EQUILIBRIUM IN SEALED HIGH BID AUCTIONS

by

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1. Introduction

There is a voluminous theoretical literature on sealed high-bid auctions (auctions in which

bids are sealed and the high bidder pays his bid.) See for example, Vickrey (1961),

Myerson (1981), Riley and Samuelson (1981), Milgrom and Weber (1982), Matthews

(1983), Maskin and Riley (1984), Holt (1980), Cox, Smith and Walker (1988). A critical

property on which this literature relies is the existence of equilibrium in which buyers'

bidding strategies are monotonic in their types. This enables the analyst to perform

comparative statics as the distribution of types changes, and to compare the welfare-

properties of the high-bid auction with those of other auction institutions.

One case in which the existence of monotonic equilibrium is readily established is that in

which buyers' types are distributed smoothly, independently and symmetrically. In this case,

there exists a symmetric equilibrium - - one in which all bidders share the same bidding

behavior as a function of their valuations - - and this bid function is the solution to a first

order differential equation.

However, many applications require consideration of asymmetric distributions (see Maskin

and Riley (1996), common values (see Milgrom and Weber (1982)), or discrete distributions.

And for these cases the issue of equilibrium existence is not as straightforward. Indeed, even

in apparently simple cases, monotonic equilibrium may fail to exist (see examples 1-3).

Therefore we attempt here to provide sufficient conditions that are general enough to cover

most applications that the literature has considered.

In section 2 we introduce the model. In section 3, we first show that, under mild assumptions

on preferences, equilibrium bidding must be monotonic, if types are distributed

independently. With affiliation we introduce the stronger assumption of log super modularity

and, in section 4, consider the case of discrete types. In general there may be no equilibrium under the standard tie-breaking rule (in which the winner is selected at random.) However we are able to establish existence of a monotonic equilibrium of a sealed high-bid auction in which the tying high bidders compete for the item in a second round Vickrey auction. In section 5 we extend our existence theorem (under the modified tie-breaking rule) to the case of continuously distributed types. Finally we establish existence of equilibrium under the standard tie-breaking rule by showing that buyers' strategies remain best responses when the tie-breaking rule in changed. We offer some concluding remarks in section 6.

2. The Model

Throughout the paper we shall make the following assumptions about auctions and participants. A single item is to be sold. All potential buyers who choose to participate submit nonnegative sealed bids simultaneously. The item is awarded to the buyer who makes the highest bid, and he pays this bid.[1] If there is more than one highest bidder, the winner is selected from them according to a specified tie-breaking rule. In particular, in the standard high-bid auction, the winner is chosen randomly, where each of the highest bidders has an equal chance of winning.

There are n potential buyers indexed by i. Buyer i has utility $u_i(b, s_i, s_{-i})$ if he wins with a bid of b and is of type $s_i \in S_i$ while the other buyers are of types $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. We shall consider two cases: (i) S_i is a finite set (the finite

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[1] If no buyer submits a bid, the item is not sold.

type-space case) and (ii) S_i is a compact interval $[\underline{s}_i, \bar{s}_i]$ (the continuous type-space case.)[2]

The joint distribution of types is given by the function $f: S_1 \times \cdots \times S_n \to \mathbb{R}$. We interpret $f(s_1, \ldots, s_n)$ as the (joint) probability of $(s_1, \ldots, s_n)$ in case (i). We suppose that, for all i and all $s_i \in S_i$, there exists $s'_{-i} \in S_{-i}$ such that $f(s_i, s'_{-i}) > 0$. This ensures that for all s_i and s_{-i} the conditional probability $g_i(s_{-i} | s_i)$ is well-defined:

$$g_i(s_{-i} | s_i) = \frac{f(s_i, s_{-i})}{\sum\limits_{s'_{-i}} f(s_i, s'_{-i})}$$

In case (ii), $f(s_1, \ldots, s_n)$ is the joint density of $(s_1, \ldots, s_n)$. In this case we suppose that f is twice differentiable and $\int\limits_{s_{-i} \in S_{-i}} f(s_i, s_{-i}) ds_{-i} > 0$ for all $s_i \in S_i$ so that the conditional density

$$g(s_{-i} | s_i) = \frac{f(s_i, s_{-i})}{\int\limits_{s'_{-i} \in S_{-i}} f(s_i, s'_{-i}) ds'_{-i}}$$

is well-defined.

Without loss of generality, we normalize payoff functions so that the utility of buyer i is zero if his bid is unsuccessful. Let $u_i(b, s_i, s_{-i})$ be decreasing in b and twice continuously differentiable for $i = 1, \ldots, n$. We shall assume that buyers are not willing to bid unbounded amounts, that is, for all i and s_{-i}, if $\bar{b}_i(s_i)$ is the biggest bid such that $\max\limits_{s_{-i}} u_i(\bar{b}_i(s_i), s_i, s_{-i}) \geq 0$, then $\bar{b}_i(s_i) < \infty$.

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[2]Although we do not demonstrate it here, our methods generalize to the case in which types are multidimensional. See also the Corollary to Proposition 3, which does not require one-dimensionality.

In the simplest case, buyer i's type s_i can be identified with buyer i's reservation price:

$$u_i(b,s_i,s_{-i}) = s_i - b.$$

Alternatively, s_i might represent information that buyer i has obtained about the objective monetary value of the item.

The bidding behavior of buyer i of type s_i can be described by the random variable $\tilde{b}_i(s_i)$, where each realization is either a nonnegative number or the null bid corresponding to the choice not to participate. The <u>bid function</u> $\tilde{b}_i(\cdot)$ is a <u>best response</u>[3] to $\tilde{b}_{-i}(\cdot)$ if, for all s_i and all $b_i \in$ real $\tilde{b}_i(s_i)$ (the set of realizations of $\tilde{b}_i(s_i)$), b_i maximizes buyer i's expected payoff, given that the other buyers are using bid functions $\tilde{b}_{-i}(\cdot)$ and their types are drawn according to the conditional distribution $g_i(s_{-i}|s_i)$.

An <u>equilibrium</u> is a vector $(\tilde{b}_1(\cdot),...,\tilde{b}_n(\cdot))$ such that, for all i, $\tilde{b}_i(\cdot)$ is a best response to $\tilde{b}_{-i}(\cdot)$.

3. Monotonicity

We are particularly interested in <u>monotonic equilibria</u>, i.e., equilibria $(\tilde{b}_1(\cdot),...,\tilde{b}_n(\cdot))$ in which, for all i and $s_i, s_i' \in S_i$, if $s_i' > s_i$, $b_i' \in$ real $\tilde{b}_i(s_i')$, and $b_i \in$ real $\tilde{b}_i(s_i)$ then $b_i' \geq b_i$. Hence we shall often invoke some widely-used assumptions on preferences and distributions to ensure monotonicity. We shall call preferences

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[3]For many tie-breaking rules (including the random rule used in the standard auction), buyer i's best response to other buyers' behavior can be captured completely by his bid function. For some rules, however, high bidders are called upon to take further action if a tie occurs, in which case the bid function by itself is not a full description of behavior (see section 4).

<u>monotonic</u> if

(1) $\dfrac{\partial u_i}{\partial s_i}(b,s_i,s_{-i}) > 0$ for all i, b, s_i and s_{-i}.

As the work of Milgrom and Shannon (1994) shows, some form of supermodularity (or "log supermodularity;" see below) is also needed to guarantee that equilibrium bid functions are increasing in types. Call preferences <u>weakly</u> <u>supermodular</u> if

(2) $\dfrac{\partial^2 u_i}{\partial b \partial s_i}(b,s_i,s_{-i}) \geq 0$ for all i, s_i and s_{-i} [4]

Note that if preferences take the commonly-used form

(3) $u_i(b,s_i,s_{-i}) = U_i(\phi_i(s_i,s_{-i})\text{-}b),$

where $U_i(\cdot)$ is an increasing von Neumann-Morgenstern utility function, then (1) and (2) are equivalent to the requirements $\dfrac{\partial \phi_i}{\partial s_i} > 0$ and $U_i'' \leq 0$ (the latter being the assumption that the buyer is risk neutral or risk averse.)

Because the high-bid auction is a discontinuous game, buyers do not always have best responses. For example, consider buyer i with utility function $u_i = 2\text{-}b$ in the standard auction. Suppose that the other buyers bid 1 with probability one. If buyer i also bids 1, then he wins with probability strictly less then one. Thus he is better off bidding slightly more than 1. However, there is no optimal choice because he would like to bid as close to 1 as possible while still remaining above 1. Nevertheless, if types are distributed independently, monotonicity and weak supermodularity together ensure that, when a best response $\tilde{b}_i(\cdot)$ <u>does</u> exist, it is monotonic in the sense that if $s_i < s_i'$, $b_i \in$ real $\tilde{b}_i(s_i)$ (where real $\tilde{b}_i(s_i)$ denotes the set of possible realizations of $\tilde{b}_i(s_i)$), and $b_i' \in$ real $\tilde{b}_i(s_i')$,

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[4]We make the qualification "weakly" because, unlike in the conventional definition, we place no restriction on the other cross partial derivatives.

then $b_i \leq b_i'$, provided that buyer i's expected payoff from b_i and b_i' are positive when

his types are s_i and s_i' respectively.[5]

Proposition 1: Monotonicity and Independence

Suppose that, for $i \in \{1,...,n\}$, buyer i's type is distributed independently of other types,

i.e., the conditional distribution $g_i(s_{-i}|s_i)$ does not depend on s_i. Assume that buyer i's

preferences are monotonic and weakly supermodular. Then if $\tilde{b}_i(\cdot)$ is a best response to

$\tilde{b}_{-i}(\cdot)$, it is monotonic.

Proof: (see the Appendix).

Proposition 1 no longer holds if we drop the independence hypothesis. To see this,

consider the following example.

Example 1: Nonmonotonic Bidding

There are two risk neutral buyers $i=1,2$ with utility functions $u_i(b,s_i) = s_i - b$. The

parameter s_i, which is buyer i's valuation, takes on the values 0, 3, or 6. If $s_i = 3$,

buyer i knows that the other buyer's valuation is also 3. If $s_i = 6$, he knows for sure

that the other buyer's valuation is 0; whereas if s_i is 0, he knows that the other

buyer's valuation is 6. In the standard high-bid auction, it is easy to see that it is an

equilibrium strategy for a buyer to bid 0 if his valuation is 6, bid 3 if his valuation is

3, and to refrain from bidding if his valuation is 0. Moreover, equilibrium is unique,

and so bidding is necessarily nonmonotonic.

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[5]We require the proviso that b_i gives a positive payoff when buyer i's type is s_i because otherwise any $\hat{b}_i$ that wins with zero probability is as good a response as b_i, since both give a zero payoff. The same is true of b_i' and s_i'.

To rule out such examples, let us assume that buyers' types are _affiliated_ (c.f. Milgrom and Weber (1982)). In the continuous type-space case, this amounts to assuming that

(4) $\dfrac{\partial^2}{\partial s_i \partial s_j} \log f(s_1,...,s_n) \geq 0$ for all i and j and all $j \neq i$.

Assume too that preferences are "log supermodular":

(5) $\dfrac{\partial^2}{\partial b \partial s_j} \log u_i(b,s_i,s_{-i}) \geq 0$ for all i,j,

and exhibit private values

(6) $\dfrac{\partial}{\partial s_j} u_i(b,s_i,s_{-i}) = 0$ for all i,j with $i \neq j$.

If preferences take the form (3), then (5) and (6) become

$$((U_i')^2 - U_i U_i'') \dfrac{\partial \phi_i}{\partial s_j} \geq 0$$

and

$$\dfrac{\partial \phi_i}{\partial s_j} = 0 \ , \text{ for } i \neq j,$$

and so sufficient conditions for (5) and (6) to be satisfied are that preferences are monotonic and exhibit either risk aversion or risk neutrality.

What are the precise hypotheses under which one can extend Proposition 1 to the case of affiliated types remains an open question. In Propositions 2 and 3 below, however, we show that as long as values are private and preferences are log supermodular, affiliation implies the existence of a monotonic equilibrium, when ties are broken appropriately.

4. Equilibrium in the Finite Type-Space Case

The discontinuities that occur in the case of ties may prevent the existence of

equilibrium in the standard version of the high-bid auction. To see this, consider the

following example:

Example 2: Non-existence in the standard high-bid auction

$$s_1 = \begin{cases} 1 & \text{with probability} = 0.5 \\ 2 & \text{with probability} = 0.5 \end{cases}$$

$$s_2 = 1$$

$$u_i(s_i, b) = s_i - b, \quad i = 1, 2$$

Given any equilibrium strategy by buyer 2, buyer 1 wins for sure with a bid of $1 + \epsilon$ for

all $\epsilon > 0$ (if not, then, in equilibrium, buyer 2 would have to make a bid more than 1

and win with positive probability, which would be irrational). Thus the maximum point in

the support of the distribution of winning bids is $b^* = 1$. Suppose that the minimum

point in the support, b_*, is strictly less than 1. Then each buyer has a strictly positive

payoff regardless of his reservation price since the latter is at least 1 and any bid between

b_* and 1 wins with positive probability. But then any equilibrium bid must win with

probability bounded away from zero, and for this to be the case, both buyers must bid b_*

with positive probability. Then, regardless of the tie-breaking rule, at least one buyer is

better off bidding a little bit more and breaking the tie. We conclude therefore that

$b_* = 1$. But this cannot correspond to an equilibrium either, since buyer 1 then wins with

probability only ½ if his reservation price is 2, and so he is strictly better off bidding a

little bit more than 1.

In view of the problem illustrated by example 2^6, let us consider an alternative to the random tie-breaking rule of the standard auction. Specifically, let us imagine that, if a tie occurs for the high bid, a <u>Vickrey</u> <u>auction</u> is conducted among the high bidders. That is, in addition to making a sealed bid b_i, each buyer i must also submit a nonnegative sealed "tie-breaker" bid c_i. If several buyers tie with a high-bid of b, the winner is the buyer i for whom c_i is highest among those bidding b, and he pays $b + \max_{j \neq i}\{c_j \mid b_j = b\}$. (If there is a tie for highest tie-breaker bid the winner is determined by randomizing among those making this bid with equal probability.)

Notice that this rule restores equilibrium to Example 2. In such an equilibrium all types bid $b_i = 1$. A buyer with valuation 1 sets $c_i = 0$ in the tie-breaking Vickrey auction whereas a buyer with valuation 2 takes $c_i > 0$. (His weakly dominant strategy is to take $c_i = 1$, but any positive c_i will do as well). Thus a valuation-2 buyer always beats a valuation-1 buyer (although he pays only 1).

In fact, except for one additional problem, the device of breaking ties using a Vickrey auction works quite generally to ensure existence of equilibrium. The additional problem turns on the issue of how a buyer's utility depends on other buyers' types. In Proposition 1 of the previous section, we needed to make no assumptions at all about $\dfrac{\partial u_i}{\partial s_j}$ for $j \neq i$. However, the following example shows that some restriction is required for existence.

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[6]In an earlier version of this paper, we established that the nonexistence problem exemplified by Example 2 is avoided if either (i) each buyer's minimum possible reservation price is zero, or (ii) the seller sets a reserve price no less than each buyer's minimum reservation price.

Example 3: Nonexistence with the Vickrey auction tie-breaking rule

There are two buyers with independently distributed types. Suppose that, for $i=1,2$,

s_i assumes the values 0 and 1 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Let

$$u_i(b,s_i,s_j) = 3 + s_i - 2s_j - b, \text{ where } j \neq i.$$

In the Appendix, we show that there exists no equilibrium for the high-bid auction in this

example, even when ties are broken using a Vickrey auction.

In view of the pathology illustrated by Example 3, we will assume henceforth that a

buyer's utility is nondecreasing in other buyers' types. For later convenience, we will

break this into two alternative cases: either that of private values (i.e. (6) holds) or else

that of common values

$$(7) \qquad \frac{\partial u_i}{\partial s_j} > 0 \quad \text{for all i and j} \neq i$$

Proposition 2: Existence with finitely many types and the Vickrey tie-breaking rule

Suppose that there are finitely many types. Assume that preferences are monotonic. If

either (i) preferences are weakly supermodular, (7) holds, and types are independently

distributed, or (ii) preferences are log supermodular, (6) holds and types are affiliated,

then there exists a monotonic equilibrium of the high-bid auction when ties are broken

using a Vickrey auction.

To establish Proposition 2, we will follow the method of Dasgupta-Maskin (1986).

Let us think of the high-bid auction as an n-player game in which each buyer i submits a

bid function $\tilde{b}_i(\cdot)$ (i.e., a random bid $\tilde{b}_i(s_i)$) for each type $s_i \in S_i$, after which a

vector of types $(s_1,...,s_n)$ is realized, and the winner is then determined according to the bids of the realized types.

For now, let us consider the following modification of the high-bid auction. Instead of breaking ties using the Vickrey auction, suppose that in case of ties, the winner is the high bidder whose expected payoff conditional on winning (given his realized type and the other buyers' bid functions) is highest. (If there is more than one buyer with the highest conditional expected payoff, the winner is determined by randomizing among them with equal probabilities.) That is, if the bid functions are $(\tilde{b}_i(\cdot), \tilde{b}_{-i}(\cdot))$ and buyer i's and j's realized types are s_i and s_j, and these two types are the high bidders with a bid of b, then buyer i beats buyer j if

$$(8) \qquad V_i(b,\tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot),s_i) > V_j(b,\tilde{b}_j(\cdot),\tilde{b}_{-j}(\cdot),s_j),$$

where the left- and right-hand sides of (8) are buyer i's and j's conditional expected payoffs.[7] If buyer i of type s_i has a higher conditional expected payoff from bidding b than any other buyer j of type s_j who (according to $\tilde{b}_j(\cdot)$) bids b, then

$$(9) \quad V_i(b,\tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot),s_i) \;=\; \frac{\displaystyle\sum_{s_{-i}} u_i(b,s_i,s_{-i})g_i(s_{-i}|s_i)\mathrm{Pr}\{\tilde{b}_{-i}(s_{-i})\le b\}}{\displaystyle\sum_{s_{-i}} g_i(s_{-i}|s_i)\mathrm{Pr}\{\tilde{b}_{-i}(s_{-i})\le b\}},$$

where $\mathrm{Pr}\{\tilde{b}_{-i}(s_{-i})\le b\}$ is the probability that all other buyers' bids are less than or equal to b, given that their types are s_{-i} and they bid according to $\tilde{b}_{-i}(s_{-i})$ (because, by assumption, buyer i of type s_i has the highest conditional expected payoff from bidding b, he wins the modified auction with a bid of b provided that all other buyers' bids are

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[7]Note that V_i depends on $\tilde{b}_i(\cdot)$, not just on b, because whether bidder i wins turns on whether V_i is bigger than V_j, and the latter depends upon $\tilde{b}_i(\cdot)$.

less than or equal to b). Similarly, if buyer i of type s_i has a <u>lower</u> conditional

expected payoff from b than any other buyer of type s_j who bids b, then his expected

payoff $V_i(b, \tilde{b}_i(\cdot), \tilde{b}_{-i}(\cdot), s_i)$ is given again by formula (9) except that $\Pr\{\tilde{b}_{-i}(s_{-i}) \leq b\}$ is

replaced by $\Pr\{\tilde{b}_{-i}(s_{-i}) < b\}$ (reflecting the fact that, given the tie-breaking rules, buyer i

wins the auction only if all other buyers bid strictly less than b). And for a type s_i

corresponding to an intermediate expected payoff, $V_i(b, \tilde{b}_i(\cdot), \tilde{b}_{-i}(\cdot), s_i)$ is defined by a

corresponding probabilistic expression intermediate between $\Pr\{\tilde{b}_{-i}(s_{-i}) \leq b\}$ and

$\Pr\{\tilde{b}_{-i}(s_{-i}) < b\}$.

Of course, this modification of the high-bid auction cannot actually be implemented

in practice because the tie-breaking rule depends on expected payoffs, which are private

information. Nevertheless, as we now show, an equilibrium of this modified auction

corresponds to an equilibrium of the auction with the Vickrey tie-breaker.

Lemma 1: Suppose that there are finitely many types. Assume that preferences satisfy

either (6) or (7). Then if there exists a monotonic equilibrium of the modified auction,

there exists a monotonic equilibrium of the auction with the Vickrey tie-breaker.

<u>Proof</u>: Let $(\tilde{b}_1(\cdot), ..., \tilde{b}_n(\cdot))$ be a monotonic equilibrium of the modified auction. We shall

construct a monotonic equilibrium of the auction with Vickrey tie-breaker. The non-tie-

breaking equilibrium behavior of buyers will be the same in the latter equilibrium. We

have only to describe their tie-breaker bids. In particular, consider buyer i of type s_i

and $b \in$ real $\tilde{b}_i(s_i)$. If there is positive probability of a tie for the high bid at b and if

(10) $\sum_{s_{-i}} u_i(b, s_i, s_{-i}) g_i(s_{-i} | s_i) \Pr\{\tilde{b}_{-i}(s_{-i}) \leq b\} > 0,$

then buyer i submits a Vickrey bid $c_i > 0$ (the precise value does not matter as long as

it is strictly positive) together with b. Otherwise, he takes $c_i = 0$.

To verify that this behavior constitutes an equilibrium for the auction with Vickrey tie-breaker, suppose first that for all $b \in$ real $\tilde{b}_i(s_i)$, the probability that b ties for the high bid is zero. Then as long as buyer i does not deviate from $\tilde{b}_i(s_i)$, the tie-breaking rules do not matter and his expected payoff is the same for the Vickrey-tie-break auction as for the modified auction. Consider, therefore, whether it would strictly pay him to deviate to some bid b'. If so then

$$(11) \quad \sum_{s_{-i}} u_i(b', s_i, s_{-i}) g_i(s_{-i}|s_i) \Pr\{\tilde{b}_{-i}(s_{-i}) \leq b'\} > 0.$$

Indeed, if (11) failed to hold then, from (6) or (7) and because other buyers' bid functions are monotonic, buyer i's payoff from b', conditional on winning, would be nonpositive, a contradiction of the assumption that it pays to deviate to b'. Now, buyer i's conditional payoff from b' can be no more than the left-hand side of (11). But in the modified auction, buyer i can obtain a payoff arbitrarily close to the left-hand side of (11) by submitting the bid $b_i = b' + \epsilon$. Hence if deviation paid in the Vickrey tie-break auction, it would also pay in the modified auction, a contradiction of equilibrium.

Next, suppose that, for some $b \in$ real $\tilde{b}_i(s_i)$, b ties for the high bid with positive probability. Assume first that, in the modified auction, buyer i wins the tie-breaker at b with probability one. Then b must satisfy (10). (If instead the left-hand side of (10) were zero, then, since the other buyers bidding b must get at least a zero conditional payoff, buyer i could not win with probability one.) We wish to argue that b still wins the tie-breaker with probability one in the Vickrey tie-break auction, so that buyer i's expected payoff from b is the same in both auctions. From our above discussion, buyer

i submits $c_i > 0$ together with b. Thus he indeed wins the Vickrey tie-break with probability one unless some other buyer j also submits $c_j > 0$. But according to our discussion, that could happen only if buyer j tied at b with positive probability and satisfied (10). Yet this would contradict the assumption that buyer j loses the tie-break with probability one in the modified auction; he would have been strictly better off submitting a bid slightly greater than b in the modified auction. Hence buyer j cannot bid $c_j > 0$. Finally, assume that in the modified auction, buyer i wins the tie-breaker at b with probability less than 1. Then b cannot satisfy (10); the left hand side of (10) must be zero (if it were positive, buyer i would be strictly better off in the modified auction bidding slightly more than b). But buyer i's conditional payoff from b must also be zero. Hence, since he loses the tie-breaker with strictly positive probability, (6) must obtain, and, from our above discussion, buyer i's expected payoff from b is also zero in the Vickrey tie-break auction.

<div align="right">Q.E.D.</div>

Proof of Proposition 2: In view of Lemma 1, it suffices to prove that a monotonic equilibrium exists for the modified auction. For each i and $s_i \in S_i$ let

$\beta_i(s_i) = [0, \bar{b}_i(s_i)]$. (Recall that $\bar{b}_i(s_i)$ is the biggest bid for which buyer i's payoff could possibly be nonnegative when his type is s_i.) For each m = 1,2,... let $\beta_i^m(s_i)$ be the finite subset $\{0, \bar{b}_i(s_i)/2^m, 2\bar{b}_i(s_i)/2^m, ..., \bar{b}_i(s_i)\}$. For each m, consider the "finite approximation" auction in which type s_i of player i is confined to making bids in the set $\beta_i^m(s_i)$.

If hypothesis (i) of Proposition 2 holds, then, from Proposition 1, each buyer i's best response to monotonic bid functions $\tilde{b}_{-i}(\cdot)$ is monotonic. Hence, from Nash's theorem,

there exists, for each m, a monotonic equilibrium for the m-th approximation auction. If hypothesis (ii) holds, then treat each player-type as a distinct player and alter the m-th approximation auction so that if a type strictly outbids a higher type of the same player, then his utility is reduced by an amount $P > 0$. From Nash's theorem, there exists an equilibrium of this altered auction where, for all i, each type s_i of player i is restricted to bid in $\beta_i^m(s_i)$. Moreover, for P sufficiently big, the equilibrium must be monotonic with high probability (i.e. the probability that a low type outbids a high type must be low.) Now send the penalty P to infinity, and we get a sequence of equilibria converging to a monotonic configuration E^* of strategies. We claim that E^* constitutes an equilibrium of the m-th approximation auction (i.e., the auction without any penalty). To see this, suppose to the contrary that (*) type s_i', of player i bids b with positive probability in the limiting configuration but that (**) in the absence of P he is strictly better off bidding b'. (Without loss of generality, we can assume that b' is type s_i''s best bid.) Then (*) and (**) hold for an equilibrium E far enough out in the sequence of equilibria of altered auctions converging to E^*. Now, with a penalty, b is an optimal bid for s_i' in E and so, if b' < b, s_i' continues to prefer b to b' (at least weakly) in E once the penalty is eliminated (the probability that b is bigger than the bid of some other type of player i is clearly at least as great as the probability that b' is bigger). Hence b' > b. Now, the fact that type s_i' is at least as well off bidding b as b' in equilibrium E when the penalty is imposed implies that there exists some $s_i'' > s_i'$ and b'' < b' such that, in E, type s_i'' bids b'' with positive probability (without loss of generality we can assume that s_i'' is the highest such type). Moreover, the fact that type s_i' is at least as well off bidding b' as b'' when the penalty is not applied implies

$$(12) \quad u_i(b',s_i') \sum_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i') \ge u_i(b'',s_i') \sum_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i') \quad ^8$$

Now if type s_i'' bids b' and the other types bid according to E, then the probability he outbids a higher type is zero, since s_i'' is the highest type who bids less than b' in E. Hence, since in E type s_i'' is at least as well off bidding b'' as b', we have

$$(13) \quad u_i(b'',s_i'') \sum_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i'') \ge u_i(b',s_i'') \sum_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i'') .$$

Dividing (13) by (12), we obtain

$$(14) \quad \frac{u_i(b'',s_i'') \sum_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i'')}{u_i(b'',s_{-i}') \sum_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i')} \ge \frac{u_i(b',s_i'') \sum_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i'')}{u_i(b',s_{-i}') \sum_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i')} .$$

But log supermodularity implies that

$$\frac{u_i(b'',s_i'')}{u_i(b'',s_i')} < \frac{u_i(b',s_i'')}{u_i(b',s_i')}$$

and affiliation implies that

$$\frac{\int_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i'')ds_{-i}}{\int_{\tilde{b}_{-i}(s_{-i}) \le b''} g(s_{-i}|s_i')ds_{-i}} \le \frac{\int_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i'')ds_{-i}}{\int_{\tilde{b}_{-i}(s_{-i}) \le b'} g(s_{-i}|s_i')ds_{-i}}$$

which together contradict (13). We conclude that E^* constitutes a monotonic equilibrium of the m-th approximation auction after all.

Let $(\tilde{b}_1^m(\cdot),...,\tilde{b}_n^m(\cdot))$ be a monotonic equilibrium for the m-th approximation auction. We can assume that the sequence $(\tilde{b}_1^m(\cdot),...,\tilde{b}_n^m(\cdot))$ weakly converges to some vector

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[8]The left- and right-hand sides of (12) may not be quite right because of the possibility of ties. But this does not affect the validity of our argument.

$(\tilde{b}_1^*(\cdot),...,\tilde{b}_n^*(\cdot))$. It remains to show that this latter vector constitutes an equilibrium of the modified high-bid auction.

We first argue that, for all i, $s_i \in S_i$,

$$(15) \quad U_i(b,\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot),s_i) \leq \limsup_{m\to\infty} U_i(\tilde{b}_i^m(s_i),\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i)$$

for all $b \in \beta_i(s_i)$, where

$$U_i(b,\tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot),s_i)$$

$$= \mathbf{Prob}\{b \text{ wins the modified auction } | \tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot),s_i\}V_i(b,\tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot),s_i).$$

Suppose, to the contrary, that for some i and s_i, there exists $\hat{b} \in \beta_i(s_i)$ for which

$$(16) \quad U_i(\hat{b},\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot),s_i) > \limsup U_i(\tilde{b}_i^m(s_i),\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i).$$

Now if, for all $k \neq i$, $\Pr\{\tilde{b}_k^*(\cdot)=\hat{b}\} = 0$ then since discontinuities occur only when there are ties, $U_i(\hat{b},\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot),s_i)$ is continuous in $(\tilde{b}_i(\cdot),\tilde{b}_{-i}(\cdot))$ at $(\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot))$, and so, by the definition of weak convergence, $U_i(\hat{b},\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i)$ converges to

$U_i(\hat{b},\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot),s_i)$. But since $(\tilde{b}_1^m(\cdot),...,\tilde{b}_n^m(\cdot))$ is an equilibrium for the mth approximation auction, we have, for all $\hat{b}^m \in \beta_i^m(s_i)$,

$$U_i(\hat{b}^m,\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i) \leq U_i(\tilde{b}_i^m(s_i),\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i) .$$

Moreover, we can find a sequence $\hat{b}^m \to \hat{b}$ such that, for all k, $\Pr\{\tilde{b}_k^*(s_k)=\hat{b}^m\} \to 0$. Hence, taking limits, we obtain,

$$U_i(\hat{b},\tilde{b}_i^*(\cdot),\tilde{b}_{-i}^*(\cdot),s_i) \leq \limsup U_i(\tilde{b}_i^m(s_i),\tilde{b}_i^m(\cdot),\tilde{b}_{-i}^m(\cdot),s_i),$$

a contradiction of (16). Hence there exists $k \neq i$ such that

$$(17) \quad \Pr\{\tilde{b}_k^*(\cdot)=\hat{b}\} > 0.$$

Now the right-hand side of (16) is non-negative, and so the left-hand side is strictly positive. In particular, this means that $\hat{b} < \bar{b}_i(s_i)$ and so there exists $b^0 > \hat{b}$ such that,

for all $b \in (\hat{b}, b^0)$,

(18) $\quad U_i(b, \tilde{b}_i^*(\cdot), \tilde{b}_{-i}^*(\cdot), s_i) > \lim \sup U_i(\tilde{b}_i^m(s_i), \tilde{b}_i^m(\cdot), \tilde{b}_{-i}^m(\cdot), s_i).$

But the above argument implies that, for each such b, there exists $k \neq i$ such that (17)

holds when $\hat{b}$ is replaced by b. Since there are uncountably many b in the interval

$(\hat{b}, b^0)$, we conclude that, for some k, $\tilde{b}_k^*(\cdot)$ has uncountably many atoms, an

impossibility.

We conclude that (15) holds after all. Hence, in particular,

(19) $\quad U_i(\tilde{b}_i^*(s_i), \tilde{b}_i^*(\cdot), \tilde{b}_{-i}^*(\cdot), s_i) \leq \lim \sup U_i(\tilde{b}_i^m(s_i), \tilde{b}_i^m(\cdot), \tilde{b}_{-i}^m(\cdot), s_i)$

for all i and s_i.

Now because, in the modified high-bid auction the winner is always the buyer with

the highest conditional payoff among those with the highest bid, the sum of payoffs across

buyers is upper semicontinuous:

$$\sum_{i=1}^{n} \sum_{s_i \in S_i} U_i(\tilde{b}_i^*(s_i), \tilde{b}_i^*(\cdot), \tilde{b}_{-i}^*(\cdot), s_i) h_i(s_i) \geq$$

(20)

$$\lim \sup \sum_{i=1}^{n} \sum_{s_i \in S_i} U_i(\tilde{b}_i^m(s_i), \tilde{b}_i^m(\cdot), \tilde{b}_{-i}^m(\cdot), s_i) h_i(s_i)$$

where $\quad h_i(s_i) = \sum_{s_{-i}} f(s_i, s_{-i}).$

From (20), we conclude that (19) holds with equality, and so, from (15),

(21) $\quad U_i(b, \tilde{b}_i^*(\cdot), \tilde{b}_{-i}^*(\cdot), s_i) \leq U_i(\tilde{b}_i^*(s_i), \tilde{b}_i^*(\cdot), \tilde{b}_{-i}^*(\cdot), s_i)$, for all b.

Thus $(\tilde{b}_1^*(\cdot), \ldots, \tilde{b}_n^*(\cdot))$ is indeed an equilibrium.

$$\text{Q.E.D.}$$

The hypotheses that the parameter space is one-dimensional and that preferences are

supermodular (or log supermodular) and that types are affiliated in Proposition 2 and

Lemma 1 are invoked simply to ensure monotonic bidding. If we are willing to dispense with monotonicity, we can drop these hypotheses while still establishing existence in the case of private values.

Corollary: Existence without monotonicity

Suppose that there are finitely many types. Assume that preferences exhibit private values. Then there exists an equilibrium of the high-bid auction when ties are broken using a Vickrey auction.

5. Equilibrium Existence with a Continuous Type-Space

In this section we extend Proposition 2 to the case of a continuous type-space. That is, we establish existence under the modified (Vickrey) tie-breaking rule. We then establish existence under the standard tie-breking rule by showing that the bidding strategies remain best responses when the Vickrey tie-breaker rule is switched. As a preliminary, we begin by establishing two general properties of monotonic equilibrium bid distributions in high-bid auctions.[9] The proofs are in the appendix.

Proposition 3: The distribution of winning bids is continuous and its support

is connected

Suppose that buyer i's utility, $u_i(b,s_i,s_{-i})$, i=1,...,n, is increasing in s and strictly increasing in his own type s_i. Given a monotonic equilibrium of the high-bid auction, the support of the distribution of winning bids is an interval $[b_*,b^*]$. Moreover the c.d.f. of

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[9] The results are derived for any tie-breaking rule for which an equilibrium exists, not just that of Proposition 2. Moreover, they hold for both the cases of discrete and continuous types.

winning bids is continuous on $[b_*, b^*]$.

Proposition 4: Payoffs at the lower endpoint

Suppose that buyer i's utility, $u_i(b, s_i, s_{-i})$, i=1,...,n, is increasing in s and strictly

increasing in his own type s_i. Given a monotonic equilibrium, let b_* be as in

Proposition 3. Suppose types s_i and s_j of buyers i and j bid b_* with positive

probability. Then if type s_i's expected payoff is strictly positive, type s_j's expected payoff

is zero.

We now use Propositions 3 and 4 to extend Proposition 2 to the case of a continuous

type-space.

For each i, let $S_i = [\underline{s}_i, \bar{s}_i]$ and let $f(s_1,...,s_n)$ be the joint density of $(s_1,...,s_n)$

as in section 2. For each i and each k=1,2,... choose $S_i^k \subseteq S_i$ such that

$$S_i^k = \{ \ \underline{s}_i, \underline{s}_i + \frac{1}{2^k}(\bar{s}_i - \underline{s}_i), \underline{s}_i + \frac{2}{2^k}(\bar{s}_i - \underline{s}_i),...,\bar{s}_i \ \}.$$ For each k, choose a probability

distribution f^k over $S_1^k \times \cdots \times S_n^k$ such that f^k converges uniformly to f and such

that if types are independent or affiliated under f, then they have the corresponding

property under f^k.

Given the hypotheses of Proposition 3, we can choose, for each k, a monotonic

equilibrium $(\tilde{b}_1^k(\cdot),...,\tilde{b}_n^k(\cdot))$ for the high-bid auction (with Vickrey auction tie-breaker)

when types are restricted to $S_1^k \times ... \times S_n^k$ and the probability distribution over types is f^k.

We will define bid functions $(b_1^*(\cdot),...,b_n^*(\cdot))$ that are the limits of $(\tilde{b}_1^k(\cdot),...,\tilde{b}_n^k(\cdot))$ as

$k \to \infty$.

Let $b_i^k(s_i) = \max \ \text{supp} \ \tilde{b}_i^k(s_i)$. Write $S \equiv \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n} S_i^k = \{s(1), s(2),...\}$.

Choose a subsequence $\{k^m(1)\}_{m=1}^{\infty}$ of $\{1,2,...\}$ such that, if $s(1) \in S_i$ for some i,

then $b_i^{k^m(1)}(s(1))$ converges to some point $\hat{b}_i(s(1))$ as $m \to \infty$. Choose a

subsequence $\{k^m(2)\}_{m=1}^{\infty}$ of $\{k^m(1)\}$ such that, for each i, if $s(2) \in S_i$ then

$b_i^{k^m(2)}(s(2))$ converges to $\hat{b}_i(s(2))$ as $m \to \infty$. Continuing in this way, we can

specify $\hat{b}_i(\cdot)$ for all points in $S \cap S_i$. Clearly, $\hat{b}_i(\cdot)$ is monotonic since each $\tilde{b}_i^k(\cdot)$

is. For any $s_i \in (\underline{s}_i, \bar{s}_i)$ define

$$b_i^*(s_i) = \begin{cases} \hat{b}_i(s_i), & \text{if } s_i \in S \\ \lim_{s \uparrow s_i} \hat{b}_i(s), & \text{if } s_i \notin S \end{cases}$$

where "$\lim_{s \uparrow s_i}$" denotes the limit with respect to an increasing sequence of points in S

converging to s_i. (Note that, because $\hat{b}_i(\cdot)$ is monotonic, the limit does not depend

upon the choice of sequence.)

Proposition 5: Existence under the standard high-bid auction rules

Consider a continuous type-space $S_1 \times ... \times S_n$ and suppose that preferences and the

distribution over types satisfy the hypotheses of Proposition 2. Then the vector

$(b_1^*(\cdot),...,b_n^*(\cdot))$ constitutes a monotonic and pure-strategy equilibrium of the standard high-

bid auction.

Proof: As in the proof of Proposition 2, consider the modified auction in which the

winner is the high bidder whose payoff conditional on winning is highest. For each m,

let $K^m = k^m(m)$, where $k^m(m)$ is as in the paragraph preceding the statement of

Proposition 5. For each i and $s_i \in S_i$ choose a sequence $\{s_i^{K^m}\}$ such that

$s_i^{K^m} \in S_i^{K^m}$ and $s_i^{K^m} \to s_i$.   Define

$$W_i(s_i) = \limsup_{m \to \infty} U_i(\tilde{b}^{K^m}(s_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), s_i^{K^m}).$$

Choose $s_i, \hat{s}_i \in S_i$ with $s_i < \hat{s}_i$ and consider the corresponding sequences $\{s_i^{K^m}\}$ and $\{\hat{s}_i^{K^m}\}$ such that $s_i^{K^m} \to s_i$ and $\hat{s}_i^{K^m} \to \hat{s}_i$. For m big enough, we have

$$U_i(\tilde{b}^{K^m}(s_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), s_i^{K^m})$$

$$\leq U_i(\tilde{b}^{K^m}(s_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), \hat{s}_i^{K^m})$$

$$\leq U_i(\tilde{b}^{K^m}(\hat{s}_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), \hat{s}_i^{K^m}) \; .$$

Hence $W_i(s_i)$ is nondecreasing in s_i. We claim that $W_i(s_i)$ is continuous. If not, then because W_i is nondecreasing, there exists s_i and $\epsilon > 0$ such that either

(22) $\displaystyle \lim_{s_i' \uparrow s_i} W_i(s_i') + \epsilon \; < \; W_i(s_i)$

or

(23) $\displaystyle W_i(s_i') \; < \; \lim_{s_i' \downarrow s_i} W_i(s_i) - \epsilon.$

Now because u_i is continuous there exists $\delta > 0$ such that if $|\hat{s}_i - s_i| < \delta$ then

$$|U_i(b, \tilde{b}_i(\cdot), \tilde{b}_{-i}(\cdot), \hat{s}_i) - U_i(b, \tilde{b}_i(\cdot), \tilde{b}_{-i}(\cdot), s_i)| \; < \; \epsilon$$

for all b, $\tilde{b}_i(\cdot)$, and $\tilde{b}_{-i}(\cdot)$. Hence, given $\hat{s}_i < s_i$ and $|\hat{s}_i - s_i| < \delta$

if $\hat{s}_i^{K^m} \to \hat{s}_i$ and $s_i^{K^m} \to s_i$, then for m big enough

$$U_i(\tilde{b}^{K^m}(\hat{s}_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), \hat{s}_i^{K^m})$$

$$\geq U_i(\tilde{b}^{K^m}(s_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), \hat{s}_i^{K^m})$$

$$\geq U_i(\tilde{b}^{K^m}(s_i^{K^m}), \tilde{b}_i^{K^m}(\cdot), \tilde{b}_{-i}^{K^m}(\cdot), s_i^{K^m}) - \epsilon$$

and so

$$W_i(\hat{s}_i) \geq W_i(s_i) - \epsilon,$$

which contradicts (22). A similar contradiction can be found to (23). We conclude that $W_i(\cdot)$ is indeed continuous.

Following the same argument that established (15), we have

$$(24) \qquad U_i(b_i, b_i^*(\cdot), b_{-i}^*(\cdot), s_i) \leq W_i(s_i)$$

for all i and $s_i \in S_i$, and b_i. From the upper semicontinuity of the sum of payoffs in the modified auction, we obtain

$$(25) \qquad \int \sum_{i=1}^{n} U_i(b_i^*(s_i), b_i^*(\cdot), b_{-i}^*(\cdot), s_i) h_i(s_i) ds_i \geq$$

$$\int \sum_{i=1}^{n} W_i(s_i) h_i(s_i) ds_i,$$

where $h_i(s_i) = \int f(s_i, s_{-i}) ds_{-i}$. Hence from (24) and (25) we have

$$(26) \qquad U_i(b_i^*(s_i), b_i^*(\cdot), b_{-i}^*(\cdot), s_i) = W_i(s_i) \quad \text{for almost all } s_i \in S_i.$$

Now if there exists s_i' such that

$$(27) \qquad U_i(b_i^*(s_i'), b_i^*(\cdot), b_{-i}^*(\cdot), s_i') < W_i(s_i')$$

then because U_i is monotonic and W_i is continuous, (27) continues to hold when s_i' is replaced by s_i, for all s_i in a left-neighborhood of s_i', a contradiction of (26). We conclude that (26) holds for <u>all</u> $s_i \in S_i$. Hence from (24),

$$U_i(b_i, b_i^*(\cdot), b_{-i}^*(\cdot), s_i) \leq U_i(b_i^*(s_i), b_i^*(\cdot), b_{-i}^*(\cdot), s_i)$$

for all b_i and s_i, and so $(b_i^*(\cdot), b_{-i}^*(\cdot))$ constitutes a monotonic and pure-strategy equilibrium for the modified auction.

It remains only to show that this constitutes an equilibrium for the standard high-bid auction. Consider buyer i of type s_i. If the bid $b_i^*(s_i) = \hat{b}$ does not tie for highest bid with positive probability, then it remains an equilibrium bid when the standard tie-breaking rule is used. Assume therefore that $\hat{b}$ ties for highest bid with positive probability. Then, for some $j \neq i$, there exists a nondegenerate interval of types (s_j^*, s_j^{**}) such that, for all s_j in this interval, $b_j^*(s_j) = \hat{b}$. From Proposition 3, $\hat{b} = b_*$, the minimum point in the support of winning bids. Now,

$$V_j(b_*, b_j^*(\cdot), b_{-j}^*(\cdot), s_j^*) \geq 0.$$

Hence

(28) $V_j(b_*, b_j^*(\cdot), b_{-j}^*(\cdot), s_j) > 0$, for all $s_j > s_j^*$.

Proposition 4 implies that

(29) $V_i(b_*, b_i^*(\cdot), b_{-i}^*(\cdot), s_i) = 0.$

Formulas (28) and (29) together imply that in the modified auction, buyer j always wins in a tie between i and j at b^*. But in view of (29), if the tie-breaker rules are changed so that the winner is determined by flipping a coin, buyer i has no incentive to change his bid.

<div align="right">Q.E.D.</div>

6. Concluding Remarks

In this paper we have established conditions under which there exists a monotonic equilibrium in a sealed high-bid auction. For the case in which types are symmetrically distributed, there are well-known conditions under which there exists a symmetric equilibrium (see, however, example 3). Our primary contribution, has been to extend the analysis to the case when types are asymmetrically distributed.

Using methods developed in Dasgupta and Maskin (1986), we are able to establish existence if values are private and types are affiliated. If values are not private we again establish existence, this time under the assumption that types are independent.

As with almost all other auction theory, our results assume that each buyer's private information can be parametrized with a single parameter[10] (see, however, the corollary to Proposition 3). Nevertheless, our methods can be generalized to include multi-dimensional type spaces.

Work by Simon and Zame (1996) establishes existence via a rather different route. For the continuous case with independent private values they show directly that in a class of games with the discontinuities in payoffs that appear in auctions, there is some tie breaking rule for which an equilibrium exists. They then characterize the equilibrium (following the approach taken here) to show that changing the tie-breaking rule to the standard rule has no effect on bidders' best replies. By contrast, Athey (1997) uses a single-crossing property to establish existence of a pure strategy equilibrium when buyers are restricted to bids in a finite set. Finally, Lebrun (1995), Bajari (1997) and Lizzeri and Persico (1997) establish directly that a solution to a suitable set of differential equations exists.

Our existence results, with heterogeneous buyers provide a foundation for our companion paper (Maskin and Riley (1999)) which compares expected revenue in the sealed high bid auction with that in the open auction when buyers' types are asymmetrically distributed.

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[10]There are a few exceptions. For an analysis of a two-signal bidding model, see for example, Cox, Smith and Walker (1988).

REFERENCES

Athey, S. (1995), "Characterizing Properties of Stochastic Objective Functions," mimeo.

Athey, S. (1997), "Single-Crossing Properties and the Existence of Pure Strategy

Equilibria in Games of incomplete Information," mimeo.

Bajari, P. (1996), "Properties of the First Price Sealed Bid Auction with Asymmetric

Bidders," mimeo.

Bikhchandani, S. and J.G. Riley, "Equilibrium in Open Common Value Auctions," Journal

of Economic Theory, 49 (1991).

Cox, J. C., V. L. Smith and J.M. Walker, "Theory and Behavior of First Price

Auctions," Journal of Risk and Uncertainty, 1 (1988) 61-99.

Dasgupta, P. and E.S. Maskin, "Existence of equilibrium in Discontinuous

Games I," Review of Economic Studies, 53 (1986), 1-26

Debreu, G., "A Social Equilibrium Existence Theorem," Proceedings of the

National Academy of Sciences (1952)

Holt C. "Competitive Bidding for Contracts under Alternative Auction Procedures,"

Journal of Political Economy, 88 (1980), 433-45 .

Lebrun B. (1995) "First Price Auctions and the Asymmetric N Bidder Case," mimeo.

Lizzeri, A. and N. Persico (1997), "Uniqueness and Existence of Equilibrium Auctions

with a Reserve Price," mimeo.

Maskin, E.S. and J.G. Riley, "Optimal Auctions with Risk Averse Buyers,"

Econometrica, 52 (1984): 1473-1518.

_____, "Asymmetric Auctions," (1999) Review of Economic Studies (forthcoming).

_____, "Uniqueness in Sealed High Bid Auctions," mimeo (1993).

Matthews, S., "Selling to Risk Averse Buyers with Unobservable Tastes," Journal of
Economic Theory, 30 (1983), 373-400.

Milgrom, P., "Rational Expectations, Information Acquisitions and Competitive Bidding,"
Econometrica, 50 (1982): 1089-1122.

Milgrom, P. and C. Shannon, "Monotone Comparative Statics," Econometrica, 62 (1994),
157-180.

Milgrom, P. and R.J. Weber, "A Theory of Auctions and Competitive Bidding,"
Econometrica, 50 (September, 1982): 1081-1122.

Myerson R. "Optimal Auction Design," Mathematics of Operations Research, 6 (1981):
58-73

Plum, M., "Characterization and Computation of Nash-Equilibria of Auctions with
Incomplete Information," International Journal of Game Theory, 20 (1992): 393-418

Riley, J.G. "Ex Post Information in Auctions," Review of Economic Studies (1988)

_____ and W.F. Samuelson, "Optimal Auctions" American Economic Review (1981)

Simon, L. and W Zame (1996), "Bertrand Competition, Timing games and Auctions:
Information and Sharing Rules Revisited," mimeo.

Vickrey, W., "Counterspeculation, Auctions and Competitive Sealed Tenders," Journal of
Finance, 16 (1961): 8-37.

Wilson, R., "A Bidder Model of Perfect Competition," Review of Economic Studies,
(1977): 511-518

_____ "Bidding," in J. Eatwell, M. Milgate and P. Newman (eds.), The New Palgrave:
A Dictionary of Economic Theory, London: MacMillan, 1987.

APPENDIX

Proposition 1: Monotonicity and Independence

Suppose that, for $i \in \{1,\ldots,n\}$, buyer i's type is distributed independently of other types, i.e., the conditional distribution $g_i(s_{-i}|s_i)$ does not depend on s_i. Assume that buyer i's preferences are monotonic and weakly supermodular. Then if $\tilde{b}_i(\cdot)$ is a best response to

$\tilde{b}_{-i}(\cdot)$, it is monotonic.

<u>Proof</u>: If $\tilde{b}_i(\cdot)$ is a best response to $\tilde{b}_{-i}(\cdot)$, then, for any s_i, any

$b_i \in$ real $\tilde{b}_i(s_i)$, and any $\hat{b}_i < b_i$,

(A1) $\displaystyle\int_{\tilde{b}_{-i}(s_{-i}) \le b_i} u_i(b_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i} \ge \int_{\tilde{b}_{-i}(s_{-i}) \le \hat{b}_i} u_i(\hat{b}_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i}.$ [11]

To complete the proof, it suffices to show that

(A2) $\displaystyle\int_{\tilde{b}_{-i}(s_{-i}) \le b_i} \frac{\partial u_i}{\partial s_i}(b_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i}$

$$> \int_{\tilde{b}_{-i}(s_{-i}) \le \hat{b}_i} \frac{\partial u_i}{\partial s_i}(\hat{b}_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i}.$$

since if it holds and $s_i' > s_i$, then any $b_i' \in$ real $\tilde{b}_i(s_i')$ must satisfy $b_i' > \hat{b}_i$. Now,

the left hand side of (A2) can be rewritten as

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[11] For simplicity, we employ integrals in (A1) even though, in the discrete case, they are actually sums. The expression $g_i(s_{-i})$ is shorthand for $g_i(s_{-i}|s_i)$ which does not depend on s_i. The left- and right-hand sides of (A1) may not be quite right because of the possibility of a tie at b_i or $\hat{b}_i$ with positive probability. However, these possible discrepancies do not affect the rest of the argument.

(A3) $\displaystyle\int_{\hat{b}_i \le \tilde{b}_{-i}(s_{-i}) \le b_i} \frac{\partial u_i}{\partial s_i}(b_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i}$

$\displaystyle + \int_{\tilde{b}_{-i}(s_{-i}) \le \hat{b}_i} \frac{\partial u_i}{\partial s_i}(b_i, s_i, s_{-i}) g_i(s_{-i}) ds_{-i}.$

Because b_i is a best response, $\hat{b}_i$ must win with strictly smaller probability than b_i (otherwise, it would pay buyer i to lower his bid to $\hat{b}_i$). Hence (1) implies that the first term in (A3) is strictly positive. But (2) implies that the second term in (A3) is at least as big as the right-hand side of (A2).   Q.E.D.

Example 3: Nonexistence with the Vickrey auction tie-breaking rule

There are two buyers with independently distributed types. Suppose that, for $i = 1, 2$, s_i assumes the values 0 and 1 with probabilities ⅔ and ⅓, respectively. Let

$$u_i(b, s_i, s_j) = 3 + s_i - 2s_j - b, \text{ where } j \ne i.$$

To verify that there is no equilibrium, we shall confine attention here to potential <u>symmetric</u> equilibria, but the same can be shown for the asymmetric case. Therefore, let $\tilde{b}(\cdot)$ be an equilibrium bid function (we omit the subscript from the bid function since we are concentrating on symmetric equilibrium.)

For $s_i \in \{0, 1\}$, let $b_*(s_i)$ be the minimum and $b^*(s_i)$ be the maximum points in the support of $\tilde{b}(s_i)$ From monotonicity (Proposition 1), we have $b^*(0) \le b_*(1)$. Suppose first that $b_*(0) \ge 3$. Then $b_*(0) = 3$ (otherwise a 0-buyer would earn a negative expected payoff from bidding $b_*(0)$ if $b_*(0)$ were bid with positive probability in equilibrium, or from bidding just above $b_*(0)$ if $b_*(0)$ were bid with zero probability

in equilibrium.) Now if $b^*(1) = 3$, there will be a tie at 3. A 0-buyer will bid zero in the tie-breaking phase. Hence, a 1-buyer will not bid zero with probability one (if both types of buyer 2 bid zero with probability 1, the type 1 of buyer 1 obtains a strictly positive payoff from making a positive bid). However, if, say, type 1 of buyer 2 sets $c_2 > \epsilon$ with positive probability then the type 1 of buyer 1 is better off setting

$c_1 \leq \epsilon$ with probability 1 (since the additional instances in which he would win by taking $c_1 > \epsilon$ would all give him a negative payoff.) Hence there can be no symmetric equilibrium in this case. If $b^*(1) > 3$, then the bid $b^*(1)$ by type 1 of buyer 2 beats type 1 of buyer 1 with positive probability. But in this event, buyer 2's payoff $(3+1-2-b^*(1))$ is negative, a contradiction. We conclude that $b_*(0) < 3$.

Now, if $b_*(0)$ is bid with probability 0, then the equilibrium payoff for a 0-buyer is 0, which implies that $3-2p(b)-b \leq 0$ for b near $b_*(0)$. Here $p(b)$ is the probability that, conditional on the buyer winning with a bid of b, the other buyer's type is 1. Hence $p(b) > 0$ for b near $b_*(0)$, which means that a 1-buyer makes bids arbitrarily close to $b_*(0)$ in equilibrium. Hence, from monotonicity, $b_*(0)$ is bid with probability 1, a contradiction.

Therefore $b_*(0)$ is bid with positive probability by a 0-buyer. Suppose first that a 1-buyer also bids $b_*(0)$ with positive probability. Then there is a tie at $b_*(0)$ with positive probability. Now, in the tie-breaking Vickrey auction, it cannot be the case that both types bid zero with probability one: if both types of buyer 2 bid zero with probability 1, then type 1 of buyer 1 gets a positive expected payoff from making a positive bid. Hence, with positive probability, 1-buyers bid more than zero in the tie-breaker. Let ϵ be the maximum bid that a 1-buyer will make. Then a 1-buyer making

this bid must obtain a non-negative payoff from beating another 1-buyer who makes this bid. But then he would have been better off making a sealed bid of $b_i = b_*(0) + \frac{1}{2}\epsilon$ in the first place (thereby avoiding the tie-breaker altogether), a contradiction. We conclude that a 1-buyer cannot bid $b_*(0)$ with positive probability after all. Hence a 0-buyer has a positive expected payoff from bidding $b_*(0)$, which means that he will not bid 0 with probability 1 in a tie-breaker. But then, as we just argued for a 1-buyer, he is better off making a sealed bid of slightly more than $b_*(0)$ in the first place, a contradiction. We conclude that no equilibrium is possible.

Proposition 3: The distribution of winning bids is continuous and its support
                   is connected

Suppose that each buyer's payoff, $u_i(b, s_i, s_{-i})$, $i = 1, \ldots, n$, is monotonic in types and strictly increasing in his own type. Given a monotonic equilibrium of the high-bid auction, the support of the distribution of winning bids is an interval $[b_*, b^*]$. Moreover the c.d.f. of winning bids is continuous on $[b_*, b^*]$.

Proof: Let b_* be the minimum point in the support of the distribution of winning bids. Then any buyer whose equilibrium bid b exceeds b_* must have a strictly positive expected payoff since his probability of winning is strictly positive on (b_*, b) (if he had a zero expected payoff, he could do strictly better, from (6) or (7), by reducing his bid slightly).

We first show that the c.d.f. of winning bids is continuous at all $b > b_*$. Suppose not. Then, for some $b > b_*$, there exists a buyer i who bids some b with strictly positive probability. We have just argued that his equilibrium payoff must be strictly

positive. (There cannot be more than one such buyer since, under any tie breaking rule, at least one buyer would be strictly better off breaking the tie by bidding slightly more than b.) Consider $b' \in (b_*, b)$. There exist δ and $\gamma > 0$ sufficiently small such that if buyer $j \neq i$ has surplus less than γ from bidding b, then he is better off bidding b' than any bid in the interval $[b-\delta, b]$. Because b is bid with strictly positive probability by buyer i, there exists $\delta' < \delta$ such that, if buyer $j \neq i$ has surplus at least γ from bidding b, then he is better off bidding slightly more than b than making any bid in the interval $[b-\delta', b)$. It follows that no buyer other than i bids in $[b-\delta', b)$ with positive probability. But then buyer i is strictly better off bidding $b-\delta'$ rather than the proposed equilibrium bid of b.

To show that the support of the distribution of winning bids is an interval, consider the interval (b', b'') where b' and b'' are in the support. Suppose contrary to the claim that no buyer bids with positive probability in this interval. Then, because the c.d.f. of winning bids is continuous, the probability of winning with a bid of b'' is the same as that with a bid of b'. It follows that any buyer bidding sufficiently close to b'' is strictly better off bidding b'.

                                                                                      Q.E.D.

Proposition 4: Payoffs at the lower endpoint

Suppose that each buyer's payoff, $u_i(b, s_i, s_{-i})$, $i = 1, \ldots, n$, is monotonic in types and strictly increasing in his own type. Given a monotonic equilibrium, let b_* be as in Proposition 3. Suppose types s_i and s_j of buyers i and j bid b_* with positive probability. Then if type s_i's expected payoff is strictly positive, type s_j's expected payoff is zero.

<u>Proof</u>: Suppose the claim is false. Regardless of the tie-breaking rule it must be the case that if s_i and s_j tie, one of them wins with a probability no greater than 0.5. This buyer can thus raise his probability of winning discontinuously by bidding slightly more than b_*. If, moreover, his expected payoff is positive from bidding b_*, then (6) or (10) imply that he is strictly better off doing so.

<div align="right">Q.E.D.</div>