t(\bar{b}) > p_t(\bar{b})$. But then the conditions of Lemma 3 hold in some left neighborhood of $\bar{b}$. and so the lower end-point condition is violated.

Q.E.D.

Suppose *k=1*, that is, only the distribution of buyer *1* shifts. Then Proposition 3 indicates that the bid distributions faced by all other bidders shift to the right. We do not have a general proof that the new distribution that buyer 1 faces stochastically dominates the old one. However, both intuition and numerical examples suggest that this is the regular case.

Definition: Suppose only buyer 1's distribution of valuations shifts to the right. An equilibrium will be described as regular if the new bid distribution faced by buyer 1 stochastically dominates the old one.

Consider Figure 1 on the next page. The shaded region denoted S is the set of points (p,q) such that $\overline{H}_1(q_1) \geq H_1(p_1)$. From the last part of Lemma 1, if Assumption 2 holds, the boundary of this region has a slope exceeding 1. Also depicted is the set of points $(p_1(b), q_1(b))$. Since the maximum bid is higher under the new distribution for buyer $1, q_1(\beta) < p_1(\beta) = 0$. Also $\overline{H}_1(q_1(\alpha)) = H_1(p_1(\alpha) = \alpha$.

Lemma 4: If Assumption 2 holds and $\overline{H}_1(q_1(b)) - H_1(p_1(b))$ changes sign only once on (α, β), the equilibrium is regular.

Proof: For b sufficiently close to $\beta, q(\beta) < p(\beta)$. If $(p_1(b), q_1(b)) \notin S, \forall b > \alpha$, then, from (9),for all b,

$$\sum_{j\neq 1} q_j'(b) = \frac{1}{\overline{H}_1(q_1) - b} > \frac{1}{H_1(p_1) - b} = \sum_{j\neq 1} p_j'(b).$$

But this is impossible since $q_j(\alpha) = p_j(\alpha), j = 2, \ldots, n$.

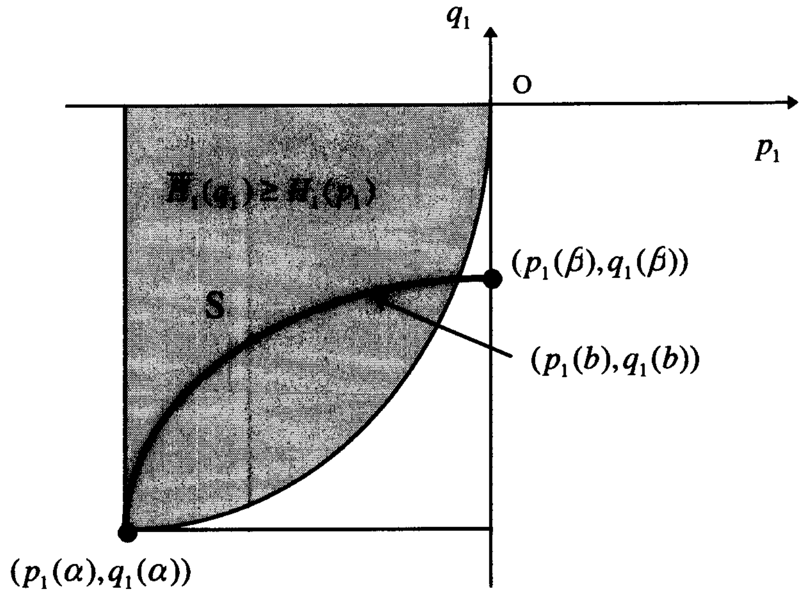

Figure 1: Conditions for the the Regular Case

Then, as depicted in Figure 1, there must be some interval over which $(p_1(b), q_1(b)) \in S$. Suppose that at $b = \bar{b}, \sum_{j\neq 1} q_j(b) - \sum_{j\neq 1} p_j(b)$ changes sign. For this to be the case, we require that at $\bar{b}$,

(10) $$\sum_{j\neq 1} q_j\text{'}(b) = \frac{1}{\bar{H}_1(q_1) - b} < \frac{1}{H_1(p_1) - b} = \sum_{j\neq 1} p_j\text{'}(b).$$

Thus $(p_1(\bar{b}), q_1(\bar{b})) \in S$. If $(p_1(b), q_1(b)) \in S, \forall b < \bar{b}$. Then (10) continues to hold for all $b \in (\alpha, \bar{b}]$. But this is impossible since $q_j(\alpha) = p_j(\alpha), j = 2, \ldots, n$.

Q.E.D.

Proposition 4: If buyer 1's distribution shifts according to Assumption 1, and the equilibrium is regular, the new distribution of high bids stochastically dominates the old one.

Proof: From Proposition 3 and the definition of a regular equilibrium, we have, for all t, and $b \in (\alpha, \beta)$,

$$\sum_{j\neq t} q_j(b) - p_j(b) < 0.$$

Summing over t,

$$(n-1)\sum q_j(b)-p_j(b)<0.$$

Q.E.D.

We now provide conditions under which the equilibrium is regular and hence the distribution of high bids shifts to the right.

Proposition 5: (Lebrun) If there are only two groups of buyers and the c.d.f. of one group's valuations shifts according to Assumption 2, the equilibrium is regular.

Proof: Let the first m_a bidders have c.d.f. $F_a(\cdot)$ and the remaining m_b bidders have c.d.f. $F_b(\cdot)$. Let $(p_a(\cdot),p_c(\cdot))$ be the equilibrium under $\{F_a,F_c\}$ and let $(q_a(\cdot),q_c(\cdot))$ be the equilibrium under $\{\overline{F}_a,F_c\}$. From Proposition 3 we know that

(11) $$m_a q_a+(m_c-1)q_c<m_a p_a+(m_c-1)p_c$$

Suppose that the new distribution that a type a buyer faces does not stochastically dominate the old distribution. Then there is some $\overline{b}$, and some left neighborhood of $\overline{b}$, over which

(12) $$(m_a-1)q_a+m_c q_c>(m_a-1)p_a+m_c p_c$$

Subtracting (12) from (11) it follows that over this left neighborhood $q_a<p_a$, and $q_c>p_c$.. Moreover, as long as $((p_a(b),q_a(b))\in S$,

$$(m_a-1)q_a'(b)+m_c q_c'(b)=\frac{1}{\overline{H}_a(q_a)-b}<\frac{1}{H_a(p_a)-b}=(m_a-1)p_a'(b)+m_c p_c'(b)$$

If this were to be true for all $b<\overline{b}$, the lower end-point condition would be violated. Then there must be some $b_o<\overline{b}$ such that $\overline{H}_a(q_a(b)-H_a(p_a(b))$ again changes sign. (In Figure 1 the heavy curve $(p_a(b),q_a(b))$ must cross the boundary of the shaded region again.) Thus at b_o it must be the case that $q_a'>p_a'$. But from (4),

$$q_a'(b_o)=\frac{m_c}{H_c(q_c)-b}-\frac{m_c-1}{\overline{H}_a(q_a)-b}<\frac{m_c}{H_c(p_c)-b}-\frac{m_c-1}{H_a(p_a)-b}=p_a'(b_o)$$

Thus there can be no such intersection after all. That the equilibrium is regular then follows from Lemma 4.

Q.E.D.

Proposition 6: If buyer 1's distribution shifts according to Assumption 1 and buyers 2,...,*n* have distributions of valuations which are sufficiently similar, then the equilibrium is regular.

Proof: From (3'), for any $s, t > 1$,

$$\sum_{j\neq t} q_j' = \frac{1}{H_t(q_t)-b} \quad \text{and} \quad \sum_{j\neq s} q_j' = \frac{1}{H_s(q_s)-b}$$

Subtracting the second from the first,

$$(13) \qquad q_t'(b) - q_s'(b) = \frac{1}{H_s(q_s)-b} - \frac{1}{H_t(q_t)-b}.$$

Since for all s, t > 1, $q_t(\overline{\beta}) = q_s(\overline{\beta}) = 0$, it follows immediately from (13) that if, for all v, $|F_t(v) - F_s(v)|$ and hence $|H_t(q) - H_s(q)|$ is sufficiently small, $q_t'(b) - q_s'(b)$ and hence $q_t(b) - q_s(b)$ is small. Then , for any $\varepsilon > 0, \exists \delta > 0$ such that, if for all v,

$$|F_t(v) - F_s(v)| < \delta \text{ then } \left|\frac{1}{H_s(q_s)-b} - \frac{1}{H_t(q_t)-b}\right| < \varepsilon.$$

The proof then follows that of the previous Proposition. Suppose the theorem is false. Then there must be two points b_o and $\overline{b}$ at which $\overline{H}_1(q_1(b)) - H_1(p_1(b))$ changes sign. Along the boundary of S,

$$\overline{H}_1'(q_1)\frac{dq_1}{dp_1} = H_1'(p_1).$$

Hence for there to be a second boundary crossing, it must be the case that

$$(14) \qquad \overline{H}_1(q_1) = H_1(p_1) \quad \text{and} \quad \frac{q_1'}{p_1'} > \frac{H_1'(p_1)}{\overline{H}_1'(q_1)}$$

By Assumption 2 and Lemma 1, the right hand side is strictly greater than 1. From (5) and the corresponding expression for $q(b)$,

$$p_1' = \frac{1}{n-1}\left(\sum_{j\neq 1}\frac{1}{H_j(p_j)-b} - \frac{n-2}{H_1(p_1)-b}\right)$$

$$q_1' = \frac{1}{n-1}\left(\sum_{j\neq 1}\frac{1}{H_j(q_j)-b} - \frac{n-2}{\overline{H}_1(q_1)-b}\right)$$

It follows immediately that if the first part of (14) holds and that if the distributions of buyers 2,...,n are sufficiently similar, the second part of (14) will be violated. Hence there can be no second boundary crossing so again the equilibrium is regular.

Q.E.D.

Proposition 7: If the distribution of valuations of buyer s stochastically dominates that of buyer t, then the equilibrium bid distribution of buyer s stochastically dominates that of t.

Proof: Given our assumptions $p_s(b)-p_t(b)=0$ at $b=\beta$. From Lemma 1, if $F_s(\cdot)$ stochastically dominates $F_t(\cdot)$, then, for all $p<0, H_s(p)>H_t(p)$. The result then follows from (13).

Q.E.D.

In the introduction it was claimed that the simple intuition of all biders becoming making higher bids is false. We now establish this claim. Define

$$(15) \qquad B(v)=\frac{F(v)}{vF'(v)} \quad \text{and} \quad \overline{B}(v)=\frac{\overline{F}(v)}{v\overline{F}'(v)},$$

Lemma 5: Suppose $\overline{F}(\cdot)$ exhibits conditional stochastic dominance over $F(\cdot)$, $\overline{F}(0)=F(0)=0$ and $\overline{F}'(0), F'(0)>0$. Then

$$(16) \qquad B(0)=\overline{B}(0)=1$$

and

$$(17) \qquad B'(0)=\frac{F''(0)}{2F'(0)}\le\frac{\overline{F}''(0)}{2\overline{F}'(0)}=\overline{B}'(0).$$

Proof: The first statement follows immediately from l'Hopital's rule. Moreover, from the definition of conditional stochastic dominance, $B(v)>\overline{B}(v), \forall v>0,$ hence $B'(0)\ge\overline{B}'(0)$. Differentiating (15) logarithmically, we obtain

$$\begin{aligned}\frac{B'}{B}&=\frac{F'}{F}-\frac{1}{v}-\frac{F''}{F'}\\&=\frac{F'(1-B)}{F}-\frac{F''}{F'}\end{aligned}$$

Applying l''Hopital's rule to the first expression on the right hand side and rearranging, we obtain

$$B'(0) = -\frac{F''}{2F'}.$$

Q.E.D

Proposition 8: Consider the case of two bidders and suppose that the c.d.f for buyer 1's distribution of valuations shifts from $F_1(\cdot)$ to $\overline{F}_1(\cdot)$. Then if the conditions of Lemma 5 hold and inequality (15) is strict, buyer 1 will bid lower than before, if his valuation is sufficiently small.

Proof: From (3), under the old distributions, the inverse equilibrium bid functions must satisfy

$$\frac{F_1'(\phi_1)}{F_1(\phi_1)}\phi_1'(b) = \frac{1}{\phi_2 - b} \quad \text{and} \quad \frac{F_2'(\phi_2)}{F_2(\phi_2)}\phi_2'(b) = \frac{1}{\phi_1 - b}.$$

Substituting from (15)

$$(18) \qquad \phi_1'(b) = \frac{B_1(\phi_1)\phi_1}{\phi_2 - b} \qquad \phi_2'(b) = \frac{B_2(\phi_2)\phi_2}{\phi_1 - b}$$

Applying l'Hopital's rule, we have

$$(19) \qquad \phi_1'(0) = \phi_2'(0) = 2.$$

Differentiating $\phi_1'(\cdot)$ as given in (18),

$$\phi_1'' = \frac{\phi_1'(B_1 + \phi_1 B_1')}{\phi_2 - b} - \frac{\phi_1 B_1(\phi_2' - 1)}{(\phi_2 - b)^2}$$

Substituting from (19), this can be rewritten as

$$\phi_1'' = \phi_1' \frac{(B_1 + \phi_1 B_1' - \phi_2' + 1)}{\phi_2 - b}$$

Applying l'Hopital's Rule we obtain,

$$\phi_1''(0) = \phi_1' \frac{(B_1' + \phi_1' B_1' - \phi_2''(0))}{\phi_2' - 1}.$$

Appealing to (19) and rearranging,

$$\tfrac{1}{2}\phi_1''(0) + \phi_2''(0) = 3B_1'(0)$$

Similarly we can differentiate $\phi_2'(\cdot)$ to obtain

$$\tfrac{1}{2}\phi_2''(0) + \phi_1''(0) = 3B_2'(0)$$

Solving these two equations we have, at last,

$$\phi_1''(0) = 2B_1'(0) - 4B_2'(0). \tag{20}$$

Let $\overline{\phi}_1(\cdot), \overline{\phi}_2(\cdot)$ be the equilibrium inverse bid functions after buyer 1's distribution of valuations shifts. Arguing exactly as above we obtain,

$$\overline{\phi}_1'(0) = \overline{\phi}_2'(0) = 2. \tag{19'}$$

and

$$\overline{\phi}_1''(0) = 2\overline{B}_1'(0) - 4B_2'(0). \tag{20'}$$

By hypothesis, inequality (17) holds strictly. It follows immediately that for sufficiently small b,

$$\overline{\phi}_1(b) > \phi_1(b)$$

Q.E.D.

To understand why stochastic dominance is not enough to generate a higher bid by buyer 1. let $G(b)$ be the distribution that he faces before his c.d.f. shifts and let $\overline{G}(b)$ be the distribution that he faces afterwards. Before the shift, if buyer 1 bids b his expected gain is

$$U_1(b,v) = G(b)(v-b)$$

We have a similar expression after the shift. Differentiating logarithmically,

$$\frac{\frac{\partial U_1}{\partial b}}{U_1} = \frac{G'(b)}{G(b)} - \frac{1}{v-b} \quad \text{and} \quad \frac{\frac{\partial \overline{U}_1}{\partial b}}{\overline{U}_1} = \frac{\overline{G}'(b)}{\overline{G}(b)} - \frac{1}{v-b}.$$

Then buyer 1 bids higher for all v if $\frac{\overline{G}'(b)}{\overline{G}(b)} > \frac{G'(b)}{G(b)}$, that is, if the new distribution exhibits <u>conditional</u> stochastic dominance (rather than stochastic dominance) over the old distribution.

APPENDIX

Lemma 1: If the stochastic dominance assumption holds,

$H_1(p) < \overline{H}(p)$ for all p<0. and if $H_1(0) = \overline{H}_1(0) = m$, then $H_1'(0) > \overline{H}_1'(0)$.

If, the conditional stochastic dominance assumption holds, then

$$H_1(p) = \overline{H}_1(q) \Rightarrow H_1'(p) > \overline{H}_1'(q)$$

Proof:

From the definitions of $H(\cdot)$ and $\overline{H}_1(\cdot)$, we have

(A.1) $\ln F_1(H_1(p)) = p.$ and $\ln \overline{F}_1(\overline{H}_1(q)) = q..$

Then, by Assumption 1, if $v < m,$ $\overline{F}(v) < F_1(v).$ and so $\overline{H}_1(p) > H_1(p).$
Differentiating (A.1), we obtain:

$$\frac{F_1'(H_1(p))}{F_1(H_1(p))} H_1'(p) = 1 \quad \text{and} \quad \frac{\overline{F}_1'(\overline{H}_1(q))}{\overline{F}_1(\overline{H}_1(q))} \overline{H}_1'(q) = 1$$

From Assumption 1, if $\overline{F}_1(m) = F_1(m) = 1, \overline{F}_1'(m) > F_1'(m).$ Hence $\overline{H}_1'(m) < H_1'(m).$ The final part of the Lemma follows directly from the definition of conditional stochastic dominance.

Q.E.D.

Lemma 2: Suppose that for some $b = \overline{b}$ and some buyer $t > k,$ a solution to the system of differential equations satisfies:

(i) $q_t(b) - p_t(b) > 0$ and (ii) $\sum_{j \neq t} q_j(b) - p_j(b) > \delta$

Then (ii) holds for all $b < \overline{b}$.

Proof: Define $r_j(b) = q_j(b) - p_j(b), j = 1,\ldots,n.$ Also, for each j =1,...,n define

(A.2) $b_j = \underset{x}{Min}\{x | r_j(b) \geq 0, \forall b \in [x, \overline{b}].$

Without loss of generality we may suppose that the smallest of these bids is b_n. From (A.2) $r_j = q_j - p_j$ is non-negative on $[b_n, \bar{b}]$ and is strictly positive in some left neighborhood of $\bar{b}$. Then, from (3) and (3'),

$$\sum_{j \neq n} r_j'(b) \leq 0, \forall b \in [b_n, \bar{b}].$$

$$\Rightarrow \sum_{j \neq n} r_j(\bar{b}) - r_j(b) = \sum_{j \neq n} \int_b^{\bar{b}} r_j'(x)dx \leq 0.$$

Hence $\sum_{j \neq n} r_j(b) > \delta, \forall b \in [b_n, \bar{b}]$. Then, for some $j = t \neq n, r_t(b_n) > 0$. Moreover, since $r_n(b)$ is non-negative over the interval $[b_n, \bar{b}]$., it follows that over this interval,

(A.3) $\sum r_j(b) > \delta.$

For each j define

$\bar{b}_j = Min\{x | r_j(b) \geq 0, \forall b \in [x, b_n].$ and $b_j^* = Min\{x | r_j(b) \leq 0, b > b_n$.

Repeating the argument above it follows that

$\sum_{j \neq t} r_j(b) > \delta, \forall b \in [b_t, b_n]$. and hence that $\sum r_j(b) > \delta, \forall b \in [b_t, \bar{b}]$.

Arguing this repeatedly we have at last,

$$\sum r_j(b) > \delta, \forall b \in [\alpha, \bar{b}].$$

Q.E.D.

Lemma 3: Suppose that Assumption 2 holds and that for some $b = \bar{b}$ and some buyer $t \neq 1$, a solution to the system of differential equations satisfies:

(i) $\overline{H}_1(q_1(b)) - H_1(p_1(b)) > 0$ and (ii) $q_t(b) - p_t(b) > 0, t \neq 1$

Then (ii) holds for all $b < \bar{b}$.

<u>Proof</u>: Suppose that (ii) is first violated at b^*. Let it be buyer i >1 for whom (ii) is first violated. That is $q_i(b) - p_i(b)$ changes sign at b^*. Suppose that (i) holds on the interval $[b^*, \bar{b}]$. Then at b^*,

$$q_i' = \frac{1}{n-1}\sum_{j\neq i}\frac{1}{\overline{H}_j(q_j)-b} - \frac{n-2}{H_i(q_i)-b} \leq \frac{1}{n-1}\sum_{j\neq i}\frac{1}{H_j(p_j)-b} - \frac{n-2}{H_i(p_i)-b} = p_i'$$

But if $q_i' - p_i' < 0$, $q_i(b) - p_i(b)$ cannot change sign at $b*$ after all. Then it must be the case that for some $b*$ condition (ii) holds and

$$\overline{H}_1(q_1(b^*)) - H_1(p_1(b^*)) = 0$$

and

$$\overline{H}_1'(q_1)q_1'(b^*) \geq H_1'(p_1)p_1'(b^*).$$

By Lemma 1, it follows that at b^*, $q_1'(b) > p_1'(b)$. But

$$q_1' = \frac{1}{n-1}\sum_{j\neq 1}\frac{1}{H_j(q_j)-b} - \frac{n-2}{\overline{H}_1(q_1)-b} \leq \frac{1}{n-1}\sum_{j\neq 1}\frac{1}{H_j(p_j)-b} - \frac{n-2}{H_1(p_1)-b} = p_1'$$

so again we have a contradiction.

Q.E.D.