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TESTABLE RESTRICTIONS ON THE EQUILIBRIUM MANIFOLD[1]

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We present a finite system of polynomial inequalities in unobservable variables and market data that observations on market prices, individual incomes, and aggregate endowments must satisfy to be consistent with the equilibrium behavior of some pure trade economy. Quantifier elimination is used to derive testable restrictions on finite data sets for the pure trade model. A characterization of observations on aggregate endowments and market prices that are consistent with a Robinson Crusoe's economy is also provided.

KEYWORDS: General equilibrium, nonparametric restrictions, quantifier elimination, representative consumer.

1. INTRODUCTION

THE CORE OF THE GENERAL EQUILIBRIUM research agenda has centered around questions on existence and uniqueness of competitive equilibria and stability of the price adjustment mechanism. Despite the resolution of these concerns, i.e. the existence theorem of Arrow and Debreu, Debreu's results on local uniqueness, Scarf's example of global instability of the tâtonnement price adjustment mechanism, and the Sonnenschein-Debreu-Mantel theorem, general equilibrium theory continues to suffer the criticism that it lacks falsifiable implications or in Samuelson' terms, "meaningful theorems."

Comparative statics is the primary source of testable restrictions in economic theory. This mode of analysis is most highly developed within the theory of the household and theory of the firm, e.g., Slutsky's equation, Shephard's lemma, etc. As is well known from the Sonnenschein-Debreu-Mantel theorem, the Slutsky restrictions on individual excess demand functions do not extend to market excess demand functions. In particular, utility maximization subject to a budget constraint imposes no testable restrictions on the set of equilibrium prices, as shown by Mas-Colell (1977). The disappointing attempts of Walras,

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Hicks, and Samuelson to derive comparative statics for the general equilibrium model are chronicled in Inagro and Israel (1990). Moreover, there has been no substantive progress in this field since Arrow and Hahn's discussion of monotone comparative statics for the Walrasian model (1971).

If we denote the market excess demand function as $F_{\hat{w}}(p)$ where the profile of individual endowments $\hat{w}$ is fixed but market prices p may vary, then $F_{\hat{w}}(p)$ is the primary construct in the research on existence and uniqueness of competitive equilibria, the stability of the price adjustment mechanism, and comparative statics of the Walrasian model. A noteworthy exception is the monograph of Balasko (1988) who addressed these questions in terms of properties of the equilibrium manifold. To define the equilibrium manifold we denote the market excess demand function as $F(\hat{w}, p)$, where both $\hat{w}$ and p may vary. The equilibrium manifold is defined as the set $\{(\hat{w}, p) \mid F(\hat{w}, p) = 0\}$. Contrary to the result of Mas-Colell, cited above, we shall show that utility maximization subject to a budget constraint does impose testable restrictions on the equilibrium manifold.

To this end we consider an alternative source of testable restrictions within economic theory: the nonparametric analysis of revealed preference theory as developed by Samuelson, Houthakker, Afriat, Richter, Diewert, Varian, and others for the theory of the household and the theory of the firm. For us, the seminal proposition in this field is Afriat's theorem (1967), for data on prices and consumption bundles. Recall that Afriat, using the Theorem of the Alternative, proved the equivalence of a finite family of linear inequalities—now called the Afriat inequalities—that contain unobservable utility levels and marginal utilities of income with his axiom of revealed preference, "cyclical consistency" —finite families of linear inequalities that contain only observables (i.e. prices and consumption bundles), and with the existence of a concave, continuous monotonic utility function rationalizing the observed data. The equivalence of the Afriat inequalities and cyclical consistency is an instance of a deep theorem in model theory, the Tarski-Seidenberg theorem on quantifier elimination.

The Tarski-Seidenberg theorem—see Van Den Dries (1988) for an extended discussion—proves that any finite system of polynomial inequalities can be reduced to an equivalent finite family of polynomial inequalities in the coefficients of the given system. They are equivalent in the sense that the original system of polynomial inequalities has a solution if and only if the parameter values of its coefficients satisfy the derived family of polynomial inequalities. In addition, the Tarski-Seidenberg theorem provides an algorithm which, in principle, can be used to carry out the elimination of the unobservable—the quantified—variables, in a finite number of steps. Each time a variable is eliminated, an equivalent system of polynomial inequalities is obtained, which contains all the variables except those that have been eliminated up to that point. The algorithm terminates in one of three mutually exclusive and exhaustive states: (i) $1 \equiv 0$, i.e. the original system of polynomial inequalities is never satisfied; (ii) $1 \equiv 1$, i.e. the original system is always satisfied; (iii) an equivalent finite family of polynomial inequalities in the coefficients of the original system which is satisfied only by some parameter values of the coefficients.

To apply the Tarski-Seidenberg theorem, we must first express the structural equilibrium conditions of the pure trade model as a finite family of polynomial inequalities. Moreover, to derive equivalent conditions on the data, the coefficients in this family of polynomial inequalities must be the market observables —in this case, individual endowments and market prices—and the unknowns must be the unobservables in the theory—in this case, individual utility levels, marginal utilities of income, and consumption bundles. A family of equilibrium conditions having these properties consists of the Afriat inequalities for each agent; the budget constraint of each agent; and the market clearing equations for each observation. Using the Tarski-Seidenberg procedure to eliminate the unknowns must therefore terminate in one of the following states: (i) $1 \equiv 0$—the given equilibrium conditions are inconsistent, (ii) $1 \equiv 1$—there is no finite data set that refutes the model, or (iii) the equilibrium conditions are testable.

Unlike Gaussian elimination—the analogous procedure for linear systems of equations—the running time of the Tarski-Seidenberg algorithm is in general not polynomial and in the worst case can be doubly exponential—see the volume edited by Arnon and Buchberger (1988) for more discussion on the complexity of the Tarski-Seidenberg algorithm. Fortunately, it is often unnecessary to apply the Tarski-Seidenberg algorithm in determining if the given equilibrium theory has testable restrictions on finite data sets. It suffices to show that the algorithm cannot terminate with $1 \equiv 0$ or with $1 \equiv 1$. In fact, as we shall show, this is the case for the pure trade model.

It follows from the Arrow-Debreu existence theorem that the Tarski-Seidenberg algorithm applied to this system will not terminate with $1 \equiv 0$. In the next section, we construct an example of a pure trade model where no values of the unobservables are consistent with the values of the observables. Hence the algorithm will not terminate with $1 \equiv 1$. Therefore the Tarski-Seidenberg theorem implies for any finite family of profiles of individual endowments $\hat{w}$ and market prices p that these observations lie on the equilibrium manifold of a pure trade economy, for some family of concave, continuous, and monotonic utility functions, if and only if they satisfy the derived family of polynomial inequalities in $\hat{w}$ and p. This family of polynomial inequalities in the data constitute the testable restrictions of the Walrasian model of pure trade.

It may be difficult, using the Tarski-Seidenberg algorithm, to derive these testable restrictions on the equilibrium manifold in a computationally efficient manner for every finite data set, although we are able to derive restrictions for two observations. If there are more than two observations, our restrictions are necessary but not sufficient. That is, if our conditions hold for every pair of observations and there are at least three observations, then the data need not lie on any equilibrium manifold. Consequently, we call our conditions the weak axiom of revealed equilibrium or WARE. Of course, if our conditions are violated for any pair of observations, then the Walrasian model of pure trade is refuted.

An important distinction between our model and Afriat's model is we do not assume individual consumptions are observed as did Afriat. As a consequence the Afriat inequalities in our model are nonlinear in the unknowns.

This paper is organized as follows. Section 2 presents necessary and sufficient conditions for observations on market prices, individual incomes, and total endowments to lie on the equilibrium manifold of some pure trade economy. Section 3 specializes the results to equilibrium manifolds corresponding to economies whose consumers have homothetic utility functions. In the final section of the paper we discuss extensions and empirical applications of our methodology. In particular, we provide a characterization of the behavior of observations on aggregate endowments and market prices that is consistent with a Robinson Crusoe economy.

2. RESTRICTIONS IN THE PURE TRADE MODEL

We consider an economy with K commodities and T traders, where the intended interpretation is the pure trade model. The commodity space is R^K and each agent has R^K_+ as her consumption set. Each trader is characterized by an endowment vector $w_t \in R^K_{++}$ and a utility function V_t: $R^K_+ \to R$. Utility functions are assumed to be continuous, monotone, and concave.

An *allocation* is a consumption vector x_t for each trader such that $x_t \in R^K_+$ and $\sum_{t=1}^T x_t = \sum_{t=1}^T w_t$. The *price simplex* $\Delta = \{p \in R^K_+ \mid \sum_{i=1}^K p_i = 1\}$. We shall restrict attention to strictly positive prices $S = \{p \in \Delta \mid p_i > 0 \text{ for all } i\}$. A *competitive equilibrium* consists of an allocation $\{x_t\}_{t=1}^T$ and prices p such that each x_t is utility maximizing for agent t subject to her budget constraint. The prices p are called *equilibrium prices*.

Suppose we observe a finite number N of profiles of individual endowment vectors $\{w_t^r\}_{t=1}^T$ and market prices p^r, where $r = 1, \ldots, N$, but we do not observe the utility functions or consumption vectors of individual agents. For each family of utility functions $\{V_t\}_{t=1}^T$ there is an equilibrium manifold, which is simply the graph of the Walras correspondence, i.e. the map from profiles of individual endowments to equilibrium prices.

We say that the pure trade model is *testable* if for every N there exists a finite family of polynomial inequalities in w_t^r and p^r for $t = 1, \ldots, T$ and $r = 1, \ldots, N$ such that observed pairs of profiles of individual endowments and market prices satisfy the given system of polynomial inequalities if and only if they lie on some equilibrium manifold.

To prove that the pure trade model is testable, we first recall Afriat's theorem (1967) (see also Varian (1982)):

AFRIAT'S THEOREM: *The following conditions are equivalent*:

(A.1) *There exists a nonsatiated utility function that "rationalizes" the data* $(p^i, x^i)_{i=1,\ldots,N}$; *i.e., there exists a nonsatiated function* $u(x)$ *such that for all* $i = 1, \ldots, N$, *and all* x *such that* $p^i \cdot x^i \geq p^i \cdot x$, $u(x^i) \geq u(x)$.

(A.2) *The data satisfies "Cyclical Consistency (CC);" i.e., for all* $\{r, s, t, \dots q\}$ $p^r \cdot x^r \geq p^r \cdot x^s$, $p^s \cdot x^s \geq p^s \cdot x^t, \dots,$ $p^q \cdot x^q \geq p^q \cdot x^r$ *implies* $p^r \cdot x^r = p^r \cdot x^s$, $p^s \cdot x^s = p^s \cdot x^t, \dots, p^q \cdot x^q = p^q \cdot x^r$.

(A3) *There exist numbers* U^i, $\lambda^i > 0$, $i = 1, \dots, n$ *such that* $U^i \leq U^j + \lambda^j p^j \cdot (x^i - x^j)$ *for* $i, j = 1, \dots, N$.

(A.4) *There exists a nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.*

Versions of Afriat's theorem for SARP (the Strong Axiom of Revealed Preference, due to Houthakker (1950)) and SSARP (the Strong SARP, due to Chiappori and Rochet (1987)) can be found in Matzkin and Richter (1991) and in Chiappori and Rochet (1987), respectively.[3]

We consider the structural equilibrium conditions for N observations on pairs of profiles of individual endowment vectors $\{w_t^r\}_{t=1}^T$ and market prices p^r for $r = 1, \dots, N$, which are:

$$\exists \{\bar{V}_t^r\}_{r=1,\dots,N;\, t=1,\dots T}, \{\lambda_t^r\}_{r=1,\dots N;\, t=1,\dots T}, \{x_t^r\}_{r=1,\dots N;\, t=1,\dots T}$$

such that

$$(1.1) \qquad \bar{V}_t^r - \bar{V}_t^s - \lambda_t^s p^s \cdot (x_t^s - x_t^s) \leq 0 \qquad (r, s = 1, \dots, N;\ t = 1, \dots, T),$$

$$(1.2) \qquad \lambda_t^r > 0,\ x_t^r \geq 0 \qquad (r = 1, \dots, N;\ t = 1, \dots, T),$$

$$(1.3) \qquad p^r \cdot x_t^r = p^r \cdot w_t^r \qquad (r = 1, \dots, N;\ t = 1, \dots, T),$$

$$(1.4) \qquad \sum_{t=1}^{T} x_t^r = \sum_{t=1}^{T} w_t^r \qquad (r = 1, \dots, N).$$

This family of conditions will be called the *equilibrium inequalities*. The observable variables in this system are the w_t^r and p^r, hence this is a nonlinear family of polynomial inequalities in unobservable utility levels, $\bar{V}_t^r$; marginal utilities of income, λ_t^r; and consumption vectors x_t^r. If we choose T concave, continuous and monotonic utility functions and N profiles of individual endowment vectors, then by the Arrow-Debreu existence theorem there exist equilibrium prices and competitive allocations such that the marginal utilities of income and utility levels of agents at the competitive allocations, together with the competitive prices and allocations and profiles of endowment vectors, satisfy the equilibrium inequalities. Therefore, the Tarski-Seidenberg algorithm applied to the equilibrium inequalities will not terminate with $1 \equiv 0$.

The following example of a pure trade economy with two goods and two traders proves that the algorithm will not terminate with $1 \equiv 1$. In Figure 1, we superimpose two Edgeworth boxes, which are defined by the aggregate endowment vectors w^1 and w^2. The first box, (I), is ABCD and the second box, (II), is

[3] Chiappori and Rochet (1987) show that SSARP characterizes demand data that can be rationalized by strictly monotone, strictly concave, C^∞ utility functions. Define the binary relationship R^0 by $x^t R^0 x$ if $p^t \cdot x^t \geq p^t \cdot x$. Let R be the transitive closure of R^0. Then, SARP is satisfied if and only if for all t, s: $[(x^t R x^s \,\&\, x^t \neq x^s) \Rightarrow (\text{not } x^s R x^t)]$; SSARP is SARP together with $[(p^s \neq \alpha p^r$ for all $\alpha > 0) \Rightarrow (x^s \neq x^r)]$.

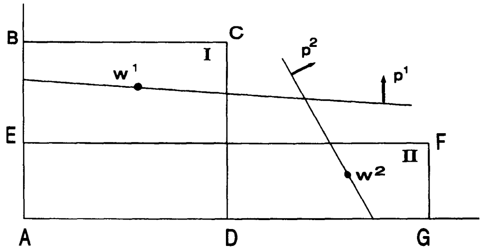

FIGURE 1.

AEFG. The first agent lives at the A vertex in both boxes and the second agent lives at vertex C in box (I) and at vertex F in box (II). The individual endowments w_1^1, w_2^1; w_1^2, w_2^2 and the two price vectors p^1 and p^2 define the budget sets of each consumer. The sections of the budget hyperplanes that intersect with each Edgeworth box are the set of potential equilibrium allocations. All pairs of allocations in box (I) and box (II) that lie on the given budget lines violate Cyclical Consistency for the first agent (the agent living at vertex A). By Afriat's theorem there is no solution to the equilibrium inequalities. This example is easily extended to pure trade models with any finite number of goods or traders.

THEOREM 1: *The pure trade model is testable.*

PROOF: The system of equilibrium inequalities is a finite family of polynomial inequalities; hence we can apply the Tarski-Seidenberg algorithm. We have shown above that the algorithm cannot terminate with $1 \equiv 0$ or with $1 \equiv 1$.

It is often difficult to observe individual endowment vectors, so in the next theorem we restate the equilibrium inequalities where the observables are the market prices, incomes of consumers, and aggregate endowments. Let I_t^r denote the income of consumer t in observation r and w^r the aggregate endowment in observation r.

THEOREM 2: *Let $\langle p^r, \{I_t^r\}_{t=1}^T, w^r \rangle$ for $r = 1, \ldots, N$ be given. Then there exists a set of continuous, concave, and monotone utility functions $\{V_t\}_{t=1}^T$ such that for each $r = 1, \ldots, N$: p^r is an equilibrium price vector for the exchange economy $\langle \{V_t\}_{t=1}^T, \{I_t^r\}_{t=1}^T, w^r \rangle$ if and only if there exists numbers $\{\overline{V}_t^r\}_{t=1,\ldots,T;\, r=1,\ldots,N}$ and $\{\lambda_t^r\}_{t=1,\ldots,T;\, r=1,\ldots,N}$ and vectors $\{x_t^r\}_{t=1,\ldots,T;\, r=1,\ldots,N}$ satisfying*

(2.1) $\overline{V}_t^r - \overline{V}_t^s - \lambda_t^s p^s \cdot (x_t^r - x_t^s) \leq 0$ $\quad (r, s = 1, \ldots, N;\ t = 1, \ldots, T)$,

(2.2) $\lambda_t^r > 0,\ x_t^r \geq 0$ $\quad (r = 1, \ldots, N;\ t = 1, \ldots, T)$,

(2.3) $p^r \cdot x_t^r = I_t^r$ $\quad (r = 1, \ldots, N;\ t = 1, \ldots, T)$,

(2.4) $\sum_{t=1}^{T} x_t^r = w^r$ $\quad (r = 1, \ldots, N)$.

PROOF: Suppose that there exists $\{\bar{V}_t^r\}$, $\{\lambda_t^r\}$, and $\{x_t^r\}$ satisfying (2.1)–(2.4). Then, (2.1)–(2.3) imply, by Afriat's Theorem that for each t, there exists a continuous, concave, and monotone utility function V_t: $R_+^K \to R$ such that for each r, x_t^r is one of the maximizers of V_t subject to the budget constraint: $p^r y \leq I_t^r$. Hence, since $\{x_t^r\}_{t=1}^T$ define an allocation, i.e. satisfy (2.4), p^r is an equilibrium price vector for the exchange economy $\langle \{V_t\}_{t=1}^T, \{w_t^r\}_{t=1}^T \rangle$ for each $r = 1, \ldots, N$.

The converse is immediate, since given continuous, concave and monotone utility functions, V_t, the equilibrium price vectors p^r and allocations $\{x_t^r\}_{t=1}^T$ satisfy (2.3) and (2.4) by definition. The existence of $\{\lambda_t^r\}_{t=1}^T$ such that (2.1) and (2.2) hold follows from the Kuhn-Tucker Theorem, where $\bar{V}_t^r = V_t(x_t^r)$.

For two observations ($r = 1, 2$) and the Chiappori-Rochet version of Afriat's theorem we use, in the proof of Theorem 3 below, quantifier elimination to derive the testable restrictions for the pure trade model with two consumers ($t = a, b$) from the equilibrium inequalities. We call the family of polynomial inequalities obtained from this process the *Weak Axiom of Revealed Equilibrium* (WARE). To describe WARE, we let $\bar{z}_t^r$ ($r = 1, 2$; $t = a, b$) denote any vector such that $\bar{z}_t^r \in \operatorname{argmax}_x \{p^s \cdot x \mid p^r \cdot x = I_t^r,\ 0 \leq x \leq w^r\}$ where $r \neq s$. Hence, among all the bundles that are feasible in observation r and are on the budget hyperplane of consumer t in observation r, $\bar{z}_t^r$ is any of the bundles that cost the most under prices p^s ($s \neq r$).

We will say that observations $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;\ t=a,b}$, $\{w^r\}_{r=1,2}$ *satisfy WARE* if

(I) $\forall r = 1, 2, \quad I_a^r + I_b^r = p^r \cdot w^r$,

(II) $\forall r, s = 1, 2\ (r \neq s),\ \forall t = a, b, \quad [(p^s \cdot \bar{z}_t^r \leq I_t^s) \Rightarrow (p^r \cdot \bar{z}_t^s > I_t^r)]$,

(III) $\forall r, s = 1, 2\ (r \neq s), \quad [(p^s \cdot \bar{z}_a^r \leq I_a^s) \& (p^s \cdot \bar{z}_b^r \leq I_b^s)] \Rightarrow (p^r \cdot w^s > p^r \cdot w^r)$.

In the next theorem we establish that WARE characterizes data that lie on some equilibrium manifold. Condition (I) says that the sum of the individuals' incomes equals the value of the aggregate endowment. Condition (II) applies when all the bundles in the budget hyperplane of consumer t in observation r that are feasible in observation r can be purchased with the income and prices faced by consumer t in observation s ($s \neq r$) (i.e., $p^s \cdot \bar{z}_t^r \leq I_t^s$). It says that it must then be the case that some of the bundles that are feasible in observation s and are in the budget hyperplane of consumer t in observation s cannot be purchased with the income and prices faced by consumer t in observation r (i.e., $p^r \cdot \bar{z}_t^s > I_t^r$). Clearly, unless this condition is satisfied, it will not be possible to find consumption bundles consistent with equilibrium and satisfying SSARP. Note that this condition is not satisfied by the observations in Figure 1. Condition (III) says that when for each of the agents it is the case that all the bundles that are feasible and affordable under observation r can be purchased with the agent's income and the price of observation s, then it must be that the aggregate endowment in observation s costs more than the aggregate endowment in observation r, with the prices of observation r. This guarantees that at least one of the pairs of consumption bundles in observation s that contain for each agent feasible and affordable bundles that could not be purchased with the income and price of observation r are such that they add up to the aggregate endowment.

THEOREM 3: *Let* $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;\, t=a,b}$, $\{w^r\}_{r=1,2}$ *be given such that* p^1 *is not a scalar multiple of* p^2. *Then the equilibrium inequalities for strictly monotone, strictly concave,* C^∞ *utility functions have a solution, i.e. the data lies on the equilibrium manifold of some economy whose consumers have strictly monotone, strictly concave,* C^∞ *utility functions, if and only if the data satisfy WARE.*

We provide in the Appendix a proof of Theorem 3 that uses the Tarski-Seidenberg theorem. A different type of proof is given in Brown and Matzkin (1993).

3. RESTRICTIONS WHEN UTILITY FUNCTIONS ARE HOMOTHETIC

In applied general equilibrium analysis—see Shoven and Whalley (1992)—utility functions are often assumed to be homothetic. We next derive testable restrictions on the pure trade model under this assumption. These restrictions can be used as a specification test for computable general equilibrium models, say in international trade, where agents have homothetic utility functions.

Afriat (1977, 1981) and Varian (1983) developed the Homothetic Axiom of Revealed Preference (HARP), which is equivalent to the Afriat inequalities for homothetic utility functions. For two observations, $\{p^r, x^r\}_{r=1\,2}$, HARP reduces to: $(p^r\cdot x^s)(p^s\cdot x^r) \geq (p^r\cdot x^r)(p^s\cdot x^s)$ for $r, s = 1,2$ $(r \neq s)$. If we substitute these for the Afriat inequalities in the equilibrium inequalities (1.1)–(1.4), we obtain a nonlinear system of polynomial inequalities where the unknowns (or unobservables) are the consumption vectors x_t^r for $r = 1,2$ and $t = a, b$. Using quantifier elimination, we derive in the proof of Theorem 4 the testable restrictions of this model on the observable variables. We call these restrictions the *Homothetic-Weak Axiom of Revealed Preference* (*H-WARE*).

Given observations $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;\, t=a,b}$, $\{w^r\}_{r=1,2}$, we define the following terms:

$$\gamma_a = I_a^1 I_a^2, \qquad \gamma_b = I_b^1 I_b^2, \qquad \gamma_w = (p^1\cdot w^2)(p^2\cdot w^1),$$

$$\psi_1 = \gamma_b - \gamma_a - \gamma_w, \qquad \psi_2 = (\gamma_b - \gamma_a - \gamma_w)^2 - 4\gamma_a\gamma_w,$$

$$r_1 = \frac{\gamma_a}{p^1\bar{z}_a^2}, \qquad r_2 = p^2 w^1 - \frac{\gamma_b}{p^1\bar{z}_b^2},$$

$$t_1 = \frac{-\psi_1 - (\psi_2)^{1/2}}{2p^1\cdot w^2}, \qquad t_2 = \frac{-\psi_1 + (\psi_2)^{1/2}}{2p^1\cdot w^2},$$

$$s_1 = \max\{r_1, t_1\}, \qquad s_2 = \min\{r_2, t_2\}.$$

Let $\underline{z}_t^r$ ($r = 1,2$; $t = a, b$) denote any vector such that $\underline{z}_t^r \in \operatorname{argmin}_x\{p^s \cdot x \mid p^r \cdot x = I_t^r, 0 \leq x \leq w^r\}$ where $r \neq s$.

Our *Homothetic Weak Axiom of Revealed Equilibrium* (*H-WARE*) is

(H.1) $\Psi_2 \geq 0$,

(H.II) $s_1 \leq s_2$,

(H.III) $s_1 \leq p^2 \cdot \bar{z}_a^1$,

(H.IV) $p^2 \cdot \underline{z}_a^1 \leq s_2$,

(H.V) $I_a^1 + I_b^1 = p^1 \cdot w^1$ and $I_a^2 + I_b^2 = p^2 \cdot w^2$.

Condition (H.I) guarantees that t_1 and t_2 are real numbers. Conditions (H.II)–(H.IV) guarantee the existence of a vector x_a^1 whose cost under prices p^2 is between s_1 and s_2. The values of s_1 and s_2 guarantee that equilibrium allocations can be found. Condition (H.V) says that the sum of the individuals' incomes equals the value of the aggregate endowment.

THEOREM 4: *Let* $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2,\, t=a,b}$, $\{w^r\}_{r=1,2}$ *be given. Then the equilibrium inequalities for homothetic utility functions have a solution, i.e. the data lie on the equilibrium manifold of some economy whose consumers have homothetic utility functions, if and only if the data satisfy H-WARE.*

In the Appendix, we provide a proof that uses the Tarski-Seidenberg theorem. See Brown and Matzkin (1993) for a different proof.

4. EMPIRICAL APPLICATIONS AND EXTENSIONS

To empirically test the pure exchange model, one might use cross-sectional data to obtain the necessary variation in market prices and individual incomes. Assuming that sampled cities or states have the same distribution of tastes but different income distributions and consequently different market prices, the observations can serve as market data for our model. In the stylized economies in our examples one should think of each "trader" as an agent type, consisting of numerous small consumers, each having the same tastes and incomes.

There is a large variety of situations that fall into the structure of a general equilibrium exchange model and for which data are available. For example, our methods can be used in a multiperiod capital market model where agents have additively separable (time invariant) utility functions, to test whether spot prices are equilibrium prices, using only observations on the spot prices and the individual endowments in each period. They can be used to test the equilibrium hypothesis in an assets markets model where agents maximize indirect utility functions over feasible portfolios of assets, using observations on the outstanding shares of the assets, each trader's initial asset holdings, and the asset prices. Or, they can be used in a household labor supply model of the type considered in Chiappori (1988), to test whether the unobserved allocation of consumption within the household is determined by a competitive equilibrium, using data on the labor supply, wages, and the aggregate consumption of the household.

To apply the methodology to large data sets, it is necessary to devise a computationally efficient algorithm for solving large families of equilibrium inequalities. A promising approach is to restrict attention to special classes of utility functions. As an example, if traders are assumed to have quasilinear utility functions—all linear in the same commodity (say the kth)—then the equilibrium inequalities can be reduced to a family of linear inequalities by choosing the kth commodity as numeraire. We can now use the simplex algorithm or the interior point algorithm of Karmarkar—which runs in polynomial time—to test for or compute solutions of the equilibrium inequalities.

The more challenging problem in economic theory is to recast the equilibrium inequalities to allow random variation in tastes. Some recent progress has been made in this area by Brown and Matzkin (1995). They consider a random utility model, which gives rise to a stochastic family of Afriat inequalities, that can be identified and consistently estimated. If their approach can be extended to random exchange models then this is a significant step in empirically testing the Walrasian hypothesis.

The methodology can also be extended to find testable restrictions on the equilibrium manifold of economies with production technologies. Only observations on the market prices, individuals' endowments, and individuals' profit shares are necessary to test the equilibrium model in production economies. In particular, for a Robinson Crusoe economy, where the consumer has a nonsatiated utility function, we have derived the following restrictions on the observable variables, for any number of observations. A direct proof of the result is given in the Appendix.

THEOREM 5: *The data* $\langle p^r, w^r \rangle$ *for* $r = 1, \ldots, N$ *lies in the equilibrium manifold of a Robinson Crusoe economy if and only if* $\langle p^r, w^r \rangle$ *for* $r = 1, \ldots, N$ *satisfy Cyclical Consistency* (*CC*).

Testable restrictions for other economic models can also be derived using the methodology that we have presented in this paper.

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APPENDIX

PROOF OF THEOREM 3: Using the Tarski-Seidenberg theorem, we need to show that WARE can be derived by quantifier elimination from the equilibrium inequalities for strictly monotone, strictly concave, C^∞ utility functions. Making use of Chiappori and Rochet (1987), these inequalities are: $\exists\{\bar{V}_t^r\}_{r=1,2;\ t=a,b}$, $\{\lambda_t^r\}_{r=1,2;\ t=a,b}$, $\{x_t^r\}_{r=1,2;\ t=a,b}$ such that

(C.1) $\bar{V}_t^2 - \bar{V}_t^1 - \lambda_t^1 p^1 \cdot (x_t^2 - x_t^1) < 0, \qquad t=a,b;$

(C.2) $\bar{V}_t^1 - \bar{V}_t^2 - \lambda_t^2 p^2 \cdot (x_t^1 - x_t^2) < 0, \qquad t=a,b;$

(C.3) $\lambda_t^r > 0, \qquad r=1,2;\ t=a,b;$

(C.4) $p^r \cdot x_t^r = I_t^r, \qquad r=1,2;\ t=a,b;$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2, \qquad t=a,b;$

(C.6) $x_t^r \geq 0, \qquad r=1,2;\ t=a,b;$

(C.7) $x_a^r + x_b^r = w^r, \qquad r=1,2.$

The equivalent expression, after eliminating $\{\lambda_t^r\}_{r=1,2;\ t=a,b}$, is: $\exists\{\bar{V}_t^r\}_{r=1,2;\ t=a,b}$, $\{x_t^r\}_{r=1,2;\ t=a,b}$ such that

(C.1′) $p^1 \cdot (x_t^2 - x_t^1) \leq 0 \Rightarrow \bar{V}_t^2 < \bar{V}_t^1, \qquad t=a,b;$

(C.2′) $p^2 \cdot (x_t^1 - x_t^2) \leq 0 \Rightarrow \bar{V}_t^1 < \bar{V}_t^2, \qquad t=a,b;$

(C.4) $p^r \cdot x_t^r = I_t^r, \qquad r=1,2;\ t=a,b;$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2, \qquad t=a,b;$

(C.6) $x_t^r \geq 0, \qquad r=1,2;\ t=a,b;$

(C.7) $x_a^r + x_b^r = w^r, \qquad r=1,2.$

Necessity is clear. Sufficiency follows by noticing that (C.1′) and (C.2′) imply, respectively, that $\exists\{\lambda_t^1\}_{t=a,b}$ satisfying (C.1) and (C.3) and $\exists\{\lambda_t^2\}_{t=a,b}$ satisfying (C.2) and (C.3). Elimination of $\{\bar{V}_t^r\}_{r=1,2;\ t=a,b}$ yields the equivalent expression: $\exists\{x_t^r\}_{r=1,2;\ t=a,b}$ such that

(C.1″) $p^1 \cdot (x_t^2 - x_t^1) \leq 0 \Rightarrow p^2 \cdot (x_t^1 - x_t^2) > 0, \qquad t=a,b;$

(C.4) $p^r \cdot x_t^r = I_t^r, \qquad r=1,2;\ t=a,b;$

(C.5) $p^1 \neq p^2 \Rightarrow x_t^1 \neq x_t^2, \qquad t=a,b;$

(C.6) $x_t^r \geq 0, \qquad r=1,2;\ t=a,b;$

(C.7) $x_a^r + x_b^r = w^r, \qquad r=1,2.$

This follows because (C.1″) is necessary and sufficient for the existence of $\{\bar{V}_t^r\}_{r=1,2;\ t=a,b}$ satisfying (C.1′)–(C.2′). Note that we have just shown how, for two observations, SSARP can be derived by quantifier elimination. Next, elimination of $\{x_b^r\}_{r=1,2}$, using (C.7), yields the equivalent expression: $\exists x_a^1, x_a^2$ such that

(C.1‴.1) $p^2 \cdot x_a^1 \leq I_a^2 \Rightarrow p^1 \cdot x_a^2 > I_a^1,$

(C.1‴.2) $p^2 \cdot (w^1 - x_a^1) \leq I_b^2 \Rightarrow p^1 \cdot (w^2 - x_a^2) > I_b^1,$

(C.4′) $p^r \cdot x_a^r = I_a^r, \qquad r=1,2,$

(C.5′) $p^1 \neq p^2 \Rightarrow [(x_a^1 \neq x_a^2) \,\&\, (w^1 - x_a^1 \neq w^2 - x_a^2)];$

(C.6′) $0 \leq x_a^r \leq w^r, \qquad r=1,2,$

(C.7′) $I_a^r + I_b^r = p^r \cdot w^r, \qquad r=1,2.$

Let $\underline{z}_t^r$ denote any vector such that $\underline{z}_t^r \in \operatorname{argmin}_x\{p^s \cdot x \mid p^r \cdot x = I_t^r,\ 0 \leq x \leq w^r\}$, where $r \neq s$. Then, after elimination of x_a^2 we get: $\exists x_a^1$ such that

(C.1'''' .1) $p^2 \cdot x_a^1 \leq I_a^2 \Rightarrow p^1 \cdot \bar{z}_a^2 > I_a^1$,

(C.1'''' .2) $p^2 \cdot (w^1 - x_a^1) \leq I_b^2 \Rightarrow p^1 \cdot (w^2 - \underline{z}_a^2) > I_b^1$,

(C.1'''' .3) $[(p^2 \cdot x_a^1 \leq I_a^2) \& (p^2 \cdot (w^1 - x_a^1) \leq I_b^2)] \Rightarrow p^1 \cdot w^2 > p^1 \cdot w^1$,

(C.4'') $p^1 \cdot x_a^1 = I_a^1$,

(C.6') $0 \leq x_a^1 \leq w^1$,

(C.7') $I_a^r + I_b^r = p^r \cdot w^r, \quad r = 1, 2.$

Necessity of (C.1''''.1) and (C.1''''.2) follows by the definitions of $\bar{z}_a^2$ and $\underline{z}_a^2$. Necessity of (C.1''''.3) follows by using (C.1'''.1), (C.1'''.2), and (C.7'). The existence of x_a^1 satisfying (C.1'''.1), (C.1'''.2), (C.4')–(C.7') follows immediately if $(p^2 \cdot x_a^1 > I_a^2) \& (p^2 \cdot (w^1 - x_a^1) > I_b^2)$; it follows using (C.1''''.1) if $(p^2 \cdot x_a^1 \leq I_a^2) \& (p^2 \cdot (w^1 - x_a^1) > I_b^2)$; it follows using (C.1''''.2) if $(p^2 \cdot x_a^1 > I_a^2) \& (p^2 \cdot (w^1 - x_a^1) \leq I_b^2)$; and it follows using (C.1''''.1)–(C.1''''.3) if $(p^2 \cdot x_a^1 \leq I_a^2) \& (p^2 \cdot (w^1 - x_a^1) \leq I_b^2)$. (C.5') can always be satisfied. Finally, elimination of x_a^1 yields, by similar arguments, the equivalent expression:

(C.1*.1) $p^1 \cdot \bar{z}_a^2 \leq I_a^1 \Rightarrow p^2 \cdot \bar{z}_a^1 > I_a^2$,

(C.1*.2) $p^1 \cdot (w^2 - \underline{z}_a^2) \leq I_b^1 \Rightarrow p^2 \cdot (w^1 - \underline{z}_a^1) > I_b^2$,

(C.1*.3) $[(p^1 \cdot \bar{z}_a^2 \leq I_a^1) \& (p^1 \cdot (w^2 - \underline{z}_a^2) \leq I_b^1)] \Rightarrow p^2 \cdot w^1 > p^2 \cdot w^2$,

(C.1*.4) $[(p^2 \cdot \bar{z}_a^1 \leq I_a^2) \& (p^2 \cdot (w^1 - \underline{z}_a^1) \leq I_b^2)] \Rightarrow p^1 \cdot w^2 > p^1 \cdot w^1$,

(C.7') $I_a^r + I_b^r = p^r \cdot w^r, \quad r = 1, 2.$

Note that $I_a^r + I_b^r = p^r \cdot w^r$ implies that $p^s \cdot \bar{z}_a^r + p^s \cdot \underline{z}_b^r = p^s \cdot w^r$ $(s \neq r)$. Hence, the above family of polynomial inequalities can be written as:

(I) $\forall r = 1, 2, \quad I_a^r + I_b^r = p^r \cdot w^r$;

(II) $\forall r, s = 1, 2\ (r \neq s), \forall t = a, b, \quad [(p^s \cdot \bar{z}_t^r \leq I_t^s) \Rightarrow (p^r \cdot \bar{z}_t^s > I_t^r)]$;

(III) $\forall r, s = 1, 2\ (r \neq s), \quad [(p^s \cdot \bar{z}_a^r \leq I_a^s) \& (p^s \cdot \bar{z}_b^r \leq I_b^s)] \Rightarrow (p^r \cdot w^s > p^r \cdot w^r)$

which is our *Weak Axiom of Revealed Equilibrium* (*WARE*).

PROOF OF THEOREM 4: Using the Tarski-Seidenberg theorem, we show that H-WARE can be derived by quantifier elimination from the equilibrium inequalities for homothetic, concave, and monotone utility functions. Hence, we have to eliminate the quantifiers in the following expression: $\exists x_a^1, x_a^2, x_b^1, x_b^2$ such that

(H.1) $(p^1 \cdot x_a^2)(p^2 \cdot x_a^1) \geq \gamma_a$,

(H.2) $(p^1 \cdot x_b^2)(p^2 \cdot x_b^1) \geq \gamma_b$,

(H.3) $p^r \cdot x_t^r = I_t^r, \quad r = 1, 2;\ t = a, b$;

(H.4) $x_t^r \geq 0, \quad r = 1, 2;\ t = a, b$;

(H.5) $x_a^r + x_b^r = w^r, \quad r = 1, 2.$

This is equivalent to: $\exists x_a^1, x_a^2$ such that

(H.1) $(p^1 \cdot x_a^2)(p^2 \cdot x_a^1) \geq \gamma_a$,

(H.2') $(p^1 \cdot (w^2 - x_a^2))(p^2 \cdot (w^1 - x_a^1)) \geq \gamma_b$,

(H.3') $p^r \cdot x_a^r = I_a^r, \quad r = 1, 2$,

(H.4') $w^r \geq x_a^r \geq 0, \quad r = 1, 2$,

(H.5') $I_a^r + I_b^r = p^r \cdot w^r, \quad r = 1, 2.$

(H.1) and (H.2′) can be expressed as:

(H.1′) $$p^1\cdot w^2-\frac{\gamma_b}{p^2\cdot(w^1-x_a^1)}\geq p^1\cdot x_a^2\geq\frac{\gamma_a}{p^2\cdot x_a^1}.$$

So, the expression: "$\exists x_a^1, x_a^2$ satisfying (H.1′)–(H.5′)" is equivalent to: $\exists x_a^1$ such that

(H.1.1) $$p^1\cdot\bar{z}_a^2\geq\frac{\gamma_a}{p^2\cdot x_a^1},$$

(H.1.2) $$p^1\cdot w^2-\frac{\gamma_b}{p^2\cdot(w^1-x_a^1)}\geq p^1\cdot\underline{z}_a^2,$$

(H.1.3) $$p^1\cdot w^2-\frac{\gamma_b}{p^2\cdot(w^1-x_a^1)}\geq\frac{\gamma_a}{p^2\cdot x_a^1},$$

(H.3″) $p^1\cdot x_a^1=I_a^1$,

(H.4″) $w^1\geq x_a^1\geq 0$,

(H.5′) $I_a^r+I_b^r=p^r\cdot w^r,\quad r=1,2,$

or, equivalently, to: $\exists x_a^1$ such that

(H.1.1′) $$\frac{\gamma_w-(p^1\cdot\underline{z}_a^2)(p^2\cdot w^1)-\gamma_b}{(p^1\cdot w^2-p^1\cdot\underline{z}_a^2)}\geq p^2\cdot x_a^1\geq\frac{\gamma_a}{p^1\cdot\bar{z}_a^2},$$

(H.1.2′) $(p^1\cdot w^2)(p^2\cdot x_a^1)^2+(\gamma_b-\gamma_w-\gamma_a)(p^2\cdot x_a^1)+\gamma_a p^2\cdot w^1\leq 0,$

(H.3″) $p^1\cdot x_a^1=I_a^1,\quad r=1,2,$

(H.4″) $w^1\geq x_a^1\geq 0,\quad r=1,2,$

(H.5′) $I_a^r+I_b^r=p^r\cdot w^r,\quad r=1,2.$

Using the fact that $p^1\cdot(w^2-\underline{z}_a^2)=p^1\cdot\bar{z}_b^2$, (H.1.1′) can be written as

$$p^2\cdot w^1-\frac{\gamma_b}{p^1\cdot\bar{z}_b^2}\geq p^2\cdot x_a^1\geq\frac{\gamma_a}{p^1\cdot\bar{z}_a^2},$$

or, equivalently, as

(H.1.1″) $r_2\geq p^2\cdot x_a^1\geq r_1.$

The necessary and sufficient conditions for the existence of x_a^1 satisfying (H.1.1″), (H.1.2′), (H.3″), (H.4″), (H.5′) are:

(H.1*) $r_1\leq p^2\cdot\bar{z}_a^1,\quad p^2\cdot\underline{z}_a^1\leq r_2,\quad r_1\leq r_2,$

(H.2*) $\Psi_2=(\Psi_1)^2-4\gamma_a\gamma_w\geq 0,$

(H.3*) $t_1\leq p^2\cdot\bar{z}_a^1,\quad p^2\cdot\underline{z}_a^1\leq t_2,$

(H.4*) $I_a^r+I_b^r=p^r\cdot w^r,\quad r=1,2,$

or, equivalently, the conditions are

(H.I) $\Psi_2\geq 0,$

(H.II) $s_1\leq s_2,$

(H.III) $s_1\leq p^2\cdot\bar{z}_a^1,$

(H.IV) $p^2\cdot\underline{z}_a^1\leq s_2,$

(H.V) $I_a^1+I_b^1=p^1\cdot w^1$ and $I_a^2+I_b^2=p^2\cdot w^2,$

which is our Homothetic Axiom of Revealed Preference. Necessity is clear. To show sufficiency, note that (H.I)–(H.IV) imply that $\exists x_a^1$ satisfying (H.3″)–(H.4″) and $\max\{r_1, t_1\} \leq p^2 \cdot x_a^1 \leq \min\{r_2, t_2\}$. That such x_a^1 satisfies (H.1.1″) is obvious. That it satisfies (H.1.2′) follows because the function $f(t) = (t - t_1)(t - t_2)$ is such that $f(t) \leq 0$ for all $t \in [t_1, t_2]$ and (H.1.2′) can be written as $(p^2 \cdot x_a^1 - t_1)(p^2 \cdot x_a^1 - t_2) \leq 0$.

PROOF OF THEOREM (5): Let x^r and y^r denote, respectively, a consumption and production plan in observation r. If $\langle p^r, w^r \rangle_{r=1}^N$ satisfy CC, then $\langle p^r, x^r = w^r,\ y^r = 0 \rangle_{r=1,\dots,N}$ satisfy the Afriat inequalities for utility maximization and profit maximization (see Varian (1984)), and markets clear. Suppose that $\langle p^r, w^r \rangle_{r=1}^N$ does not satisfy CC but lies in the equilibrium manifold. Let x^r and y^r denote, respectively, any equilibrium consumption and equilibrium production plan in observation r. Since CC is violated, there exists $\{s, v, f, \dots, e\}$ such that

$$(5.1) \qquad p^s \cdot w^v \leq p^s \cdot w^s, \qquad p^v \cdot w^f \leq p^v \cdot w^v, \qquad \dots, \qquad p^e \cdot w^s \leq p^e \cdot w^e$$

where at least one of the inequalities is strict. Profit maximization ($p^s \cdot y^v \leq p^s \cdot y^s$, $p^v \cdot y^f \leq p^v \cdot y^v, \dots, p^e \cdot y^s \leq p^e \cdot y^e$) and markets clearing ($x^v = w^v + y^v$, $x^s = w^s + y^s$, $x^f = w^f + y^f, \dots,$ $x^e = w^e + y^e$) imply with (5.1) that

$$(5.2) \qquad p^s \cdot x^v \leq p^s \cdot x^s, \qquad p^v \cdot x^f \leq p^v \cdot x^v, \qquad \dots, \qquad p^e \cdot x^s \leq p^e \cdot x^e$$

where at least one of the inequalities is strict. Since (5.2) is inconsistent with utility maximization, a contradiction has been found.

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