# Identification of consumers' preferences when their choices are unobservable* 

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Summary. We provide conditions under which the heterogenous, deterministic preferences of consumers in a pure exchange economy can be identified from the equilibrium manifold of the economy. We extend those conditions to consider exchange economies, with two commodities, where consumers' preferences are random. For the latter, we provide conditions under which consumers' heterogenous random preferences can be identified from the joint distribution of equilibrium prices and endowments. The results can be applied to infer consumers' preferences when their demands are unobservable.

Keywords and Phrases: Preferences, Random utility, Pure exchange economies, Identification, Equilibrium correpsondence

JEL Classification Numbers: D12, D51
*Section 2 of this paper is joint work with Donald J. Brown; it is included here for publication with his permission. Those results were presented at the 1990 Workshop on Mathematical Economics at the University of Bonn, the 1992 SITE Workshop on Empirical Implications of General Equilibrium Models at Stanford University, and, more recently, at the June 2000 Conference in Honor of Rolf Mantel, in Buenos Aires, Argentina. The comments of the participants at those conferences and workshops are much appreciated. I am very grateful to an anonymous referee, Donald Brown, and Daniel McFadden for their detailed comments and insightful suggestions. The research presented in this paper was supported by NSF grants SES-8900291, SBR-9410182, SES-0241858, and BCS-0433990. This paper is dedicated to Marcel K. Richter, who has inspired much of my research.

## 1. Introduction

A large body of work in economics has dealt with aggregation of agents' behavior. The use of a representative consumer has been common in macroeconomics, due to its tractability, but, at the same time, it has been recognized that only very strong assumptions on the preferences of the consumers or on the distribution of incomes are consistent with such a model (See Gorman (1953), Samuelson (1956), Eisenberg (1961), Chipman (1974), Chipman and Moore (1979), and Polemarchakis (1983), for theoretical results. For relevant empirical approaches, see Lewbel (1989), Stoker (1993), and their references.) When such strong conditions are not satisfied, one may consider studying conditions that will guarantee only a particular aggregate behavior of interest. For example, the research initiated by Hildenbrand (1983), and followed by Chiappori (1985), Grandmont (1987, 1992), Marhuenda (1995), and Quah (2000), among others, provides conditions on the shape of the distribution of income or the shape of agents' characteristics under which aggregate demand is monotone in prices. Another alternative, is, of course, to study the full disaggregated model, which specifies an individual demand function for each consumer. This allows for general types of consumer demands and distributions of incomes, but requires much more knowledge about the consumers. In particular, most predictions derived from such a model would require being able to first identify the demand functions of each of the individuals in the economy.

The identification of underlying behavior from observable behavior has attracted the attention of economic theorists and econometricians for a long time. The theory of revealed preference, which studies whether an individual's choices are generated by the maximization of preferences within a certain type, integrability theory, which provides conditions under which one can identify individual preferences from individual demand functions, and the econometric problem of identifying structural equations from reduced form equations, are all examples of questions of this type that have attracted the attention of many economists. Thanks to their work, we now have methods that allow us to identify preferences of consumers and production technologies of firms purely from their individual market behavior, and we have methods to identify aggregate demand functions and aggregate supply functions from only equilibrium observations. This identification is essential if, for example, one wants to evaluate the change in a consumer's welfare due to some new income tax or a new price policy, or if one wants to predict changes in the production plans of a firm due to some new legislation, or if one wants to predict a new market equilibrium in some new environment. The fact that these underlying functions and relationships can be identified making use of restrictions derived from economic models, such as optimization behavior by the individuals and the firms, or market equilibrium conditions, provides strong evidence about the usefulness of economic models.

For some time, the power of optimization and equilibrium restrictions, in the sense described above, had been contrasted with the weakness of aggregation restrictions. The path-breaking work of Sonnenschein (1973,1974), together with Debreu (1974), Mantel (1974, 1976), McFadden, Mas-Colell, Mantel and Richter (1974), and later results by Mas-Colell (1977a), Geanakoplos and Polemarchakis (1980), Andreu (1983) and, more recently, Chiappori and Ekeland (1999) have been widely interpreted to mean that aggregate observable
behavior contains no strong implications, derived from the individual behavior that generated it. More specifically, their result is that, if the number of consumers is sufficiently large, any function satisfying some weak properties can be the aggregate demand function of an economy, or, in other words, the restrictions on individual demand that are generated by the optimization of individual preferences essentially vanish when these demand functions are aggregated. As Mas-Colell (1977a) showed, these results imply that any set of prices can be the equilibrium prices for some economy. This interpretation was challenged by Brown and Matzkin (1996), who showed that the restrictions of consumer demand that are generated by preference optimization are effectively translated into the equilibrium manifold of the economy. Brown and Matzkin (1996) showed that if individual endowments are observed, the aggregate behavior of the consumers satisfies restrictions that are derived from individual preference maximization. ${ }^{1}$ In an unpublished paper, Brown and Matzkin (1990) showed that given the equilibrium manifold of a pure exchange economy, one can identify the demand functions of all the consumers in the economy. Later work by Balasko (1999) and by Chiappori, Ekeland, Kubler, and Polemarchakis (2002) provided constructive ways of identifying the individual demand functions from the equilibrium manifold. Unlike Brown and Matzkin (1990), Balasko (1999) used the condition that one can observe equilibrium prices when the endowments of all but one individual are zero, and Chiappori, Ekeland, Kubler, and Polemarchakis (2002) restricted the individual demand functions to be differentiable in income.

These identification results show that, without observing the choices that individuals make, one can still identify their preferences, as long as their endowments are observed and the aggregation of their behavior is also observed. In fact, a stronger statement is true. To identify individual preferences one only needs to observe the incomes of the individuals and the aggregate behavior, e.g. the aggregate endowment and equilibrium price. Since in most economic situations it is much easier to observe the incomes of the consumers than to observe their endowments, these results are important for empirical work. One could combine these identification results with prior results about the existence of representative consumers to derive a model with a small number of community groups. In such a model, the behavior of each community group could be required to be consistent with the existence of a representative consumer for the group, but no restrictions would be imposed across representative consumers. The identification results in Brown and Matzkin (1990) could then be used to identify the preferences of each of the representative consumers using only observations on the aggregate endowment, the equilibrium prices, and the aggregate income of each community.

When one is interested in using observational data to apply these results, however, one typically encounters the problem that it is rarely the case that the primitives of an economy stay fixed across observations. Some unobservable random shock may affect consumer preferences, generating a distribution of equilibrium prices, instead of a deterministic set of prices. The relevant question of interest in this context is then whether one can identify the random demand or random preferences of the individuals from the distribution of equilibrium prices, when the distributions of the individual demands are not observable For the

[^0]case where a distribution of demand is observable, McFadden (1975, 2002), McElroy (1981), Brown and Walker (1989), Lewbel (1996), and Brown and Calsamiglia (2003) consider restrictions on the distribution of demand generated from a distribution of preferences, and, starting from Barten (1968), there is a substantial literature on the identification of the distribution of preferences from an observable distribution of demand. The latter literature includes Heckman (1974), Dubin and McFadden (1984), McElroy (1987), and recent work by Brown and Matzkin (1998) and Beckert (2000) ${ }^{2}$. For the case where the distribution of demands are unobservable, Carvajal (2002) considers restrictions on the distribution of equilibrium prices.

This paper has two objectives. The first objective is to present and develop the identification results for pure exchange deterministic economies of Brown and Matzkin (1990). The second objective is to develop identification results for stochastic economies, where the preferences of consumers are random. We present the results for deterministic economies in the next section. In Section 3, we present those for stochastic economies.

## 2. Deterministic Economies

In this section, we present identification results for pure exchange economies with nonrandom preferences. Since in many situations, consumers' incomes are easier to observe than consumer's endowments, we first express our identification results in terms of incomes. This requires defining the aggregate demand and the equilibrium correspondence over income tuples and aggregate endowments. Later on, we show that similar results can be obtained when the aggregate demand and the equilibrium correspondence are defined over tuples of individual endowments.

We consider an economy with $J$ consumers and $K$ commodities. To each commodity $k$, there corresponds a price $p_{k}$. We let $\Delta=\left\{p=\left(p_{1}, \ldots, p_{K}\right) \in R_{+}^{K} \mid \sum_{k=1}^{K} p_{k}=1\right\}$ denote the set of normalized prices, $\Upsilon \subset R_{+}$denote a set of incomes, and $\Upsilon^{J}=\prod_{j=1}^{J} \Upsilon$ denote a set of $J$-tuples of incomes. We will assume that to each consumer $j$, there corresponds a demand function $D_{j}: \Delta \times \Upsilon \rightarrow R_{+}^{K}$, which, for the time being, is defined just as a function that assigns to each price vector $p \in \Delta$ and income $I_{j} \in \Upsilon$, an element of the budget hyperplane $B(p, I)=\left\{x \in R_{+}^{K} \mid p \cdot x=I\right\}$. We let $Đ=\left(D_{1}, \ldots, D_{J}\right)$ denote the $J$-tuple of demand functions, and denote the aggregate demand function generated by $Đ$ by a function $\underline{D}$ : $\Delta \times \Upsilon^{J} \rightarrow R_{+}^{K}$, defined for each $\left(p, I_{1}, \ldots, I_{J}\right) \in \Delta \times \Upsilon^{J}$ by

[^1]$$
\underline{D}\left(p, I_{1}, \ldots, I_{J} ; Ð\right)=\sum_{j=1}^{J} D_{j}\left(p, I_{j}\right)
$$

The vector $\underline{\omega} \in R_{+}^{K}$ will denote the aggregate endowment. An equilibrium price for an economy with demand functions $Đ$ and aggregate endowment $\underline{\omega}$ is defined to be any $p \in \Delta$ such that for some $J$-tuple of endowment vectors $\left(\omega_{1}, \ldots, \omega_{J}\right) \in R_{+}^{J K}$
(1) $\quad \sum_{j=1}^{J} \omega_{j}=\underline{\omega} \quad \& \quad \underline{D}\left(p, p \cdot \omega_{1}, \ldots, p \cdot \omega_{J} ; Ð\right)=\underline{\omega}$

The equilibrium correspondence generated by $Đ$, which assigns to each vector of aggregate endowments and J-tuple of incomes the set of equilibrium prices, will be denoted by $\Gamma$ : $R_{+}^{K} \times \Upsilon^{J} \rightarrow \Delta$, and defined for all $\left(\underline{\omega}, I_{1}, \ldots, I_{J}\right) \in R_{+}^{K} \times \Upsilon^{J}$ by $\Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right)=\left\{p \in \Delta \mid\right.$ for $\left(\omega_{1}, \ldots, \omega_{J}\right) \in R_{+}^{J K}$ with $p \cdot \omega_{j}=I_{j}(j=1, \ldots, J),(1)$ is satisfied $\}$
(We allow for the possibility that $\Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right)$ is empty-valued, for some $\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right)$.)
The analysis of any type of identification result for an underlying function requires a specification of the set to which this function belongs. The specification may require properties such as continuity and differentiability, or some other type of restrictions. For example, any result about the identification of a utility function from a demand function will need to specify the utility function to belong to a set such that no two utility functions in that set are strictly increasing transformations of each other. For our identification result of individual demand functions, we will specify a set of $J$-tuples of demand functions $Đ=\left(D_{1}, \ldots, D_{J}\right)$ to be such that, for each consumer $j$, if $D_{j}$ and $D_{j}^{\prime}$ are in two $J$-tuples within this set and for some $\left(\widetilde{p}, \widetilde{I}_{j}\right)$ in their domain, $D_{j}\left(\widetilde{p}, \widetilde{I}_{j}\right) \neq D_{j}^{\prime}\left(\widetilde{p}, \widetilde{I}_{j}\right)$, then the income expansion path generated from $D_{j}$, when the price is $\widetilde{p}$, is not a translation of the income expansion path of $D_{j}^{\prime}$, when the price is $\widetilde{p}$. Clearly, we need to impose such a restriction to be able to identify each individual demand from observable variables that only depend on the sum of these individual demands. To see this, suppose that the set of allowable $J$ - tuples of demand functions includes $Đ=\left(D_{1}, \ldots, D_{J}\right)$ and $Ð^{\prime}=\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right)$ where $Ð^{\prime}$ is exactly the same as Đ , except that, at some value of $(p, I)$, and for some vector $a, D_{1}^{\prime}(p, I)=D_{1}(p, I)+a$ and $D_{2}^{\prime}(p, I)=D_{2}^{\prime}(p, I)-a$. Then, the aggregate demand generated from $Đ$ will be identical to that generated from $Ð^{\prime}$, even though $Đ \neq Ð^{\prime}$. We next formally specify this set, and then, in Theorem 1, we show that this condition is sufficient to identify the individual demand functions from either the aggregate demand or from the equilibrium correspondence, as defined above.

DEFINITION: $\Phi_{I}$ will denote the set of all $J$-tuples of demand functions, $\left(D_{1}, \ldots, D_{J}\right)$, such that for all $\left(D_{1}, \ldots, D_{J}\right),\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right)$ in $\Phi_{I}$, for all $j$ and all $p \in \Delta$, either there exits $I_{j} \in \Upsilon$ such that $D_{j}\left(p, I_{j}\right)=D_{j}^{\prime}\left(p, I_{j}\right)$ or there exist values $I_{j}, I_{j}^{\prime} \in \Upsilon$ such that

$$
\Phi_{I}(i): \quad D_{j}\left(p, I_{j}\right)-D_{j}\left(p, I_{j}^{\prime}\right) \neq D_{j}^{\prime}\left(p, I_{j}\right)-D_{j}^{\prime}\left(p, I_{j}^{\prime}\right)
$$

To see the restriction that this definition implies on the elements of $\Phi_{I}$, consider, for example, the subset of individual demand functions of the Gorman type

$$
\Theta_{j}=\left\{D_{j}(\cdot, \cdot ; a, b) \mid D_{j}\left(p, I_{j} ; a, b\right)=a(p)+b(p) I_{j}, \text { for some functions } a(\cdot) \in A, b(\cdot) \in B\right\}
$$

where $A$ and $B$ are set of functions defined on the set of prices. The definition of $\Phi_{I}$ implies that the subset of admissible demand functions $D_{j}$ within $\Theta_{j}$ is

$$
\bar{\Theta}_{j}=\left\{D_{j}\left(\cdot, \cdot ; a^{*}, b\right) \mid D_{j}\left(p, I_{j} ; a^{*}, b\right)=a^{*}(p)+b(p) I_{j}, \text { for some function } b(\cdot) \in B\right\}
$$

where $a^{*}$ is an element of $A$. In contrast to the standard results on aggregation of demand, the restriction is not imposed across the demands of the different consumers; instead, it is imposed across the set of demands permissible for any particular consumer.

In Theorem 1, we show that different demand tuples in $\Phi_{I}$ generate different aggregate demand functions and different equilibrium correspondences. In other words, this theorem shows that given an aggregate demand or an equilibrium correspondence, there is a unique $J$-tuple of demand functions in $\Phi_{I}$ that could have generated it.

THEOREM 1: If $Đ, Ð^{\prime} \in \Phi_{I}$ and $Đ \neq Ð^{\prime}$,
(1.i) there exists $\left(p, I_{1}, \ldots, I_{J}\right) \in \Delta \times I^{J}$ such that

$$
\underline{D}\left(p, I_{1}, \ldots, I_{J} ; Ð\right) \neq \underline{D}\left(p, I_{1}, \ldots, I_{J} ; Ð^{\prime}\right), \text { and }
$$

(1.ii) there exists $\left(\underline{\omega}, I_{1}, \ldots, I_{J}\right) \in R_{+}^{K} \times I^{J}$ such that

$$
\Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right) \neq \Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð^{\prime}\right)
$$

PROOF: Let $\mathrm{Đ}=\left(D_{1}, \ldots, D_{J}\right)$ and $\mathrm{\Xi}^{\prime}=\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right)$. Since $\mathrm{Đ} \neq \mathrm{Đ}^{\prime}$, there exists $j \in$ $\{1, \ldots, J\}, \widetilde{p} \in \Delta$, and $I_{j} \in \Upsilon$ such that $D_{j}\left(\widetilde{p}, I_{j}\right) \neq D_{j}^{\prime}\left(\widetilde{p}, I_{j}\right)$. Suppose, without loss of generality, that $j=1$. Then,
(1.1) $\quad D_{1}\left(\widetilde{p}, I_{1}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)$

Since $\left(D_{1}, \ldots, D_{J}\right),\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right) \in \Phi_{I}$, either for some $\widetilde{I}_{2} \in \Upsilon$

$$
\begin{equation*}
D_{2}\left(\widetilde{p}, \widetilde{I}_{2}\right)=D_{2}^{\prime}\left(\widetilde{p}, \widetilde{I}_{2}\right) \tag{1.2}
\end{equation*}
$$

or there exist $I_{2}, I_{2}^{\prime}$ such that

$$
\begin{equation*}
D_{2}\left(\widetilde{p}, I_{2}\right)-D_{2}\left(\widetilde{p}, I_{2}^{\prime}\right) \neq D_{2}^{\prime}\left(\widetilde{p}, I_{2}\right)-D_{2}^{\prime}\left(\widetilde{p}, I_{2}^{\prime}\right) \tag{1.3}
\end{equation*}
$$

If (1.2) holds, then

$$
D_{1}\left(\widetilde{p}, I_{1}\right)+D_{2}\left(\widetilde{p}, \widetilde{I}_{2}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)+D_{2}^{\prime}\left(\widetilde{p}, \widetilde{I}_{2}\right)
$$

If (1.3) holds, then either

$$
\begin{equation*}
D_{2}\left(\widetilde{p}, I_{2}\right)-D_{2}^{\prime}\left(\widetilde{p}, I_{2}\right) \neq D_{1}\left(\widetilde{p}, I_{1}\right)-D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right) \tag{1.4}
\end{equation*}
$$

or

$$
\begin{equation*}
D_{2}\left(\widetilde{p}, I_{2}^{\prime}\right)-D_{2}^{\prime}\left(\widetilde{p}, I_{2}^{\prime}\right) \neq D_{1}\left(\widetilde{p}, I_{1}\right)-D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right) \tag{1.5}
\end{equation*}
$$

Suppose w.l.o.g. that (1.4) holds, then

$$
\begin{equation*}
D_{1}\left(\widetilde{p}, I_{1}\right)+D_{2}\left(\widetilde{p}, I_{2}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)+D_{2}^{\prime}\left(\widetilde{p}, I_{2}\right) \tag{1.6}
\end{equation*}
$$

Hence, by (1.3) and (1.6), we have established the existence of $I_{2} \in \Upsilon$ such that

$$
D_{1}\left(\widetilde{p}, I_{1}\right)+D_{2}\left(\widetilde{p}, I_{2}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)+D_{2}^{\prime}\left(\widetilde{p}, I_{2}\right)
$$

Using the same argument, we can establish that there exist $I_{3}$ such that

$$
D_{1}\left(\widetilde{p}, I_{1}\right)+D_{2}\left(\widetilde{p}, I_{2}\right)+D_{3}\left(\widetilde{p}, I_{3}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)+D_{2}^{\prime}\left(\widetilde{p}, I_{2}\right)+D_{3}^{\prime}\left(\widetilde{p}, I_{3}\right)
$$

Continuing in this fashion, we can determine the existence of $I_{2}, I_{3}, \ldots, I_{J}$ and $I_{2}^{\prime}, I_{3}^{\prime}, \ldots, I_{J}^{\prime}$ such that

$$
\begin{equation*}
D_{1}\left(\widetilde{p}, I_{1}\right)+\sum_{j=2}^{J} D_{j}\left(\widetilde{p}, I_{j}\right) \neq D_{1}^{\prime}\left(\widetilde{p}, I_{1}\right)+\sum_{j=2}^{J} D_{j}^{\prime}\left(\widetilde{p}, I_{j}\right) \tag{1.7}
\end{equation*}
$$

Hence,

$$
\underline{D}\left(\widetilde{p}, I_{1}, \ldots, I_{J} ; Ð\right) \neq \underline{D}\left(\widetilde{p}, I_{1}, \ldots, I_{J} ; Ð^{\prime}\right)
$$

This proves (1.i).
To prove (1.ii), let $\omega_{j}=D_{j}\left(p, I_{j}\right) \quad(j=1, \ldots, J)$ and $\underset{\underline{\omega}}{ }=\sum_{j=1}^{J} D_{j}\left(p, I_{j}\right)$. Then, $\underline{\omega}=$ $\sum_{j=1}^{J} \omega_{j}$. By (1.7),

$$
\sum_{j=1}^{J} D_{j}\left(p, p \cdot \omega_{j}\right)=\underset{-}{\omega} \neq \sum_{j=1}^{J} D_{j}^{\prime}\left(p, p \cdot \omega_{j}\right)
$$

This implies that $p \in \Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right)$ and $p \notin \Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð^{\prime}\right)$. Hence

$$
\Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð\right) \neq \Gamma\left(\underline{\omega}, I_{1}, \ldots, I_{J} ; Ð^{\prime}\right)
$$

This completes the proof.

The above theorem does not require any particular assumptions on the preferences of the consumers. In fact, the individual demand functions are not even required to be generated by the maximization of some preferences. All that is required is that to each budget set, each individual demand assigns only one element, and that element is on the budget hyperplane.

In particular, and in contrast to the result in Chiappori, Ekeland, Kubler, and Polemarchakis (2002), demand functions do not need to be differentiable.

To compare our result with that in Balasko (1999), we note that, in the above theorem, the set $\Upsilon$ of incomes over which the demand functions are defined is not required to include the value 0 . In contrast, Balasko's argument is based on the fact that, when only one consumer has positive income, the equilibrium manifold is the inverse demand function of that consumer. Hence, being able to observe the equilibrium manifold when consumers are given 0 income is critical for his argument. Balasko's argument does not require restricting the set of possible demand functions like we did above when we defined the set $\Phi_{I}$. But, if we allow 0 to belong to $\Upsilon$, then the restriction $\left(\Phi_{I}(i)\right)$ is trivially satisfied, and $\Phi_{I}$ becomes the set of all demand functions. To see this, note that when $I_{j}^{\prime}=0$, condition $\left(\Phi_{I}(i)\right)$ is satisfied by any demand functions that are different. Hence, Theorem 1 implies that when $0 \in \Upsilon$, the individual demands can be identified from either the aggregate demand or from the equilibrium correspondence, without imposing any other restrictions.

It is easy to obtain a result analogous to that of Theorem 1, when the individual demands, the aggregate demand, and the equilibrium correspondence are defined over the set of individual endowments, instead of over incomes. Let $W \subset R_{+}^{K}$ denote a set of individual endowments and denote $\Pi_{j=1}^{J} W$ by $W^{J}$. Abusing notation, we will now define for each consumer $j$, the demand function $D_{j}: \Delta \times W \rightarrow R_{+}^{K}$, as a function which assigns to each price vector $p \in \Delta$ and endowment vector $\omega_{j} \in W$, an element of the budget hyperplane $B\left(p, \omega_{j}\right)=\left\{x \in R_{+}^{K} \mid p \cdot x=p \cdot \omega_{j}\right\}$. We let $Đ=\left(D_{1}, \ldots, D_{J}\right)$ denote the $J$-tuple of demand functions, and define the aggregate demand function generated by $Đ$ by

$$
\underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; Ð\right)=\sum_{j=1}^{J} D_{j}\left(p, \omega_{j}\right)
$$

We say that $p$ is an equilibrium price if

$$
\underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; Ð\right)=\sum_{j=1}^{J} \omega_{j}
$$

and we define the (possibly empty-valued) equilibrium correspondence $\Gamma: W^{J} \rightarrow \Delta$ for all $\left(\omega_{1}, \ldots, \omega_{J}\right) \in W^{J}$ by

$$
\Gamma\left(\omega_{1}, \ldots, \omega_{J} ; Ð\right)=\left\{p \in \Delta \mid \underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; Ð\right)=\sum_{j=1}^{J} \omega_{j}\right\}
$$

DEFINITION: $\Phi_{\omega}$ will denote the set of all $J$-tuples of demand functions, $\left(D_{1}, \ldots, D_{J}\right)$, such that for all $\left(D_{1}, \ldots, D_{J}\right),\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right)$ in $\Phi_{\omega}$, for all $j$ and $\widetilde{p} \in \Delta$, either there exists $\omega_{j} \in W$ such that $D_{j}\left(\widetilde{p}, \omega_{j}\right)=D_{j}^{\prime}\left(\widetilde{p}, \omega_{j}\right)$ or there exist vectors $\omega_{j}, \omega_{j}^{\prime} \in W$, such that

$$
\Phi_{\omega}(i): \quad D_{j}\left(\widetilde{p}, \omega_{j}\right)-D_{j}\left(\widetilde{p}, \omega_{j}^{\prime}\right) \neq D_{j}^{\prime}\left(\widetilde{p}, \omega_{j}\right)-D_{j}^{\prime}\left(\widetilde{p}, \omega_{j}^{\prime}\right)
$$

(Note that if $0 \in W$, then $\Phi_{\omega}$ is the set all demand functions, since then for all $D_{j}, D_{j}^{\prime}$, $\left.D_{j}(\widetilde{p}, 0)=D_{j}^{\prime}(\widetilde{p}, 0).\right)$.

Then, using the arguments in the proof of Theorem 1 and the fact that if we let $\omega_{j}^{\prime}=$ $D_{j}\left(p, \omega_{j}\right)$, then $D_{j}\left(p, \omega_{j}\right)=D_{j}\left(p, \omega_{j}^{\prime}\right)$ and $D_{j}^{\prime}\left(p, \omega_{j}^{\prime}\right)=D_{j}^{\prime}\left(p, \omega_{j}\right)$, for any $p, \omega_{j}, D_{j}$, and $D_{j}^{\prime}$, we have

THEOREM 2: If $Đ, \boxplus^{\prime} \in \Phi_{\omega}$ and $Đ \neq Ð^{\prime}$,
(2.i) there exists $\left(p, \omega_{1}, \ldots, \omega_{J}\right) \in \Delta \times W^{J}$ such that

$$
\underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; Ð\right) \neq \underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; Ð^{\prime}\right), \quad \text { and }
$$

(2.ii) there exists $\left(\omega_{1}, \ldots, \omega_{J}\right) \in R_{+}^{K} \times W^{J}$ such that

$$
\Gamma\left(\omega_{1}, \ldots, \omega_{J} ; Ð\right) \neq \Gamma\left(\omega_{1}, \ldots, \omega_{J} ; Ð^{\prime}\right)
$$

Theorems 1 and 2 show that consumers demands can be identified from aggregate behavior. From the individual demands we can identify the preferences of each of the individuals, imposing additional restrictions. We establish this in the following theorem. Assume that $W=R_{+}^{K}$. Let $\Phi_{\succcurlyeq}$ denote the set of all $J-$ tuples $\succcurlyeq=\left(\succcurlyeq_{1}, \ldots, \succcurlyeq_{J}\right)$ of preference relations on $R_{+}^{K} \times R_{+}^{K}$ that generate a $J$-tuple in $\Phi_{\omega}$ and are such that (i) for each $j, \succcurlyeq_{j}$ can be represented by a monotone, continuous, concave, and strictly quasiconcave utility function, and (ii) the set of all bundles in the range of the demand function generated by $\succcurlyeq_{j}$ is $R_{+}^{K}$. For each $\succeq \in \Phi_{\succcurlyeq}$, let $\underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; \succcurlyeq\right)$ and $\Gamma\left(\omega_{1}, \ldots, \omega_{J} ; \succcurlyeq\right)$ denote, respectively, the aggregate demand and the value of the equilibrium correspondence generated by $\succeq$. Then

THEOREM 3: If $\succeq, \succcurlyeq^{\prime} \in \Phi_{\succcurlyeq}$ and $\succeq \neq \succcurlyeq^{\prime}$,
(3.i) there exists $\left(p, \omega_{1}, \ldots, \omega_{J}\right) \in \Delta \times W^{J}$ such that

$$
\underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; \succcurlyeq\right) \neq \underline{D}\left(p, \omega_{1}, \ldots, \omega_{J} ; \succcurlyeq^{\prime}\right), \quad \text { and }
$$

(3.ii) there exists $\left(\omega_{1}, \ldots, \omega_{J}\right) \in R_{+}^{K} \times W^{J}$ such that

$$
\Gamma\left(\omega_{1}, \ldots, \omega_{J} ; \succcurlyeq\right) \neq \Gamma\left(\omega_{1}, \ldots, \omega_{J} ; \succcurlyeq \prime\right)
$$

PROOF: Let $\succeq, \succeq^{\prime} \in \Phi_{\succcurlyeq}$ be such that $\succeq \neq \succeq^{\prime}$, and for each $j$, let $D_{j}$ and $D_{j}^{\prime}$ denote the demand functions generated, respectively, by $\succcurlyeq_{j}$ and $\succcurlyeq_{j}^{\prime}$. Since $\succcurlyeq \neq \succcurlyeq^{\prime}$, for at least one $j \in\{1, \ldots, J\}, \succcurlyeq_{j} \nsucccurlyeq_{j}^{\prime}$. Since, concavifiable preferences are lipschitizian (see Corollary in Mas-Colell (1977b, pp. 1412)) and since $\succcurlyeq_{j}$ and $\succcurlyeq_{j}^{\prime}$ can be represented by monotone, continuous, concave, and strictly quasiconcave utility functions, it follows from Theorem 2 in Mas-Colell (1977b, pp. 1413) that $D_{j} \neq D_{j}^{\prime}$. Hence, $\left(D_{1}, \ldots, D_{J}\right) \neq\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right)$, and, by assumption, $\left(D_{1}, \ldots, D_{J}\right),\left(D_{1}^{\prime}, \ldots, D_{J}^{\prime}\right) \in \Phi_{\omega}$. The statements in (3.i) and (3.ii) then follow by Theorem 2 .

A similar result can be obtained, using Theorem 1, when the aggregate demand and the equilibrium correspondence are defined over a vector of aggregate endowments and a $J$-tuple of incomes.

To illustrate the results in this section, we next consider the equilibrium function of an economy with two commodities, 1 and 2 , and two individuals, A and B , which possess Cobb-Douglas utility functions. We show that, when it is known a-priori that the utility functions are Cobb-Douglas, but the values of the parameters of the Cobb-Douglas utilities are unknown, one can identify the values of those parameters when the equilibrium price is observed at only two points of its domain. More specifically, suppose that it is known that the utility function of individual $A$ is $U^{A}\left(x_{1}, x_{2}\right)=x_{1}^{\alpha} x_{2}^{1-\alpha}$, for some unknown value of $\alpha$, and the utility function of individual $B$ is $U^{B}\left(x_{1}, x_{2}\right)=x_{1}^{\beta} x_{2}^{1-\beta}$, for some unknown value of $\beta$. Suppose that the equilibrium price is observed at two points, $\left(I^{A}, I^{B}, \omega\right)$ and $\left(\bar{I}^{A}, \bar{I}^{B}, \bar{\omega}\right)$, such that

$$
\operatorname{det}\left[\begin{array}{cc}
I^{A} & I^{B} \\
\bar{I}^{A} & \bar{I}^{B}
\end{array}\right] \neq 0
$$

where $I^{A}$ and $\bar{I}^{A}$ denote values for the income of individual A, $I^{B}$ and $\bar{I}^{B}$ denote values for the income of individual $B$, and $\omega$ and $\bar{\omega}$ denote values for the vector of aggregate endowment. Normalize the price of the second commodity to 1 . Let $p_{1}$ denote the equilibirum price at $\left(I^{A}, I^{B}, \omega\right)$ and let $\bar{p}_{1}$ denote the equilibrium price at $\left(\bar{I}^{A}, \bar{I}^{B}, \bar{\omega}\right)$. Using the properties of Cobb-Douglass utility functions, it is easy to verify that $p_{1},\left(I^{A}, I^{B}, \omega\right)$ and $\bar{p}_{1},\left(\bar{I}^{A}, \bar{I}^{B}, \bar{\omega}\right)$ satisfy

$$
p_{1} \cdot \omega_{1}=\alpha \cdot I^{A}+\beta \cdot I^{B}
$$

and

$$
\bar{p}_{1} \cdot \bar{\omega}_{1}=\alpha \cdot \bar{I}^{A}+\beta \cdot \bar{I}^{B}
$$

It is then clear that there is a unique solution for $\alpha$ and $\beta$.

## 3. Random economies

In many cases, and in particular when one is dealing with real data, there are random elements that affect the preferences of the consumers and which therefore generate a distribution of prices, for a same vector of endowments. In this section, we show how the results in the previous section can be extended to guarantee identification of preferences from the equilibrium prices when these preferences are random. We consider economies with 2 commodities, where we normalize the price of the second commodity to equal 1 . We will restrict the set of individual demand functions that we consider to be such that the demand for the first commodity is strictly decreasing in its price, $p$. Under these conditions, for every realization of the random elements, an equilibrium price, if it exists, is unique. Hence, we will be dealing with an equilibrium function instead of an equilibrium correspondence, as in Section 2. We concentrate on the case where the individual demand functions, the aggregate demand function, and the equilibrium correspondence are all defined on a set of $J$-tuples of individual endowments. To incorporate randomness into the model, we will assume that the preferences of the consumers in an economy depend on unobservable variables. We will show that, in this case, one can still identify the demand functions of each of the individual consumers, when their choices are not observed. This will require either specifying the distribution of the unobservable variables, or restricting the way in which the demand functions depend on these unobservable variables.

We consider first the case where the random shock is univariate and affects, in not necessarily the same way, the demand of all the consumers in the economy. Let $E \subset R$ denote the support of an unobservable random variable, $\varepsilon$. We define a random demand function $\widetilde{D}: R_{+} \times W \times E \rightarrow R_{+}^{K}$ to be any function that assigns, to each price $p \in R_{+}$, endowment vector $\omega_{j} \in W$, and realization of $\varepsilon$, an element in the budget hyperplane $B(p, I)$ $=\left\{x \in R_{+}^{K} \mid p \cdot x=I\right\}$. We let $\widetilde{Ð}=\left(\widetilde{D}_{1}, \ldots, \widetilde{D}_{J}\right)$ denote the $J$-tuple of random demand functions, and we denote the aggregate random demand function generated by Ð by the function $\underline{\widetilde{D}}: R_{+} \times W^{J} \times E \rightarrow R_{+}^{K}$, defined for each $\left(p, \omega_{1}, \ldots, \omega_{J}, \varepsilon\right) \in R_{+} \times W^{J} \times E$ by

$$
\underline{\widetilde{D}}\left(p, \omega_{1}, \ldots, \omega_{J}, \varepsilon ; \widetilde{Ð}\right)=\sum_{j=1}^{J} \widetilde{D}_{j}\left(p, \omega_{j}, \varepsilon\right)
$$

An equilibrium price for an economy with demand functions $\widetilde{Ð}$, J-tuple of endowment vectors $\left(\omega_{1}, \ldots, \omega_{J}\right)$, and realization of the unobservable random variable $\varepsilon$, is defined to be any $p \in R_{+}$such that

$$
\underline{\widetilde{D}}\left(p, \omega_{1}, \ldots, \omega_{J,} \varepsilon ; \widetilde{Ð}\right)=\sum_{j=1}^{J} \omega_{j}
$$

The (possibly empty-valued) random equilibrium correspondence generated by $\widetilde{\text { Đ , which to }}$ each $\left(p, \omega_{1}, \ldots, \omega_{J,} \varepsilon\right)$ assigns the set of equilibrium prices will be denoted by $\widetilde{\Gamma}: W^{J} \times E \rightarrow R_{+}$. This correspondence, together with any specified distribution $F_{\varepsilon,\left(\omega_{1}, \ldots, \omega_{J}\right)}$ of the unobservable random shock and the $J$-tuple of endowment vectors, generates a distribution $F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}$ of the equilibrium price and the $J$-tuple of endowment vectors. We want to determine whether,
from $F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}$, we can identify the $J$-tuple of individual random demand functions that generated $F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}$. Clearly, this is a more demanding result than the one in Section 2, where $\varepsilon$ and therefore $p$ had degenerate distributions, conditional on $\left(\omega_{1}, \ldots, \omega_{J}\right)$. In this new case, $\varepsilon$ is unobservable, and the individual demand functions depend on it. We will assume, throughout, that $\varepsilon$ is distributed independently of $\left(\omega_{1}, . ., \omega_{J}\right)$ and that the support of the distribution of $\left(\omega_{1}, . ., \omega_{J}\right)$ is $W^{J}$. Since even for the case where individual choices are observed and monotonicity properties are imposed, one can not jointly identify the distribution of $\varepsilon$ and the demand function without any further restrictions (Matzkin (2003)), we will need to make some additional assumptions to achieve our results. For any $J$-tuple of random demand functions $\widetilde{Ð}$, let $F_{p \mid\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}, F_{\varepsilon}\right)$ denote the conditional distribution of the equilibrium price $p$ generated by $\widetilde{Ð}$ and $F_{\varepsilon}$. We will impose restrictions that will either specify the distribution for $\varepsilon$, or will specify a restriction in the way that the demand functions depend on the unobservable shock. Our first identification result will assume that $\varepsilon$ is distributed independently of $\left(\omega_{1}, \ldots, \omega_{J}\right)$, with a specified distribution $F_{\varepsilon}$ that possesses a continuous density $f_{\varepsilon}$.

DEFINITION: $\Phi_{\omega, \varepsilon}$ will denote the set of $J$-tuples of continuous random demand functions, $\widetilde{Ð}=\left(\widetilde{D}_{1}, \ldots, \widetilde{D}_{J}\right)$, such that
$\Phi_{\omega, \varepsilon}(i)$ : For each $j$, the first coordinate of $\widetilde{D}_{j}$ is continuous in $\left(p, \omega_{j}, \varepsilon\right)$, strictly decreasing in $p$, and strictly increasing in $\varepsilon$, and
$\Phi_{\omega, \varepsilon}(i i):$ For all $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$, for all $j$, for all $(p, \varepsilon)$, either for some $\omega_{j} \in W$,

$$
\widetilde{D}_{j}\left(p, \omega_{j}, \varepsilon\right)=\widetilde{D}_{j}^{\prime}\left(p, \omega_{j}, \varepsilon\right)
$$

or there exist $\omega_{j}, \omega_{j}^{\prime} \in W$ such that

$$
\widetilde{D}_{j}\left(p, \omega_{j}, \varepsilon\right)-\widetilde{D}_{j}\left(p, \omega_{j}^{\prime}, \varepsilon\right) \neq \widetilde{D}_{j}^{\prime}\left(p, \omega_{j}, \varepsilon\right)-\widetilde{D}_{j}^{\prime}\left(p, \omega_{j}^{\prime}, \varepsilon\right)
$$

Condition $\left(\Phi_{\omega, \varepsilon}(i)\right)$ is made to guarantee the uniqueness of the equilibrium price, for any given $J$-tuple of endowments and value of $\varepsilon$, and the monotonicity in $\varepsilon$ of the equilibrium price, for any given $J$-tuple of endowments. Condition $\left(\Phi_{\omega, \varepsilon}(i i)\right)$ is a condition similar to $\left(\Phi_{\omega}(i)\right)$. It is used to eliminate from the set of possible $J$-tuples those that possess demand functions that generate income expansion paths that are translations of each other. Note that when $0 \in W, \Phi_{\omega, \varepsilon}$ consists of the set of all $J$-tuples of random demand functions satisfying only $\left(\Phi_{\omega, \varepsilon}(i)\right)$, since $\left(\Phi_{\omega, \varepsilon}(i i)\right)$ will always be satisfied by letting $\omega_{j}^{\prime}=0$. The following theorem shows that, under these conditions, we can identify the random demand functions of each of the consumers in an economy from the distribution of the equilibrium prices, conditional on the vector of individual endowments.

THEOREM 4: Suppose that $\varepsilon$ is distributed independently of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ with a specified distribution $F_{\varepsilon}$, which possesses a continuous density, $f_{\varepsilon}$, and whose support is the bounded set E. Suppose that the distribution of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ has support $W^{J}$. Then, if $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$ and $\widetilde{Ð} \neq \widetilde{Ð}^{\prime}$

$$
F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)
$$

PROOF: Suppose that $\widetilde{Đ}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$ and $\widetilde{Đ} \neq \widetilde{Ð}^{\prime}$. Then, for some $j$ and some $\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right) \in$ $R_{+} \times W \times E, \widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right) \neq \widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)$. By the continuity of the demand functions in $\varepsilon$, we can assume that $f_{\varepsilon}(\widetilde{\varepsilon})>0$. Suppose, without loss of generality that $j=1$. Then,

$$
\widetilde{D}_{1}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}\right) \neq \widetilde{D}_{1}^{\prime}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}\right)
$$

By the definition of $\Phi_{\omega, \varepsilon}$, either for some $\omega_{2}, \widetilde{D}_{2}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right)=\widetilde{D}_{2}^{\prime}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right)$, or for some $\omega_{2}, \omega_{2}^{\prime}$, $\widetilde{D}_{2}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right)-\widetilde{D}_{2}\left(\widetilde{p}, \omega_{2}^{\prime}, \widetilde{\varepsilon}\right) \neq \widetilde{D}_{2}^{\prime}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right)-\widetilde{D}_{2}^{\prime}\left(\widetilde{p}, \omega_{2}^{\prime}, \widetilde{\varepsilon}\right)$. In either case, we can establish the existence of a $\omega_{2}$ such that

$$
\widetilde{D}_{1}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}\right)+\widetilde{D}_{2}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right) \neq \widetilde{D}_{1}^{\prime}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}\right)+\widetilde{D}_{2}\left(\widetilde{p}, \omega_{2}, \widetilde{\varepsilon}\right)
$$

Continuing in this fashion, we can find $\omega_{1}, \omega_{2}, \ldots, \omega_{J}$ such that

$$
\sum_{j=1}^{J} \widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right) \neq \sum_{j=1}^{J} \widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)
$$

Suppose, without loss of generality, that

$$
\sum_{j=1}^{J} \widetilde{D}_{j}^{(1)}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)<\sum_{j=1}^{J} \widetilde{D}_{j}^{\prime(1)}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)
$$

where $\widetilde{D}_{j}^{(1)}$ and $\widetilde{D}_{j}^{\prime(1)}$ denote the first coordinates of, respectively, $\widetilde{D}_{j}$ and $\widetilde{D}_{j}^{\prime}$. For each $j$, let $\widetilde{\omega}_{j}=\widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)$. Then, since $\widetilde{p} \cdot \widetilde{\omega}_{j}=\widetilde{p} \cdot \widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)=\widetilde{p} \cdot \omega_{j}$,

$$
\widetilde{D}_{j}\left(\widetilde{p}, \widetilde{\omega}_{j}, \widetilde{\varepsilon}\right)=\widetilde{\omega}_{j}=\widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right) \quad \text { and } \quad \widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \widetilde{\omega}_{j}, \widetilde{\varepsilon}\right)=\widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}\right)
$$

Hence, when the endowment vector is $\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)$ and the value of the random shock is $\widetilde{\varepsilon}, \widetilde{p}$ is an equilibrium price when the $J$-tuple of demand functions is $\widetilde{Ð}$ and $\widetilde{p}$ is not an equilibrium price when the $J$-tuple of demand functions is $\widetilde{Ð}^{\prime}$. Since for any $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$, the first coordinate of the random aggregate demand function generated from any $J$-tuple of demand functions in $\Phi_{x, \varepsilon}$ is strictly decreasing in the price of the first commodity, the equilibrium price, if it exists, is unique, given any J-tuple of endowment vectors and any value of the unobservable random term. By the continuity of the $\widetilde{D}_{j}^{\prime(1)}$ functions in $\varepsilon$, the continuity of $f_{\varepsilon}$, and the fact that $f_{\varepsilon}(\widetilde{\varepsilon})>0$, it follows that there exists a neighborhood of $\widetilde{\varepsilon}$ in $E$ such that for all values $\varepsilon^{\prime}$ in that neighborhood, $f_{\varepsilon}\left(\varepsilon^{\prime}\right)>0$ and

$$
\sum_{j=1}^{J} \widetilde{\omega}_{j}^{(1)}<\sum_{j=1}^{J} \widetilde{D}_{j}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{j}, \varepsilon^{\prime}\right)
$$

For any $\widetilde{Ð} \in \Phi_{\omega, \varepsilon}$ and any $p \in R_{+}$and $\omega=\left(\omega_{1}, \ldots, \omega_{J}\right)$ define
$e(p, \omega ; \widetilde{Ð})=\left\{\begin{array}{c}\sup \left\{\varepsilon \in E \mid \sum_{j=1}^{J} \widetilde{D}_{j}^{(1)}\left(p, \omega_{j}, \varepsilon\right) \leq \sum_{j=1}^{J} \omega_{j}\right\} \\ \text { if }\left\{\varepsilon \in E \mid \sum_{j=1}^{J} \widetilde{D}_{j}^{(1)}\left(p, \omega_{j}, \varepsilon\right) \leq \sum_{j=1}^{J} \omega_{j}\right\} \neq \varnothing \\ \inf (E) \\ \text { otherwise }\end{array}\right.$

Then, $e(p, \omega ; \widetilde{Ð})$ denotes the value of $\varepsilon$ for which $p$ is an equilibrium price when the vector of endowments is $\omega$, if such a value exists; it equals $\inf (E)$ if for all values of $\varepsilon$ in $E$, $\sum_{j=1}^{J} \omega_{j}<\sum_{j=1}^{J} \widetilde{D}_{j}^{(1)}\left(\widetilde{p}, \omega_{j}, \varepsilon\right) ;$ and it equals $\sup \left\{\varepsilon \in E \mid \sum_{j=1}^{J} \widetilde{D}_{j}^{(1)}\left(p, \omega_{j}, \varepsilon\right)<\sum_{j=1}^{J} \omega_{j}\right\}$ otherwise.

Since $f_{\varepsilon}(\widetilde{\varepsilon})>0$, the first coordinate of the aggregate demand generated by $\widetilde{Ð}^{\prime}$ is strictly increasing in the value of the unobservable variable, and, from above, $\sum_{j=1}^{J} \widetilde{\omega}_{j}^{(1)}<\sum_{j=1}^{J} \widetilde{D}_{j}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{j}, \varepsilon^{\prime}\right)$ for all $\varepsilon^{\prime}$ in a neighborhood of $\widetilde{\varepsilon}$, it follows that $e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right)<\varepsilon^{\prime}<\widetilde{\varepsilon}$ for all $\varepsilon^{\prime}$ in a neighborhood that possesses positive probability. By the definition of $e(p, \omega ; \widetilde{Ð})$ and the fact that $\widetilde{p}$ is the equilibrium price when the endowment vector is $\widetilde{\omega}$, the value of $\varepsilon$ is $\widetilde{\varepsilon}$, and the vector of demand functions is $\widetilde{Ð}$, it follows that $e(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð})=\widetilde{\varepsilon}$. Hence,

$$
\widetilde{\varepsilon}=e(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð})>e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right)
$$

and

$$
\operatorname{Pr}(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}))>\operatorname{Pr}\left(\varepsilon \leq e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right)\right)
$$

Note that

$$
\begin{aligned}
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right) & =\operatorname{Pr}\left(p \leq \widetilde{p} \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}) \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon \leq \widetilde{\varepsilon} ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =F_{\varepsilon}(\widetilde{\varepsilon})
\end{aligned}
$$

and

$$
\begin{aligned}
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) & =\operatorname{Pr}\left(p \leq \widetilde{p} \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon \leq e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right) \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon \leq e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =F_{\varepsilon}\left(e\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right)\right)
\end{aligned}
$$

where $F_{\left.p \mid \widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right)$ and $F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)$ are the conditional distributions of the equilibrium price, given $\omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)$, when the $J$-tuple of demand functions are, respectively, $\widetilde{Ð}$ and $\widetilde{Ð}^{\prime}$. Hence, it follows that

$$
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \tilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \tilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)
$$

This completes the proof.

Theorem 4 showed that when the distribution of $\varepsilon$ is specified, we can identify the individual random demand functions from the distribution of the equilibrium price, conditional on the $J$-tuple of endowment vectors. The next theorem relaxes this assumption. Instead, a restriction on the demand functions is imposed. To describe this restriction, we will denote for each $j, \omega_{j}=\left(\omega_{1, j}, \omega_{2, j}\right)$, and we will define the set $\widetilde{W}$ by

$$
\widetilde{W}=\left\{\left(\omega_{1, j}, t\right) \mid \text { for some } \varepsilon \in E \text { and } \omega_{2, j}, \quad\left(\omega_{1, j}, \omega_{2, j}\right) \in W \text { and } t=\omega_{2, j}-\varepsilon\right\}
$$

DEFINITION: $\Phi_{\omega, \varepsilon}^{\prime}$ will denote the set of $J$-tuples $\widetilde{Ð}=\left(\widetilde{D}_{1}, \ldots, \widetilde{D}_{J}\right)$ of continuous random demand functions, $\widetilde{D}_{j}: R_{+} \times \widetilde{W} \rightarrow R_{+}^{K}$ such that
$\Phi_{\omega, \varepsilon}^{\prime}(i)$ : For each $j$, the first coordinate of $\widetilde{D}_{j}$ is strictly decreasing in $p$ and strictly increasing in $\omega_{2, j}-\varepsilon$,
$\Phi_{\omega, \varepsilon}^{\prime}(i i)$ : For all $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$, all $\underset{\widetilde{D}}{ }$, and all $(p, \varepsilon)$, either for some $\omega_{j} \in W$,

$$
D_{j}\left(p, \omega_{1, j}, \omega_{2, j}-\varepsilon\right)=\widetilde{D}_{j}^{\prime}\left(p, \omega_{1, j}, \omega_{2, j}-\varepsilon\right),
$$

or there exist $\omega_{j}, \omega_{j}^{\prime} \in W$ such that

$$
\widetilde{D}_{j}\left(p, \omega_{1, j}, \omega_{2, j}-\varepsilon\right)-\widetilde{D}_{j}\left(p, \omega_{1, j}^{\prime}, \omega_{2, j}^{\prime}-\varepsilon\right) \neq \widetilde{D}_{j}^{\prime}\left(p, \omega_{1, j}, \omega_{2, j}-\varepsilon\right)-\widetilde{D}_{j}^{\prime}\left(p, \omega_{1, j}^{\prime}, \omega_{2, j}^{\prime}-\varepsilon\right)
$$

$\Phi_{\omega, \varepsilon}^{\prime}(i i i):$ For some $\bar{p} \in R_{+}$, there exist, for all $j,\left(\bar{\omega}_{1, j}, \bar{t}_{j}\right) \in \widetilde{W}$ such that for all $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$,

$$
\widetilde{D}_{j}\left(\bar{p}, \bar{\omega}_{1 j}, \bar{t}_{j}\right)=\widetilde{D}_{j}^{\prime}\left(\bar{p}, \bar{\omega}_{1 j}, \bar{t}_{j}\right)
$$

Condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i)\right)$ is analogous to condition $\left(\Phi_{\omega, \varepsilon}(i)\right)$. It is made to guarantee the uniqueness of the equilibrium price, for any given $J$-tuple of endowments and value of $\varepsilon$, and the monotonicity in $\varepsilon$ of the equilibrium price, for any given $J$-tuple of endowments. Note that the monotonicity of the equilibrium price in $\varepsilon$ is decreasing. Condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i i)\right)$ involves two types of restrictions. The first is analogous to condition $\left(\Phi_{\omega, \varepsilon}(i i)\right)$ in that it eliminates from the set $\Phi_{\omega, \varepsilon}^{\prime}$ any $J$-tuples with demand functions that generate income expansion paths that are translations of each other. The second restriction imposes a particular type of weak separability in the demand function. If, for example, the preferences of each consumer $j$ are represented by a utility function of the form $U_{j}\left(x_{1}, x_{2}-\varepsilon\right)$, then, it is easy to verify that when the price of $x_{2}$ is normalized to 1 , the demand function generated from this utility function will satisfy the special type of weak separability required in condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i i)\right)$. Condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i i i)\right)$ fixes the values of the demand functions of each consumer at one point. If, for each $j$, we could observe the distribution of choices made by $j$, given $p$ and $\omega_{j}$, then condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i i i)\right)$ together with the special type of weak separability condition imposed in $\left(\Phi_{\omega, \varepsilon}(i i)\right)$ and the monotonicity with respect to $\omega_{2, j}-\varepsilon$ imposed in condition $\left(\Phi_{\omega, \varepsilon}^{\prime}(i)\right)$ would be enough to identify the distribution of $\varepsilon$ and the demand function $\widetilde{D}_{j}$ (see Matzkin (2003)). Since, in our case, the distribution of consumer $j^{\prime} s$ choices is not observed, we need to require the additional conditions on the set $\Phi_{\omega, \varepsilon}^{\prime}$. The following theorem establishes that, from the joint distribution of equilibrium prices and $J$-tuples of
endowment vectors, we can identify the distribution of $\varepsilon$ and the random demand functions of each of the consumers in the economy.

THEOREM 5: Suppose that $\varepsilon$ is distributed independently of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ with an unknown distribution function, $F_{\varepsilon}$, which possesses a continuous density, $f_{\varepsilon}$, and whose support is the bounded set $E$. Suppose that the the distribution of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ has support $W^{J}$. Then, if $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}^{\prime}$ and either $\widetilde{Ð} \neq \widetilde{Ð}^{\prime}$ or $F_{\varepsilon} \neq F_{\varepsilon}^{\prime}$ (or both)

$$
F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}^{\prime}, F_{\varepsilon}^{\prime}\right)
$$

PROOF: Suppose first that $F_{\varepsilon} \neq F_{\varepsilon}^{\prime}$. Then, for some $\widetilde{\varepsilon} \in E, F_{\varepsilon}(\widetilde{\varepsilon}) \neq F_{\varepsilon}^{\prime}(\widetilde{\varepsilon})$. By $\Phi_{\omega, \varepsilon}^{\prime}(\mathrm{iii})$,

$$
\sum_{j=1}^{J} \widetilde{D}_{j}\left(\bar{p}, \bar{\omega}_{1, j}, \bar{t}_{j}\right)=\sum_{j=1}^{J} \widetilde{D}_{j}^{\prime}\left(\bar{p}, \bar{\omega}_{1, j}, \bar{t}_{j}\right)
$$

Hence, $\bar{p}$ is an equilibrium price generated from both, $\widetilde{Ð}, \widetilde{Ð}^{\prime}$, given $\widetilde{\varepsilon}$ and the endowment vec$\operatorname{tor}\left(\omega_{1}, \ldots, \omega_{J}\right)=\left(\left(\bar{\omega}_{1,1}, \bar{\omega}_{2,1}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{\omega}_{2, J}\right)\right)=\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right) . \operatorname{By} \Phi_{\omega, \varepsilon}^{\prime}(\mathrm{i})$ the equilibrium price is unique and decreasing in the value of $\varepsilon$. Hence,

$$
\begin{aligned}
F_{p \mid\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right)}\left(\bar{p} ; \widetilde{Ð}, F_{\varepsilon}\right) & =\operatorname{Pr}\left(p \leq \bar{p} \mid \omega=\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right) ; \widetilde{Đ}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon \geq \widetilde{\varepsilon} ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =1-F_{\varepsilon}(\widetilde{\varepsilon})
\end{aligned}
$$

and

$$
\begin{aligned}
F_{p \mid\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J,}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right)}\left(\bar{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}^{\prime}\right) & =\operatorname{Pr}\left(p \leq \bar{p} \mid \omega=\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right) ; \widetilde{Ð}^{\prime}, F_{\varepsilon}^{\prime}\right) \\
& =\operatorname{Pr}\left(\varepsilon \geq \widetilde{\varepsilon} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}^{\prime}\right) \\
& =1-F_{\varepsilon}^{\prime}(\widetilde{\varepsilon})
\end{aligned}
$$

Since $F_{\varepsilon}(\widetilde{\varepsilon}) \neq F_{\varepsilon}^{\prime}(\widetilde{\varepsilon})$,

$$
F_{p \mid\left(\left(\bar{\omega}_{1,1}, \bar{t}_{1}+\tilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\tilde{\varepsilon}\right)\right)}\left(\bar{p} ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p \mid\left(\left(\bar{\omega}_{1,1,}, \bar{t}_{1}+\widetilde{\varepsilon}\right), \ldots,\left(\bar{\omega}_{1, J}, \bar{t}_{J}+\widetilde{\varepsilon}\right)\right)}\left(\bar{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}^{\prime}\right)
$$

Suppose, next, that $F_{\varepsilon}=F_{\varepsilon}^{\prime}$. Then, $\widetilde{Ð} \neq \widetilde{Ð}^{\prime}$, where $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon}$. Hence, for some $j$ and some $\left(\widetilde{p}, \omega_{1, j}, \omega_{2, j}-\widetilde{\varepsilon}\right), \quad \widetilde{D}_{j}\left(\widetilde{p}, \omega_{1, j}, \omega_{2, j}-\widetilde{\varepsilon}\right) \neq \widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \omega_{1, j}, \omega_{2, j}-\widetilde{\varepsilon}\right)$. Then, using $\left(\Phi_{\omega, \varepsilon}^{\prime}(i i)\right)$ and following arguments very similar to those used in the proof of Theorem 4, we can show that

$$
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \tilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)
$$

Hence, different distributions of $\varepsilon$ generate different conditional distributions of equilibrium prices given endowment vectors. This completes the proof.

Theorems 4 and 5 establish that in 2-commodity economies where the individual demands of the consumers are monotone in an unobservable random term, one can identify these individual demands, and, under some additional restrictions, also the distribution of the random term, solely from the conditional distribution of the equilibrium price, given the $J$ - tuples of individual endowments. These results assumed that a common unobservable variable was an argument in each of the individual demand functions. In many situations, however, it may be more reasonable to assume that to each individual consumer there corresponds a different unobservable random term. We next show that, restricting the demand functions further, we can still identify the individual demand functions also in this situation.

DEFINITION: $\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}$ will denote the set of $J$-tuples, $\widetilde{Ð}=\left(\widetilde{D}_{1}, \ldots, \widetilde{D}_{J}\right)$, of continuous random demand functions $\widetilde{D}_{j}: R_{+} \times W \times E \rightarrow R_{+}^{K}$ such that
$\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i)$ : For all all $p \in R_{+}$and all $j$, there exists $\bar{\omega}_{j} \in W$ and $\alpha_{j}$ such that for all $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}, \quad$ and all $\varepsilon_{j} \in E$,

$$
\widetilde{D}_{j}\left(p, \bar{\omega}_{j}, \varepsilon_{j}\right)=\widetilde{D}_{j}^{\prime}\left(p, \bar{\omega}_{j}, \varepsilon_{j}\right)={\underset{\sim}{j}}_{j}
$$

$\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i i):$ For each $j$, the first coordinate of $\widetilde{D}_{j}$ is strictly decreasing in $p$ and, except at vectors $\left(p, \omega_{j}, \varepsilon_{j}\right)$ such that $p \cdot \omega_{j}=p \cdot \bar{\omega}_{j}$, where $\bar{\omega}_{j}$ is as specified in $\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i)$, the first coordinate of $\widetilde{D}_{j}$ is strictly increasing in $\varepsilon_{j}$.

The effect of condition $\left(\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i)\right)$ is to eliminate the randomness of $\varepsilon_{j}$ at some points. Note that when $0 \in W$, condition $\left(\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i)\right)$ is always satisfied by letting $\bar{\omega}_{j}=0$. Condition $\left(\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i i)\right)$ plays a role similar to that played by condition $\left(\Phi_{\omega, \varepsilon}(i)\right)$ in Theorem 4. For each $j$, let $\varepsilon_{-j}$ denote the $J-1$ dimensional vector $\left(\varepsilon_{1}, . ., \varepsilon_{j-1}, \varepsilon_{j+1}, \ldots, \varepsilon_{J}\right)$. Assuming that the $\varepsilon_{j}^{\prime} s$ are independent across $j$ and, for each $j, F_{\varepsilon_{j}}$ is a specified distribution, we can show that the demand functions of each of the individual consumers can be identified from the distribution of prices.

THEOREM 6: Suppose that for each $j, \varepsilon_{j}$ is distributed independently of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ and of $\varepsilon_{-j}$ with a specified distribution, $F_{\varepsilon_{j}}$, which possesses a continuous density, $f_{\varepsilon_{j}}$, and whose support is the bounded set $E$. Suppose that the distribution of $\left(\omega_{1}, \ldots, \omega_{J}\right)$ has support $W^{J}$. Then, if $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}$ and $\widetilde{Ð} \neq \widetilde{Ð}^{\prime}$

$$
F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p,\left(\omega_{1}, \ldots, \omega_{J}\right)}\left(\cdot ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)
$$

PROOF: Suppose that $\widetilde{Ð}, \widetilde{Ð}^{\prime} \in \Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}$ and $\widetilde{Ð} \neq \widetilde{Ð}^{\prime}$. Then, for some $j$ and some $\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}_{j}\right) \in R_{+} \times W \times E, \widetilde{D}_{j}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}_{j}\right) \neq \widetilde{D}_{j}^{\prime}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}_{j}\right)$. By the continuity of $\widetilde{D}_{j}$ and $\widetilde{D}_{j}^{\prime}$, we can assume that $f_{\varepsilon_{j}}\left(\widetilde{\varepsilon}_{j}\right)>0$. Suppose, w.l.o.g. that $j=1$ and $\widetilde{D}_{j}^{(1)}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}_{j}\right)<\widetilde{D}_{j}^{\prime(1)}\left(\widetilde{p}, \omega_{j}, \widetilde{\varepsilon}_{j}\right)$, where, as in the proofs of previous theorems, $\widetilde{D}_{j}^{(1)}$ and $\widetilde{D}_{j}^{\prime(1)}$ denote the first coordinate of $\widetilde{D}_{j}$ and $\widetilde{D}_{j}^{\prime}$, respectively. Then,

$$
\widetilde{D}_{1}^{(1)}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)<\widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)
$$

By the definition of $\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}$, there exists $\bar{\omega}_{2}, \ldots, \bar{\omega}_{J}$ and $\alpha_{2}, \ldots, \alpha_{J}$ such that for all $k=2, \ldots, J$ and all $\varepsilon_{k} \in E, \widetilde{D}_{k}\left(\widetilde{p}, \bar{\omega}_{k}, \varepsilon_{k}\right)=\widetilde{D}_{k}^{\prime}\left(\widetilde{p}, \bar{\omega}_{k}, \varepsilon_{k}\right)=\alpha_{k}$. Hence, for all $\widetilde{\varepsilon}_{2}, \ldots, \widetilde{\varepsilon}_{J}$

$$
\begin{aligned}
& \widetilde{D}_{1}^{(1)}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{(1)}\left(\widetilde{p}, \bar{\omega}_{k}, \widetilde{\varepsilon}_{k}\right) \\
= & \widetilde{D}_{1}^{(1)}\left(\widetilde{p}, \bar{\omega}_{1}, \widetilde{\varepsilon}_{1}\right)+\sum_{k=1}^{J} \alpha_{k}^{(1)} \\
< & \widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)+\sum_{k=1}^{J} \alpha_{k}^{(1)} \\
= & \widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{\prime(1)}\left(\widetilde{p}, \bar{\omega}_{k}, \widetilde{\varepsilon}_{k}\right)
\end{aligned}
$$

Let $\widetilde{\omega}_{1}=\widetilde{D}_{1}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)$ and for each $k=2, \ldots, J$, let $\widetilde{\omega}_{k}=\alpha_{k}$. Then, since $\widetilde{p} \cdot \widetilde{\omega}_{k}=\widetilde{p}$. $\widetilde{D}_{k}\left(\widetilde{p}, \bar{\omega}_{k}, \widetilde{\varepsilon}_{k}\right)=\widetilde{p} \cdot \bar{\omega}_{k}$,

$$
\widetilde{D}_{k}\left(\widetilde{p}, \widetilde{\omega}_{k}, \widetilde{\varepsilon}_{k}\right)=\widetilde{\omega}_{k}=\widetilde{D}_{k}\left(\widetilde{p}, \bar{\omega}_{k}, \widetilde{\varepsilon}_{k}\right) \quad \text { and } \quad \widetilde{D}_{1}^{\prime}\left(\widetilde{p}, \widetilde{\omega}_{1}, \widetilde{\varepsilon}_{1}\right)=\widetilde{D}_{1}^{\prime}\left(\widetilde{p}, \omega_{1}, \widetilde{\varepsilon}_{1}\right)
$$

Hence, when the endowment vector is $\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)$ and the value of $\varepsilon_{1}$ is $\widetilde{\varepsilon}_{1}, \widetilde{p}$ is an equilibrium price for all values of $\left(\varepsilon_{2}, \ldots, \varepsilon_{J}\right)$, when the $J$-tuple of demand functions is $\widetilde{Ð}$, and $\widetilde{p}$ is not an equilibrium price, for any value of $\left(\varepsilon_{2}, \ldots, \varepsilon_{J}\right)$, when the $J$-tuple of demand functions is $\widetilde{\mathrm{Đ}}^{\prime}$. By $\Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}(i i), \widetilde{p}$ is the unique such equilibrium price, when the $J$-tuple of demand functions is $\widetilde{Ð}$. By the continuity of $\widetilde{D}_{1}^{\prime(1)}$ and $f_{\varepsilon_{1}}$ in $\varepsilon_{1}$ and the fact that $f_{\varepsilon_{1}}\left(\widetilde{\varepsilon}_{1}\right)>0$, it follows that there exists a neighborhood of $\widetilde{\varepsilon}_{1}$ such that for all values $\varepsilon_{1}^{\prime}$ in that neighborhood, $f_{\varepsilon_{1}}\left(\varepsilon_{1}^{\prime}\right)>0$ and

$$
\sum_{j=1}^{J} \widetilde{\omega}_{j}^{(1)}<\widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{1}, \varepsilon_{1}^{\prime}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{k}, \widetilde{\varepsilon}_{k}\right)
$$

For any $\widetilde{Ð} \in \Phi_{\omega, \varepsilon_{1}, \ldots, \varepsilon_{J}}$, define

$$
e_{1}\left(\widetilde{p}, \omega_{1} ; \widetilde{Ð}\right)=\left\{\begin{array}{c}
\sup \left\{\varepsilon_{1} \in E \mid \widetilde{D}_{1}^{(1)}\left(\widetilde{p}, \omega_{1}, \varepsilon_{1}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{(1)}\left(\widetilde{p}, \widetilde{\omega}_{k}, \varepsilon_{k}\right) \leq \omega_{1}+\sum_{k=2}^{J} \widetilde{\omega}_{k}\right\} \\
\quad \text { if }\left\{\varepsilon_{1} \in E \mid \widetilde{D}_{1}^{(1)}\left(\widetilde{p}, \omega_{1}, \varepsilon_{1}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{(1)}\left(\widetilde{p}, \widetilde{\omega}_{k}, \varepsilon_{k}\right) \leq \omega_{1}+\sum_{k=2}^{J} \widetilde{\omega}_{k}\right\} \neq \varnothing \\
\inf (E) \\
\text { otherwise }
\end{array}\right.
$$

Since $f_{\varepsilon_{1}}\left(\widetilde{\varepsilon}_{1}\right)>0, \widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{1}, \varepsilon_{1}\right)$ is strictly increasing in the value of the unobservable variable, and, from above, $\sum_{k=1}^{J} \widetilde{\omega}_{k}^{(1)}<\widetilde{D}_{1}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{1}, \varepsilon_{1}^{\prime}\right)+\sum_{k=2}^{J} \widetilde{D}_{k}^{\prime(1)}\left(\widetilde{p}, \widetilde{\omega}_{k}, \widetilde{\varepsilon}_{k}\right)$ for all $\varepsilon_{1}^{\prime}$ in a neighborhood of $\widetilde{\varepsilon}_{1}$, it follows that $e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}^{\prime}\right)<\varepsilon_{1}^{\prime}<\widetilde{\varepsilon}_{1}$ for all $\varepsilon_{1}^{\prime}$ in a neighborhood that possesses positive probability. By the definition of $e_{1}\left(\widetilde{p}, \omega_{1} ; \widetilde{Ð}\right)$ and the fact that $\widetilde{p}$ is the equilibrium price when the endowment vector is $\widetilde{\omega}$, the value of $\varepsilon_{1}$ is $\widetilde{\varepsilon}_{1}$, and the vector of demand functions is $\widetilde{Ð}$, it follows that $e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}\right)=\widetilde{\varepsilon}_{1}$. Hence,

$$
\widetilde{\varepsilon}_{1}=e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}\right)>e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}^{\prime}\right)
$$

and

$$
\operatorname{Pr}\left(\varepsilon_{1} \leq e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}\right)\right)>\operatorname{Pr}\left(\varepsilon_{1} \leq e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}^{\prime}\right)\right)
$$

Since

$$
\begin{aligned}
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right) & =\operatorname{Pr}\left(p \leq \widetilde{p} \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon_{1} \leq e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}\right) \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon_{1} \leq \widetilde{\varepsilon_{1}} ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =F_{\varepsilon_{1}}\left(\widetilde{\varepsilon}_{1}\right) \\
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) & =\operatorname{Pr}\left(p \leq \widetilde{p} \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon_{1} \leq e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}^{\prime}\right) \mid \omega=\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right) ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right) \\
& =\operatorname{Pr}\left(\varepsilon_{1} \leq e_{1}\left(\widetilde{p}, \widetilde{\omega}_{1} ; \widetilde{Ð}^{\prime}\right) ; \widetilde{Ð}, F_{\varepsilon}\right) \\
& =F_{\varepsilon_{1}}\left(e_{1}\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{\omega}^{\prime}\right)\right)
\end{aligned}
$$

and

$$
F_{\varepsilon_{1}}\left(\widetilde{\varepsilon}_{1}\right)>F_{\varepsilon_{1}}\left(e_{1}\left(\widetilde{p}, \widetilde{\omega} ; \widetilde{Ð}^{\prime}\right)\right)
$$

it follows that

$$
F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}, F_{\varepsilon}\right) \neq F_{p \mid\left(\widetilde{\omega}_{1}, \ldots, \widetilde{\omega}_{J}\right)}\left(\widetilde{p} ; \widetilde{Ð}^{\prime}, F_{\varepsilon}\right)
$$

This completes the proof.

If instead of specifying each distributions $F_{\varepsilon_{j}}$ we would have required that, for each $j$, $\widetilde{D}_{j}$ satisfies properties such as those in $\Phi_{\omega, \varepsilon}^{\prime}$, then we would have been able to establish the identification of also the distributions $F_{\varepsilon_{j}}$.

In Theorems 4-6, we established that from the distribution of equilibrium prices and $J$ - tuples of endowment vectors, we can identify the demand function of each of the $J$ individual consumers. Using each of these demand functions, $\widetilde{D}_{j}$, together with the results in Mas-Colell (1977b), we can identify, for each value of $\varepsilon$, a unique preference relation
generating the demand function $\widetilde{D}_{j}(\cdot, \cdot, \varepsilon)$, in the same way that in the proof of Theorem 3 we used the results in Mas-Colell (1977b) to establish the identification of the preferences of the $J$ consumers from the demand functions of these consumers. An alternative way of identifying these preferences, which might be preferred in some circumstances, would be to first identify from the distribution of equilibrium prices the distribution of the demand function of each individual consumer, and then use results in Brown and Matzkin (1998) to identify the random utility functions that generate each of the demand distributions.

## 4. Conclusions

We have provided very weak conditions under which either from the aggregate demand function or from the equilibrium correspondence of a pure exchange economy one can identify the preferences of the consumers in the economy. We considered the case where the preferences of the consumers are deterministic, and cases where they are stochastic. In the latter case, we provided conditions under which from the conditional distribution of equilibrium prices, given endowments, one can identify both, the random demand functions and the distribution of an unobservable random terms which generate the randomness in the demand functions.

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[^0]:    ${ }^{1}$ Earlier works that presented restrictions are McFadden, Mas-Colell, Mantel and Richter (1974), McFadden (1975), Diewert (1977), Mas-Colell and Neuefeind (1977), and Hildenbrand (1983), among others.

[^1]:    ${ }^{2}$ A large literature exists also for the case where observed individual behavior is generated by the maximization of a random preference over a finite, discrete choice set. The study of the recoverability of preferences, in this case, was introduced by McFadden (1974). (See Matzkin (1992, 1993) for later work on recoverability results under weak conditions.) McFadden and Richter (1991) characterized the restrictions that random optimization generates in this case. (See McFadden (2002) and the references mentioned in that paper for other work along this line.)

