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# IDENTIFICATION IN NONPARAMETRIC SIMULTANEOUS EQUATIONS MODELS 

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#### Abstract

This paper provides conditions for identification of functionals in nonparametric simultaneous equations models with nonadditive unobservable random terms. The conditions are derived from a characterization of observational equivalence between models. We show that, in the models considered, observational equivalence can be characterized by a restriction on the rank of a matrix. The use of the new results is exemplified by deriving previously known results about identification in parametric and nonparametric models as well as new results. A stylized method for analyzing identification, which is useful in some situations, is also presented.


Keywords: Nonparametric methods, nonadditive models, nonseparable models, identification, simultaneous equations, endogeneity.

## 1. INTRODUCTION

THE INTERPLAY BETWEEN ECONOMETRICS AND ECONOMIC THEORY comes to its full force when analyzing the identification of underlying functions and distributions in structural models. Identification in structural models that are linear in variables and parameters and have additive unobservable variables has been studied for a long time. On the other hand, identification in structural models that do not impose parametric assumptions in the functions and distributions in the model, or do not impose additivity in the unobservable variables, has not yet been completely understood. The objective of this paper is to provide insight into these latter cases. Starting from a characterization of observational equivalence, this paper provides new conditions that can be used to determine the identification of the underlying functions and distributions in simultaneous equations models.

The study of identification is a key element in the econometric analysis of many structural models. Such study allows one to determine conditions under which, from the distribution of observable variables, one can recover fea-

[^0]tures of the primitive functions and distributions in the model. These features are needed, for example, for the analysis of counterfactuals, where one wants to calculate the outcomes that would result when some of the elements in the model change. The analysis of identification dates back to the works of H. Working (1925), E. J. Working (1927), Tinbergen (1930), Frisch (1934, 1938), Haavelmo (1943, 1944), Hurwicz (1950a), Koopmans and Reiersol (1950), Koopmans, Rubin, and Leipnik (1950), Wald (1950), Fisher (1959, 1961, 1965, 1966), Wegge (1965), Rothenberg (1971), and Bowden (1973). While the importance of identification in models with nonparametric functions and with nonadditive unobservable random terms has been recognized since the early years (see Hurwicz (1950a, 1950b)), most works at the time concentrated on providing conditions for linear models with additive unobservable random terms or for nonlinear parametric models.

More recently, nonparametric models with nonadditive unobservable variables have received increasing attention, with new theoretical developments and application possibilities frequently appearing. This has motivated researchers to revisit older studies armed with new tools. In the context of identification in simultaneous equations models, Benkard and Berry (2006) recently revisited the path-breaking results by Brown (1983), and their extension by Roehrig (1988), on the identification of nonlinear and nonparametric simultaneous equations, and found some arguments to be controversial. A contribution of this paper is to provide a different set of conditions for the identification of such models.

The current literature on identification of nonparametric models with endogenous regressors is very large. Within this literature, Ng and Pinske (1995), Newey, Powell, and Vella (1999), and Pinske (2000) considered nonparametric triangular systems with additive unobservable random terms. Altonji and Ichimura (2000) considered models with latent variables. Altonji and Matzkin (2001, 2005) provided estimators for average derivatives in nonseparable models, using conditional independence, and for nonparametric nonseparable functions, using exchangeability. Altonji and Matzkin (2003) extended their 2001 results to discrete endogenous regressors. Chesher (2003) considered local identification of derivatives in triangular systems with nonadditive random terms. Imbens and Newey (2003) studied global identification and estimation of derivatives and average derivatives in triangular systems with nonadditive random terms. Matzkin (2003, 2004) considered estimation under conditional independence, with normalizations and restrictions on nonadditive functions. Vytlacil and Yildiz (2007) studied estimation of average effects in models with weak separability and dummy endogenous variables. For nontriangular systems, Newey and Powell (1989, 2003), Darolles, Florens, and Renault (2003), and Hall and Horowitz (2005) considered estimation using conditional moment conditions between additive unobservables and instruments. Brown and Matzkin (1998), Ai and Chen (2003), Chernozhukov and Hansen (2005), and Chernozhukov, Imbens, and Newey (2007) allowed for the unobservable variables to be nonadditive. The latter two articles exploited an independence as-
sumption between the unobservable variables and an instrument, to study identification. Matzkin (2004) considered identification using instruments and an independence condition across the unobservable variables. Blundell and Powell (2003) analyzed several nonparametric and semiparametric models, and provided many references. Matzkin (2007a) provided a partial survey of recent results on nonparametric identification. A parallel approach has considered partial identification in structural triangular and nontriangular models (see Chesher (2005, 2007)).

The outline of the paper is as follows. In the next section, we describe the model and its main assumptions. In Section 3, we derive several characterizations of observational equivalence. We demonstrate how these characterizations can be used to determine identification in a linear and an additively separable model, in Section 4, and how they can be used to obtain the already known results for single and triangular nonadditive models, in Section 5. A more stylized method for analyzing identification is presented in Section 6. Section 7 presents the main conclusions of the paper.

## 2. THE MODEL

We consider a system of structural equations, described as

$$
\text { (2.1) } \quad U=r(Y, X)
$$

where $r: R^{G+K} \rightarrow R^{G}$ is an unknown, twice continuously differentiable function, $Y$ is a vector of $G$ observable endogenous variables, $X$ is a vector of $K$ observable exogenous variables, and $U$ is a vector of $G$ unobservable variables, which is assumed to be distributed independently of $X$. Let $f_{U}$ denote the density of $U$, assumed to be continuously differentiable. Our objective is to determine conditions under which the function $r$ and the density $f_{U}$ are identified within a set of functions and densities to which $r$ and $f_{U}$ belong. We will assume that the vector $X$ has a continuous, known density $f_{X}$ that has support $R^{K}$. Assuming that $f_{X}$ is known does not generate a loss of generality, for the purpose of the analysis of identification, because $X$ is observable.

A typical example of a system (2.1) is a demand and supply model,

$$
\begin{align*}
& Q=D\left(P, I, U_{1}\right)  \tag{2.2}\\
& P=S\left(Q, W, U_{2}\right)
\end{align*}
$$

where $Q$ and $P$ denote the quantity and price of a commodity, $I$ denotes consumers' income, $W$ denotes producers' input prices, $U_{1}$ denotes an unobservable demand shock, and $U_{2}$ denotes an unobservable supply shock. If the demand function, $D$, is strictly increasing in $U_{1}$ and the supply function, $S$, is strictly increasing in $U_{2}$, one can invert these functions and write this system as in (2.1), with $Y=(P, Q), X=(I, W)$, and $U=\left(U_{1}, U_{2}\right), r_{1}$ denoting the
inverse of $D$ with respect to $U_{1}$, and $r_{2}$ denoting the inverse of $S$ with respect to $U_{2}$ :

$$
\begin{aligned}
& U_{1}=r_{1}(Q, P, I) \\
& U_{2}=r_{2}(Q, P, W)
\end{aligned}
$$

We will assume that the system of structural equations (2.1) possesses a unique reduced form system

$$
\begin{equation*}
Y=h(X, U) \tag{2.3}
\end{equation*}
$$

where $h: R^{K+G} \rightarrow R^{G}$ is twice continuously differentiable. In particular, conditional on $X, r$ is one-to-one in $Y$. In the supply and demand example, this reduced form system is expressed as

$$
\begin{align*}
& Q=h_{1}\left(I, W, U_{1}, U_{2}\right)  \tag{2.4}\\
& P=h_{2}\left(I, W, U_{1}, U_{2}\right)
\end{align*}
$$

where the values of $Q$ and $P$ are the unique values satisfying (2.2).
To determine conditions for identification, we start out from a characterization of observational equivalence, within a set of functions and distributions to which $r$ and $f_{U}$ are, respectively, assumed to belong. We will let $\Gamma$ denote the set of functions to which $r$ belongs and let $\Phi$ denote the set of densities to which $f_{U}$ belongs.

The functions $\tilde{r}: R^{G+K} \rightarrow R^{G}$, in the set $\Gamma$, satisfy the following properties.
(i) $\widetilde{r}$ is twice continuously differentiable on $R^{G+K}$.
(ii) For each $x \in R^{K}, \widetilde{r}(\cdot, x): R^{G} \rightarrow R^{G}$ is one-to-one and onto $R^{G}$.
(iii) For each $(y, x) \in R^{G+K}$, the Jacobian determinant $|\tilde{\partial} \tilde{r}(y, x) / \partial y|$ is strictly positive.
Note that to each such $\widetilde{r}$ there corresponds a function $\widetilde{h}: R^{K+G} \rightarrow R^{G}$, which assigns to each value $(x, u) \in R^{K+G}$ the unique value $y$ satisfying $u=\widetilde{r}(y, x)$. The function $\widetilde{h}$ is twice continuously differentiable on $R^{K+G}$. For each $x \in R^{K}$, $\widetilde{h}(x, \cdot): R^{G} \rightarrow R^{G}$ is one-to-one and onto $R^{G}$.

The set $\Phi$ will be defined to be the set of densities $f_{\tilde{U}}: R^{G} \rightarrow R$ such that (i) $f_{\tilde{U}}$ is continuously differentiable on $R^{G}$ and (ii) the support of $f_{\tilde{U}}$ is $R^{G}$.

The differentiability of $\tilde{r}$ and $f_{\tilde{U}}$ will allow us to express conditions in terms of derivatives. The support conditions on $f_{\tilde{U}}$, the density of $X$, and on the density of $Y$ conditional on $X$ will allow us to guarantee that all densities converge to 0 as the value of one of their arguments tends to infinity. The condition on the sign of the Jacobian determinant $|\partial \widetilde{r}(y, x) / \partial y|$ is a normalization.

Given $f_{X}$, we can derive, for each $\left(\widetilde{r}, f_{\tilde{U}}\right) \in(\Gamma \times \Phi)$, a unique distribution function, $F_{Y, X}\left(\cdot ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right)$ for the vector of observable variables $(Y, X)$. Under
our conditions, if $f_{X}$ is differentiable, $F_{Y, X}\left(\cdot ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right)$ is characterized by a differentiable density $f_{Y, X}\left(\cdot ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right)$, which is defined, for all $y, x$, by

$$
\begin{aligned}
f_{Y, X}\left(y, x ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right) & =f_{Y \mid X=x}\left(y ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right) f_{X}(x) \\
& =f_{\widetilde{U}}(\widetilde{r}(y, x))\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right| f_{X}(x) .
\end{aligned}
$$

## 3. OBSERVATIONAL EQUIVALENCE

Following the standard definition, we will say that two elements of $(\Gamma \times \Phi)$ are observationally equivalent if they generate the same distribution of the observable variables. Formally, this is stated as follows.

DEFINITION 3.1: $\left(\tilde{r}, f_{\tilde{U}}\right),\left(\bar{r}, f_{\bar{U}}\right) \in(\Gamma \times \Phi)$ are observationally equivalent, if for all $(y, x) \in R^{G+K}$,

$$
F_{Y, X}\left(y, x ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right)=F_{Y, X}\left(y, x ;\left(\bar{r}, f_{\bar{U}}\right)\right) .
$$

The standard, closely related, definition of identification is given by the next statement:

DEFINITION 3.2: $\left(r, f_{U}\right) \in(\Gamma \times \Phi)$ is identified in $(\Gamma \times \Phi)$, if for all $\left(\widetilde{r}, f_{\tilde{U}}\right) \in$ $(\Gamma \times \Phi)$ such that $\left(r, f_{U}\right) \neq\left(\tilde{r}, f_{\tilde{U}}\right),\left(\tilde{r}, f_{\tilde{U}}\right)$ is not observationally equivalent to $\left(r, f_{U}\right)$.

More generally, we might be interested in the identification of the value of some functional, $\mu\left(r, f_{U}\right)$. Let $\Omega=\left\{\mu\left(\widetilde{r}, f_{\tilde{U}}\right) \mid\left(\widetilde{r}, f_{\tilde{U}}\right) \in(\Gamma \times \Phi)\right\}$ denote the set of all possible values that $\mu$ can attain over pairs $\left(\widetilde{r}, f_{\tilde{U}}\right)$ in $(\Gamma \times \Phi)$, given $f_{X}$. Then we have the following definition.

DEFINITION 3.3: The value $\widetilde{\omega} \in \Omega$ is observationally equivalent to $\bar{\omega} \in \Omega$ if there exist $\left(\widetilde{r}, f_{\tilde{U}}\right),\left(\bar{r}, f_{\bar{U}}\right) \in(\Gamma \times \Phi)$ such that $\widetilde{\omega}=\mu\left(\widetilde{r}, f_{\tilde{U}}\right), \bar{\omega}=\mu\left(\bar{r}, f_{\bar{U}}\right)$, and ( $\widetilde{r}, f_{\tilde{U}}$ ) is observationally equivalent to ( $\bar{r}, f_{\bar{U}}$ ).

The value $\omega$ of any functional $\mu$ at $\left(r, f_{U}\right)$ is identified if all pairs $\left(\tilde{r}, f_{\tilde{U}}\right) \in$ $(\Gamma \times \Phi)$ that are observationally equivalent to $\left(r, f_{U}\right)$ are assigned, by $\mu$, the same value, $\omega$; that is, $\mu\left(\tilde{r}, f_{\tilde{U}}\right)=\omega$. The formal statement follows.

DEFINITION 3.4: The value $\omega=\mu\left(r, f_{U}\right) \in \Omega$ is identified within $\Omega$, with respect to $(\Gamma \times \Phi)$, if for any $\left(\widetilde{r}, f_{\tilde{U}}\right) \in(\Gamma \times \Phi)$ and $\widetilde{\omega} \in \Omega$ such that $\widetilde{\omega}=$ $\mu\left(\widetilde{r}, f_{\tilde{U}}\right) \neq \mu\left(r, f_{U}\right)=\omega,\left(\widetilde{r}, f_{\tilde{U}}\right)$ is not observationally equivalent to $\left(r, f_{U}\right)$; that is,

$$
F_{Y, X}\left(\cdot ;\left(\widetilde{r}, f_{\tilde{U}}\right)\right) \neq F_{Y, X}\left(\cdot ;\left(r, f_{U}\right)\right)
$$

Since, in our model, the continuous marginal density $f_{X}$, whose support is $R^{K}$, does not depend on $\left(\widetilde{r}, f_{\tilde{U}}\right)$ or $\left(\bar{r}, f_{\bar{U}}\right)$, we can state that $\left(\widetilde{r}, f_{\widetilde{U}}\right),\left(\bar{r}, f_{\bar{U}}\right) \in$
$(\Gamma \times \Phi)$ are observationally equivalent if for all $y, x$,

$$
\begin{equation*}
f_{\bar{U}}(\bar{r}(y, x))\left|\frac{\partial \bar{r}(y, x)}{\partial y}\right|=f_{\tilde{U}}(\widetilde{r}(y, x))\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right| . \tag{3.1}
\end{equation*}
$$

Note that, under our conditions, if $r$ is identified, so is $f_{U}$. This is easy to see, since for any $u$,

$$
f_{U}(u)=f_{Y \mid X=x}(h(x, u))\left|\frac{\partial h(x, u)}{\partial u}\right|
$$

and if $r$ is identified, so is $h$.
We will analyze the identification of functionals, $\mu$, of $\left(r, f_{U}\right)$. The approach used will be to first determine conditions for observational equivalence between $\left(r, f_{U}\right)$ and any $\left(\widetilde{r}, f_{\tilde{U}}\right) \in(\Gamma \times \Phi)$, and then verify that for any such $\left(\widetilde{r}, f_{\tilde{U}}\right)$ that is observationally equivalent to $\left(r, f_{U}\right)$,

$$
\mu\left(\tilde{r}, f_{\tilde{U}}\right)=\mu\left(r, f_{U}\right)
$$

Our starting point is equation (3.1). Given the true ( $r, f_{U}$ ) and an alternative function $\widetilde{r} \in \Gamma$, this equation can be used to derive a density, $f_{\tilde{U}}$, such that ( $\widetilde{r}, f_{\tilde{U}}$ ) is observationally equivalent to $\left(r, f_{U}\right)$. For this, we study the relationship between the mapping which assigns to any $(y, x)$, the value $\widetilde{u}$ of $\widetilde{U}$, satisfying

$$
\begin{equation*}
\widetilde{u}=\widetilde{r}(y, x) \tag{3.2}
\end{equation*}
$$

and the mapping which assigns to that same $(y, x)$, the value $u$ of $U$, satisfying

$$
\begin{equation*}
u=r(y, x) \tag{3.3}
\end{equation*}
$$

Since $\tilde{r} \in \Gamma$, (3.2) implies that

$$
y=\widetilde{h}(x, \widetilde{u})
$$

where $\tilde{h}$ is the reduced form function corresponding to $\tilde{r}$. Substituting in (3.3), we get that ${ }^{2}$

$$
u=r(\tilde{h}(x, \widetilde{u}), x)
$$

Hence, we can write (3.1) as

$$
f_{\tilde{U}}(\widetilde{u})\left|\frac{\partial \widetilde{r}(\tilde{h}(x, \widetilde{u}), x)}{\partial y}\right|=f_{U}(r(\tilde{h}(x, \tilde{u}), x))\left|\frac{\partial r(\tilde{h}(x, \widetilde{u}), x)}{\partial y}\right|
$$

[^1]or, after dividing both sides by the first determinant, as
\[

$$
\begin{equation*}
f_{\tilde{U}}(\widetilde{u})=f_{U}(r(\tilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\tilde{h}(x, \widetilde{u}), x)}{\partial y} \| \frac{\partial \widetilde{r}(\tilde{h}(x, \widetilde{u}), x)}{\partial y}\right|^{-1} \tag{3.4}
\end{equation*}
$$

\]

Two important implications can be derived from expression (3.4). First, (3.4) implies that $f_{\tilde{U}}(\widetilde{u})$ is completely determined by $\widetilde{r}, r$, and $f_{U}$. That is, once we know $\widetilde{r}, r$, and $f_{U}$, we can, when it exists, determine the distribution of $\widetilde{U}=$ $\widetilde{r}(Y, X)$ such that $\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$. Second, since the left-hand side of (3.4) does not depend on $x$, the right-hand side should not depend on $x$ either. As we next show, the latter is a condition for independence between $\widetilde{U}$ and $X$.

### 3.1. Independence

Consider deriving, for each $x$, the conditional density, $f_{\widetilde{U} \mid X=x}$ of $\widetilde{U}=\widetilde{r}(Y, X)$ given $X=x$. Under our assumptions, this conditional density always exists and belongs to $\Phi$. Since $U=r(\widetilde{h}(X, \widetilde{U}), X)$, and $\widetilde{h}$ and $r$ are one-to-one and onto, conditional on $X=x$, it follows by the standard formula for transformation of variables that

$$
\begin{equation*}
f_{\tilde{U} \mid X=x}(\widetilde{u})=f_{U}(r(\widetilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\widetilde{h}(x, \tilde{u}), x)}{\partial \widetilde{u}}\right| \tag{3.5}
\end{equation*}
$$

Differentiating with respect to $\tilde{u}$ the identity

$$
\tilde{u}=\widetilde{r}(\tilde{h}(x, \widetilde{u}), x),
$$

one gets that

$$
\frac{\partial \widetilde{h}(x, \tilde{u})}{\partial \widetilde{u}}=\left(\frac{\partial \widetilde{r}(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right)^{-1}
$$

Hence, the density of $\tilde{U}$ conditional on $X=x$ is given by

$$
\begin{aligned}
f_{\widetilde{U} \mid X=x}(\widetilde{u}) & =f_{U}(r(\widetilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\tilde{h}(x, \widetilde{u}), x)}{\partial \widetilde{u}}\right| \\
& =f_{U}(r(\widetilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\widetilde{h}(x, \widetilde{u}), x)}{\partial y} \frac{\partial \widetilde{h}(x, \widetilde{u})}{\partial \widetilde{u}}\right| \\
& =f_{U}(r(\widetilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\widetilde{h}(x, \widetilde{u}), x)}{\partial y}\right|\left|\frac{\partial \widetilde{r}(\widetilde{h}(x, \widetilde{u}), x)}{\partial y}\right|^{-1} .
\end{aligned}
$$

Under our assumptions, the random variable $\widetilde{U}$ is independent of $X$ if and only if for all $x$,

$$
f_{\widetilde{U} \mid X=x}(\widetilde{u})=f_{\widetilde{U}}(\widetilde{u}) .
$$

Note that this is exactly the same condition as in (3.4). Hence, requiring observational equivalence between $\left(\widetilde{r}, f_{\tilde{U}}\right)$ and $\left(r, f_{U}\right)$ is equivalent to requiring that $f_{\tilde{U} \mid X=x}$ in (3.5) equals, for all $x$, the marginal density of $\widetilde{U}$. Making use of our support and differentiability assumptions, the condition for independence between $\widetilde{U}$ and $X$ can be expressed as the condition that for all $x, \tilde{u}$,

$$
\frac{\partial \tilde{\widetilde{U}}_{\mid} \mid X=x}{}(\widetilde{u}),
$$

### 3.2. Characterization of Independence

To obtain a more practical characterization of the independence condition, we proceed to express it in terms of the derivatives of the functions $\widetilde{r}$ and $r$. Let $\partial \log f_{U}(r(y, x)) / \partial u$ denote the $G \times 1$ gradient of $\log \left(f_{U}(u)\right)$ with respect to $u$, evaluated at $u=r(y, x)$. Since $f_{\tilde{U} \mid X=x}(\widetilde{u})>0$, the condition that for all $x, \widetilde{u}, \partial f_{\tilde{U} \mid X=x}(\widetilde{u}) / \partial x=0$ is equivalent to the condition that for all $x, \widetilde{u}$,

$$
\frac{\partial \log f_{\tilde{U} \mid X=x}(\widetilde{u})}{\partial x}=0
$$

Since

$$
f_{\tilde{U} \mid X=x}(\widetilde{u})=f_{U}(r(\tilde{h}(x, \widetilde{u}), x))\left|\frac{\partial r(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right|\left|\frac{\partial \widetilde{r}(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right|^{-1}
$$

the above is equivalent to the condition that

$$
\begin{aligned}
0= & \left(\frac{\partial \log \left(f_{U}(r(\tilde{h}(x, \tilde{u}), x))\right)}{\partial u}\right)^{\prime} \\
& \times\left[\frac{\partial r(\widetilde{h}(x, \widetilde{u}), x)}{\partial y} \frac{\partial \tilde{h}(x, \widetilde{u})}{\partial x}+\frac{\partial r(\tilde{h}(x, \widetilde{u}), x)}{\partial x}\right] \\
& +\left(\frac{\partial}{\partial x}\left[\log \left|\frac{\partial r(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right|-\log \left|\frac{\partial \widetilde{r}(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right|\right]\right)^{\prime} \\
& +\left(\frac{\partial}{\partial y}\left[\log \left|\frac{\partial r(\widetilde{h}(x, \tilde{u}), x)}{\partial y}\right|-\log \left|\frac{\partial \widetilde{r}(\tilde{h}(x, \widetilde{u}), x)}{\partial y}\right|\right]\right)^{\prime} \frac{\partial \tilde{h}(x, \tilde{u})}{\partial x}
\end{aligned}
$$

or, substituting $\widetilde{h}(x, \widetilde{u})$ by $y$, to

$$
\begin{align*}
& \left(\frac{\partial \log f_{U}(r(y, x))}{\partial u}\right)^{\prime}\left[\frac{\partial r(y, x)}{\partial x}+\frac{\partial r(y, x)}{\partial y} \frac{\partial \widetilde{h}(x, \tilde{u})}{\partial x}\right]  \tag{3.6}\\
& \quad+\left(\frac{\partial}{\partial x}\left(\log \left|\frac{\partial r(y, x)}{\partial y}\right|\right)\right)^{\prime}+\left(\frac{\partial}{\partial y}\left(\log \left|\frac{\partial r(y, x)}{\partial y}\right|\right)\right)^{\prime} \frac{\partial \widetilde{h}(x, \widetilde{u})}{\partial x} \\
& \quad=\left(\frac{\partial}{\partial x} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)^{\prime}+\left(\frac{\partial}{\partial y}\left(\log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)\right)^{\prime} \frac{\partial \widetilde{h}(x, \tilde{u})}{\partial x}
\end{align*}
$$

This may be interpreted as stating that the proportional change in the conditional density of $Y$ given $\underset{\sim}{X}$, when the value of $X$ changes and $Y$ responds to that change according to $\widetilde{h}$, has to equal the proportional change in the value of the determinant determined by $\tilde{r}$ when $X$ changes and $Y$ responds to that change according to $\widetilde{h}$.

To obtain an equivalent expression for (3.6) in terms of only the structural functions, $r$ and $\widetilde{r}$, and the density $f_{U}$, we note that differentiating the identity

$$
y=\widetilde{h}(x, \widetilde{r}(y, x))
$$

with respect to $x$ gives

$$
0=\frac{\partial \widetilde{h}(x, \widetilde{u})}{\partial x}+\frac{\partial \widetilde{h}(x, \widetilde{r}(y, x))}{\partial \widetilde{u}}\left(\frac{\partial \widetilde{r}(y, x)}{\partial x}\right)
$$

Using the relationship, derived above, that

$$
\frac{\partial \widetilde{h}(x, \tilde{u})}{\partial \widetilde{u}}=\left(\frac{\partial \widetilde{r}(\tilde{h}(x, \tilde{u}), x)}{\partial y}\right)^{-1}
$$

and substituting $y$ for $\widetilde{h}(x, \widetilde{u})$ gives an expression for the derivative with respect to $x$ of the reduced form function $\widetilde{h}$, in terms of derivatives with respect to $y$ and $x$, of the structural function $\widetilde{r}$ :

$$
\frac{\partial \widetilde{h}(x, \widetilde{u})}{\partial x}=-\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{-1} \frac{\partial \widetilde{r}(y, x)}{\partial x}
$$

where $\tilde{h}(x, \tilde{u})$ is the reduced form function of the alternative model evaluated at $\widetilde{u}=\widetilde{r}(y, x)$. Hence, a different way of writing condition (3.6), in terms of the structural functions $r$ and $\tilde{r}$ of the observable variables, and the density $f_{U}$, is

$$
\begin{equation*}
\left(\frac{\partial \log f_{U}(r(y, x))}{\partial u}\right)^{\prime}\left[\frac{\partial r(y, x)}{\partial x}-\frac{\partial r(y, x)}{\partial y}\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{-1} \frac{\partial \widetilde{r}(y, x)}{\partial x}\right] \tag{3.7}
\end{equation*}
$$

$$
\begin{aligned}
& +\left(\frac{\partial}{\partial x} \log \left|\frac{\partial r(y, x)}{\partial y}\right|-\frac{\partial}{\partial x} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)^{\prime} \\
& -\left[\left(\frac{\partial}{\partial y} \log \left|\frac{\partial r(y, x)}{\partial y}\right|-\frac{\partial}{\partial y} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)^{\prime}\right. \\
& \\
& \left.\times\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{-1} \frac{\partial \widetilde{r}(y, x)}{\partial x}\right] \\
& =0
\end{aligned}
$$

We can express condition (3.7) in a more succinct way. Define, for any $y, x$, the $G \times K$ matrix $A(y, x ; \partial r, \partial \widetilde{r})$, the $K \times 1$ vector $b\left(y, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)$, and the $G \times 1$ vector $s\left(y, x ; f_{U}, r\right)$ by

$$
\begin{aligned}
& A(y, x ; \partial r, \partial \widetilde{r})=\left[\frac{\partial r(y, x)}{\partial x}-\frac{\partial r(y, x)}{\partial y}\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{-1} \frac{\partial \widetilde{r}(y, x)}{\partial x}\right] \\
& b\left(y, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)^{\prime} \\
& =-\left(\frac{\partial}{\partial x} \log \left(\left|\frac{\partial r(y, x)}{\partial y}\right|\right)-\frac{\partial}{\partial x} \log \left(\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)\right)^{\prime} \\
& \quad+\left[\left(\frac{\partial}{\partial y} \log \left(\left|\frac{\partial r(y, x)}{\partial y}\right|\right)-\frac{\partial}{\partial y} \log \left(\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)\right)^{\prime}\right. \\
& \left.\quad \times\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{-1} \frac{\partial \widetilde{r}(y, x)}{\partial x}\right]
\end{aligned}
$$

and

$$
s\left(y, x ; f_{U}, r\right)=\frac{\partial \log \left(f_{U}(r(y, x))\right)}{\partial u}
$$

Condition (3.7) can then be expressed as stating that for all $y, x$,

$$
s\left(y, x ; f_{U}, r\right)^{\prime} A(y, x ; \partial r, \partial \widetilde{r})=b\left(y, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)^{\prime}
$$

We index the $G \times K$ matrix $A(y, x)$ by $(\partial r, \partial \widetilde{r})$, the $K \times 1$ vector $b(y, x)$ by ( $\partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}$ ), and the $G \times 1$ vector $s(y, x)$ by $\left(f_{U}, r\right)$ to emphasize that the value of $A$ depends on the first order derivatives of the functions $r$ and $\widetilde{r}$, the value of $b$ depends on the first and second order derivatives of the functions $r$ and $\tilde{r}$, and the value of $s$ depends on the function $f_{U}$ and the value of the function $r$. Our arguments above lead to the following result:

Theorem 3.1: Suppose that $\left(r, f_{U}\right) \in(\Gamma \times \Phi)$ and that $\tilde{r} \in \Gamma$. Define the density of $\widetilde{U}$ conditional on $X=x$ as in (3.5). Then

$$
\frac{\partial f_{\tilde{U} \mid X=x}(\widetilde{u})}{\partial x}=0 \quad \text { for all } x, \widetilde{u}
$$

if and only if for all $y, x$,

$$
\begin{equation*}
s\left(y, x ; f_{U}, r\right)^{\prime} A(y, x ; \partial r, \partial \widetilde{r})=b\left(y, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)^{\prime} \tag{3.8}
\end{equation*}
$$

### 3.3. Characterization of Observational Equivalence as an Independence Condition

Making use of the connection between independence and observational equivalence, we can use (3.8) to provide a characterization of observational equivalence. This is established in the next theorem.

Theorem 3.2: Suppose that $\left(r, f_{U}\right) \in \Gamma \times \Phi$ and $\tilde{r} \in \Gamma$. There exists $f_{\tilde{U}} \in \Phi$ such that $\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$ if and only if for all $y, x$, (3.8) is satisfied.

The proof of Theorem 3.2 follows, again, by the previous arguments. Observational equivalence between $\left(\tilde{r}, f_{\tilde{U}}\right)$ and $\left(r, f_{U}\right)$, as in (3.1) and (3.4), implies that $f_{\tilde{U} \mid X=x}$ defined by (3.5) satisfies $\partial f_{\tilde{U} \mid X=x}(\widetilde{u}) / \partial x=0$ for all $\widetilde{u}, x$. By Theorem 3.1, this implies (3.8). Conversely, given $\left(r, f_{U}\right)$ and $\widetilde{r}$, define $f_{\tilde{U} \mid X=x}$ by (3.5). The condition in (3.8) implies, by Theorem 3.1, that $\partial \tilde{\tilde{U}} \mid X=x(\widetilde{u}) / \partial x=0$ for all $\tilde{u}, x$. Hence, $\widetilde{U}$ is independent of $X$. This together with (3.4) and (3.5) implies that $\left(\widetilde{r}, f_{\tilde{U}}\right)$ and $\left(r, f_{U}\right)$ are observational equivalent.

We next provide some intuition about condition (3.8) by means of a particular example. Note that (3.8) is a set of $K$ restrictions on the density $f_{U}$, the function $r$, and the alternative function $\widetilde{r}$. These restrictions highlight the power of the density $f_{U}$ to restrict the set of observationally equivalent values of functionals. Suppose, for example, that the model has the form

$$
U=m(Y, Z)+B X
$$

where $Y$ is the vector of observable endogenous variables and $(Z, X) \in R^{K_{1}+K_{2}}$ is a vector of observable exogenous variables ( $K_{2} \geq G ; K_{1}$ may be 0 ), and where $B$ is a $G \times K_{2}$ matrix of constants. Let an alternative model be

$$
\tilde{U}=\tilde{m}(Y, Z)+\widetilde{B} X
$$

Consider determining the implications of observational equivalence for the relationship between $\partial m(y, z) / \partial(y, z)$ and $\partial \widetilde{m}(y, z) / \partial(y, z)$ at some specified value $(y, z)$ of $(Y, Z)$. Assume that the range of the function
$\partial \log \left(f_{U}(r(y, z, \cdot))\right) / \partial u: R^{K_{2}} \rightarrow R^{G}$ contains an open neighborhood. Note that as $y, z$ stay fixed and $x$ varies, the matrix $A(y, z, x ; \partial r, \partial \widetilde{r})$ and the vector $b\left(y, z, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)$ stay constant, since the derivatives of $m$ and $\widetilde{m}$ do not depend on $x$, and the derivatives $B$ and $\widetilde{B}$ with respect to $X$ are constant. On the other hand, the value of $\partial \log \left(f_{U}(r(y, z, \cdot))\right) / \partial u$ will, by assumption, vary. When multiplied by nonzero elements of $A(y, x ; \partial r, \partial \widetilde{r})$, these different values of $\partial \log \left(f_{U}(r(y, z, \cdot))\right) / \partial u$ should cause the equality in (3.8) not to be satisfied. Hence, observational equivalence will force elements of the matrix $A(y, z, x ; \partial r, \partial \widetilde{r})$ to be zero. Let $a_{i j}$ denote the element in the $i$ th row and $\left(K_{1}+j\right)$ th column of $A(y, z, x ; \partial r, \partial \widetilde{r})$. It is possible to show (see, e.g., Brown (1983), Roehrig (1988), or Matzkin (2005)) that $a_{i j} \neq 0$ if and only if the rank of the matrix

$$
\left(\begin{array}{cc}
\frac{\partial r^{i}(y, z, x)}{\partial(y, z)} & \frac{\partial r^{i}(y, x)}{\partial x_{j}} \\
\frac{\partial \widetilde{r}(y, z, x)}{\partial(y, z)} & \frac{\partial \widetilde{r}(y, x)}{\partial x_{j}}
\end{array}\right)
$$

is $G+1$, where $r=\left(r^{1}, \ldots, r^{G}\right)$. Hence, observational equivalence together with variation in the value of the vector $\partial \log \left(f_{U}(r(y, z, x))\right) / \partial u$ will imply restrictions on the rank of matrices whose elements are derivatives of $\widetilde{r}$ and of $r$. The next subsection provides a rank condition on a matrix that depends also on the vector $\partial \log \left(f_{U}(r(y, z, x))\right) / \partial u$ and on $b\left(y, z, x ; \partial r, \partial \widetilde{r}, \partial^{2} r, \partial^{2} \widetilde{r}\right)$, and from which all particular cases can be derived.

### 3.4. Rank Conditions for Observational Equivalence

The condition for independence between $\widetilde{U}$ and $X$, or alternatively, the condition for observational equivalence, can be expressed in terms of a condition about the rank of a matrix. To see this, recall the equation determining the distribution of $\widetilde{U}$ conditional on $X=x$. By our assumptions, this distribution always exists. Its density is defined by the condition that for all $y, x$,

$$
f_{\widetilde{U} \mid X=x}(\widetilde{r}(y, x))\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|=f_{U}(r(y, x))\left|\frac{\partial r(y, x)}{\partial y}\right| .
$$

Taking logs on both sides and differentiating the expression first with respect to $y$ and then with respect to $x$, one gets that

$$
\begin{align*}
& \left(\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{r}(y, x))}{\partial \widetilde{u}}\right)^{\prime} \frac{\partial \widetilde{r}(y, x)}{\partial y}+\left(\frac{\partial}{\partial y} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)^{\prime}  \tag{3.9}\\
& =\left(\frac{\partial \log \left(f_{U}(r(y, x))\right)}{\partial u}\right)^{\prime} \frac{\partial r(y, x)}{\partial y}+\left(\frac{\partial}{\partial y} \log \left|\frac{\partial r(y, x)}{\partial y}\right|\right)^{\prime}
\end{align*}
$$

and

$$
\begin{align*}
& \left(\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{r}(y, x))}{\partial \widetilde{u}}\right)^{\prime} \frac{\partial \widetilde{r}(y, x)}{\partial x}+\left(\frac{\partial}{\partial x} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)^{\prime}  \tag{3.10}\\
& \quad+\left(\left.\frac{\partial \log f_{\widetilde{U} \mid X=x}(t)}{\partial x}\right|_{t=\widetilde{r}(y, x)}\right)^{\prime} \\
& =\left(\frac{\partial \log \left(f_{U}(r(y, x))\right)}{\partial u}\right)^{\prime} \frac{\partial r(y, x)}{\partial x}+\left(\frac{\partial}{\partial x} \log \left|\frac{\partial r(y, x)}{\partial y}\right|\right)^{\prime},
\end{align*}
$$

where $\partial \log \widetilde{f}_{U}(\widetilde{r}(y, x)) / \partial \widetilde{u}$ and $\partial \log f_{U}(r(y, x)) / \partial u$ are $G \times 1$ vectors, $\partial \widetilde{r}(y$, $x) / \partial y$ and $\partial r(y, x) / \partial y$ are $G \times G$ matrices, whose $i, j$ th entries are, respectively, $\partial \widetilde{r}^{i}(y, x) / \partial y_{j}$ and $\partial r^{i}(y, x) / \partial y_{j} ; \partial \widetilde{r}(y, x) / \partial x$ and $\partial r(y, x) / \partial x$ are $G \times K$ matrices, whose $i, j$ th entries are, respectively, $\partial \widetilde{r}^{i}(y, x) / \partial x_{j}$ and $\partial r^{i}(y, x) / \partial x_{j} ; \partial \log (|\partial \widetilde{r}(y, x) / \partial y|) / \partial y, \partial \log (|\partial r(y, x) / \partial y|) / \partial y$ are $G \times 1$ vectors, and $\partial \log (|\partial \widetilde{r}(y, x) / \partial y|) / \partial x, \partial \log (|\partial r(y, x) / \partial y|) / \partial x$ are $K \times 1$ vectors, where $\widetilde{r}=\left(\widetilde{r}^{1}, \ldots, \widetilde{r}^{G}\right)$ and $r=\left(r^{1}, \ldots, r^{G}\right)$.

The critical term in these expressions, whose value determines the dependence between $\widetilde{U}$ and $X$, is $\partial \log f_{\widetilde{U} \mid X=x}(t) / \partial x$. Given $r, f_{U}$, and $\widetilde{r}$, one can view (3.9) and (3.10) as a system of equations with unknown vectors

$$
\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{r}(y, x))}{\partial \widetilde{u}} \quad \text { and }\left.\quad \frac{\partial \log f_{\widetilde{U} \mid X=x}(t)}{\partial x}\right|_{t=\widetilde{r}(y, x)}
$$

We may ask under what conditions a solution exists and satisfies for all $t$,

$$
\left.\frac{\partial \log f_{\widetilde{U} \mid X=x}(t)}{\partial x}\right|_{t=\widetilde{r}(y, x)}=0
$$

The following theorem establishes a rank condition that guarantees this, and hence it provides an alternative characterization of observational equivalence.
Let

$$
\Delta_{y}\left(y, x ; \partial r, \partial^{2} r, \partial \widetilde{r}, \partial^{2} \widetilde{r}\right)=\frac{\partial}{\partial y} \log \left|\frac{\partial r(y, x)}{\partial y}\right|-\frac{\partial}{\partial y} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|
$$

and

$$
\Delta_{x}\left(y, x ; \partial r, \partial^{2} r, \partial \widetilde{r}, \partial^{2} \widetilde{r}\right)=\frac{\partial}{\partial x} \log \left|\frac{\partial r(y, x)}{\partial y}\right|-\frac{\partial}{\partial x} \log \left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|
$$

ThEOREM 3.3: Suppose that $\left(r, f_{U}\right) \in \Gamma \times \Phi$ and $\tilde{r} \in \Gamma$. There exists $f_{\tilde{U}} \in \Phi$ such that $\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$ if and only if for all $y, x$, the rank of the matrix

$$
\left(\begin{array}{cc}
\left(\frac{\partial \widetilde{r}(y, x)}{\partial y}\right)^{\prime}\left(y, x ; \partial r, \partial^{2} r, \partial \widetilde{r}, \partial^{2} \widetilde{r}\right)  \tag{3.11}\\
+\left(\frac{\partial r(y, x)}{\partial y}\right)^{\prime} \frac{\partial \log \left(f_{U}(r(y, x))\right)}{\partial u} \\
\left(\frac{\partial \widetilde{r}(y, x)}{\partial x}\right)^{\prime} & \Delta_{x}\left(y, x ; \partial r, \partial^{2} r, \partial \widetilde{r}, \partial^{2} \widetilde{r}\right) \\
+\left(\frac{\partial r(y, x)}{\partial x}\right)^{\prime} \frac{\partial \log \left(f_{U}(r(y, x))\right)}{\partial u}
\end{array}\right)
$$

is $G$.
Proof: Let

$$
\begin{aligned}
& \widetilde{r}_{y}=\frac{\partial \widetilde{r}(y, x)}{\partial y}, \quad r_{y}=\frac{\partial r(y, x)}{\partial y}, \quad \widetilde{r}_{x}=\frac{\partial \widetilde{r}(y, x)}{\partial x}, \quad r_{x}=\frac{\partial r(y, x)}{\partial x}, \\
& \widetilde{s}_{u}=\frac{\partial \log f_{\tilde{U} \mid X=x}(\widetilde{r}(y, x))}{\partial u}, \quad s_{u}=\frac{\partial \log f_{U}(r(y, x))}{\partial u}, \\
& \widetilde{s}_{x}=\left.\frac{\partial \log f_{\tilde{U} \mid X=x}(t)}{\partial x}\right|_{t=\widetilde{r}(y, x)}, \\
& \Delta_{y}=\frac{\partial \log \left(\left|\frac{\partial r(y, x)}{\partial y}\right|\right)}{\partial y}-\frac{\partial \log \left(\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)}{\partial y}, \quad \text { and } \\
& \Delta_{x}=\frac{\partial \log \left(\left|\frac{\partial r(y, x)}{\partial y}\right|\right)}{\partial x}-\frac{\partial \log \left(\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|\right)}{\partial x} .
\end{aligned}
$$

Equations (3.9) and (3.10) can be written as

$$
\begin{aligned}
& \widetilde{s}_{u}^{\prime} \widetilde{r}_{y}=s_{u}^{\prime} r_{y}+\Delta_{y}^{\prime}, \\
& \widetilde{s}_{u}^{\prime} \widetilde{r}_{x}+\widetilde{s}_{x}^{\prime}=s_{u}^{\prime} r_{x}+\Delta_{x}^{\prime}
\end{aligned}
$$

or, after transposing, as

$$
\begin{equation*}
\tilde{r}_{y}^{\prime} \tilde{s}_{u}=r_{y}^{\prime} s_{u}+\Delta_{y} \tag{3.12}
\end{equation*}
$$

$$
\text { (3.13) } \widetilde{r}_{x}^{\prime} \widetilde{s}_{u}+\widetilde{s}_{x}=r_{x}^{\prime} s_{u}+\Delta_{x} .
$$

Equation (3.12) states that $r_{y}^{\prime} s_{u}+\Delta_{y}$ is a linear combination of the columns of $\widetilde{r}_{y}^{\prime}$, with the coefficients given by $\widetilde{s}_{u}$. Since $\widetilde{r}_{y}$ is invertible, this vector of coefficients is unique. Suppose that $\widetilde{s}_{x}=0$. Then equation (3.13) states that $r_{x}^{\prime} s_{u}+\Delta_{x}$
is a linear combination of the columns of $\widetilde{r}_{x}^{\prime}$ and that the vector of coefficients is $\widetilde{s}_{u}$ also. Consider the $(G+K) \times(G+1)$ matrix

$$
\left(\begin{array}{cc}
\widetilde{r}_{y}^{\prime} & r_{y}^{\prime} s_{u}+\Delta_{y}  \tag{3.14}\\
\widetilde{r}_{x}^{\prime} & r_{x}^{\prime} s_{u}+\Delta_{x}
\end{array}\right) .
$$

The rank of this matrix must be at least $G$, because $\widetilde{r}_{y}^{\prime}$ is invertible. When $\widetilde{s}_{x}=0$, the last column is a linear combination of the other $G$ columns. Hence, when $\tilde{s}_{x}=0$, the rank of this matrix is $G$. But observational equivalence implies that $\widetilde{s}_{x}=0$. (This can also be seen by using (3.12) to solve for $\widetilde{s}_{u}$, substituting the result in (3.13), and obtaining then that

$$
\tilde{s}_{x}=\left(r_{x}^{\prime}-\widetilde{r}_{x}^{\prime}\left(\widetilde{r}_{y}^{\prime}\right)^{-1} r_{y}^{\prime}\right) s_{u}+\Delta_{x}-\widetilde{r}_{x}^{\prime}\left(\widetilde{r}_{y}^{\prime}\right)^{-1} \Delta_{y},
$$

which is exactly the transpose of the expression in (3.7).) Hence, observational equivalence implies that the rank of the matrix in (3.11) and (3.14) is $G$.

Conversely, suppose that the matrix in (3.11) and (3.14) has rank $G$ for all $y, x$. Then since $\widetilde{r}_{y}$ is invertible, it must be that the last column is a linear combination of the first $G$ columns. Let $\lambda \in R^{G}$ be the vector of coefficients such that

$$
\begin{equation*}
\widetilde{r}_{y}^{\prime} \lambda=r_{y}^{\prime} s_{u}+\Delta_{y} . \tag{3.15}
\end{equation*}
$$

Note that $\lambda$ is unique. Since $\widetilde{s}_{u}$ satisfies (3.12), it must be that $\lambda=\widetilde{s}_{u}$, and since the rank of the matrix being $G$ implies that $\lambda$ satisfies

$$
\begin{equation*}
\widetilde{r}_{x}^{\prime} \lambda=r_{x}^{\prime} s_{u}+\Delta_{x}, \tag{3.16}
\end{equation*}
$$

it must be also that

$$
\begin{equation*}
\tilde{r}_{x}^{\prime} \tilde{s}_{u}=r_{x}^{\prime} s_{u}+\Delta_{x} \tag{3.17}
\end{equation*}
$$

This implies that $\tilde{s}_{x}=0$, which, as shown above, is just (3.7). Hence, if the rank of the matrix is $G,\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$.
Q.E.D.

## 4. IDENTIFICATION IN LINEAR AND SEPARABLE MODELS

We next provide examples that use the results derived in the previous sections to determine the identification of functionals of $\left(r, f_{U}\right)$. Recall that a functional of ( $r, f_{U}$ ) is identified if whenever $\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$, the value of the functional at $\left(\widetilde{r}, f_{\tilde{U}}\right)$ equals its value at $\left(r, f_{U}\right)$.

### 4.1. A Linear Simultaneous Equations Model

Suppose that $r$ and $\widetilde{r}$ are specified to be linear:

$$
r(y, x)=B y+C x \quad \text { and } \quad \widetilde{r}(y, x)=\widetilde{B} y+\widetilde{C} x
$$

where $B$ and $\widetilde{B}$ are $G \times G$ nonsingular matrices, and $C$ and $\widetilde{C}$ are $G \times K$ matrices. Since the functions are linear, for all $y, x, \Delta_{y}=\Delta_{x}=0$. Let $F=[B, C]$ and $\widetilde{F}=[\widetilde{B}, \widetilde{C}]$ denote the matrices of all coefficients. Consider identification of the first row, $F_{1}$, of $F$. Suppose that there exists a value of $(y, x)$ such that the gradient of $\log f_{U}$ evaluated at $r(y, x)$ is $\left(s_{1}\left(r(y, x), \ldots, s_{G}(r(y, x))\right)^{\prime}=(1,0, \ldots, 0)^{\prime}\right.$; that is, $\partial \log f_{U}(r(y, x)) / \partial u_{1} \neq 0$ and for $j=2, \ldots, G, \partial \log f_{U}(r(y, x)) / \partial u_{j}=0$. Observational equivalence then implies that the matrix

$$
\left(\widetilde{F}^{\prime} \quad F_{1}^{\prime}\right)
$$

has rank $G$. Consider linear restrictions on $F_{1}$, denoted by $\phi F_{1}^{\prime}=0$, where $\phi$ is a constant matrix. The rank condition for identification is then ${ }^{3}$

$$
\operatorname{rank}\left(\phi F^{\prime}\right)=G-1
$$

To see this, note that since the rank of $\widetilde{F}^{\prime}$ is $G, F_{1}^{\prime}=\widetilde{F}^{\prime} c$ for some $c$. Then, premultiplying by $\phi$ gives

$$
\phi \widetilde{F}^{\prime} c=\phi F_{1}^{\prime}=0
$$

The rank condition for $\widetilde{F}$ says that $\operatorname{rank}\left(\phi \widetilde{F}^{\prime}\right)=G-1$. Since the first column of $\phi \widetilde{F}^{\prime}$ is zero, this rank condition implies that the other $G-1$ columns of $\widetilde{F}$ must be linearly independent, so that all the elements of $c$ other than the first element, $c_{1}$, must be zero. Therefore,

$$
F_{1}=c_{1} \widetilde{F}_{1} .
$$

By the usual normalization that one of the elements of $F_{1}$ and $\widetilde{F}_{1}$ is equal to 1 , we have $c=1$. That is, we must have that $F_{1}=\widetilde{F}_{1}$. Hence, if $\Gamma$ is the set of linear functions whose coefficients are characterized by $F$, the linear restrictions on the first row, $\phi F_{1}^{\prime}=0$, satisfy $\operatorname{rank}\left(\phi F^{\prime}\right)=G-1$, one coefficient of $F_{1}$ is normalized to 1 , and for some $(y, x), s_{1}(r(y, x)) \neq 0$ while for $j=2, \ldots, G$, $s_{j}(r(y, x))=0$, then $F_{1}$ is identified.

### 4.2. A Demand and Supply Example

Consider a demand and supply model specified as

$$
\begin{aligned}
& u_{1}=D(p, q)+m(I) \\
& u_{2}=S(p, q)+v(w)
\end{aligned}
$$

[^2]and an alternative model specified as
\[

$$
\begin{aligned}
& \widetilde{u}_{1}=\widetilde{D}(p, q)+\widetilde{m}(I), \\
& \widetilde{u}_{2}=\widetilde{S}(p, q)+\widetilde{v}(w)
\end{aligned}
$$
\]

where $p$ and $q$ are, respectively, price and quantity, and $I$ and $w$ are, respectively, income and wages. Suppose that for all $I, m_{I}(I)=\partial m(I) / \partial I>0$, and for all $w, v_{W}(w)=\partial v(w) / \partial w>0$. Assume that $(I, w)$ is independent of $\left(u_{1}, u_{2}\right)$ and the supports of $(m(I), v(w))$ and of $(D(p, q), S(p, q))$ are $R^{2}$. Further assume that there exists $\left(u_{1}^{0}, u_{2}^{0}\right)$, and that for all $u_{1}^{1}, u_{2}^{2}$, there exist $u_{2}^{1}, u_{1}^{2}$ such that

$$
\begin{aligned}
& \frac{\partial f_{U}\left(u_{1}^{0}, u_{2}^{0}\right)}{\partial u_{1}}=\frac{\partial f_{U}\left(u_{1}^{0}, u_{2}^{0}\right)}{\partial u_{2}}=0, \\
& \frac{\partial f_{U}\left(u_{1}^{1}, u_{2}^{1}\right)}{\partial u_{1}} \neq 0, \quad \frac{\partial f_{U}\left(u_{1}^{1}, u_{2}^{1}\right)}{\partial u_{2}}=0, \\
& \frac{\partial f_{U}\left(u_{1}^{2}, u_{2}^{2}\right)}{\partial u_{1}}=0, \quad \frac{\partial f_{U}\left(u_{1}^{2}, u_{2}^{2}\right)}{\partial u_{2}} \neq 0 .
\end{aligned}
$$

We will show that the derivatives of the demand and supply functions are identified up to scale. That is, for any alternative function $\widetilde{r}=(\widetilde{D}+\widetilde{m}, \widetilde{S}+\widetilde{v})$ for which there exists $f_{\tilde{U}}$ such that $\left(\widetilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$, there exists $\lambda_{1}, \lambda_{2} \in R$ such that for all $p, q, I, w$,

$$
\begin{aligned}
& D_{p}(p, q)=\lambda_{1} \widetilde{D}_{p}(p, q) \\
& D_{q}(p, q)=\lambda_{1} \widetilde{D}_{q}(p, q) \\
& m_{I}(I)=\lambda_{1} \widetilde{m}_{I}(I)
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{p}(p, q)=\lambda_{2} \widetilde{S}_{p}(p, q), \\
& S_{q}(p, q)=\lambda_{2} \widetilde{S}_{q}(p, q), \\
& v_{W}(w)=\lambda_{2} \widetilde{v}_{W}(w)
\end{aligned}
$$

Note that, because of the additive separability, $\Delta_{p}$ and $\Delta_{q}$ depend only on $(p, q)$, and $\Delta_{I}=\Delta_{w}=0$. (We suppress arguments for simplicity.) Let $p, q$ be arbitrary. By our assumptions, there exists $\left(I^{0}, w^{0}\right)$, and for all values $I_{1}, w_{2}$, there exist values $I_{2}$ and $w_{1}$ such that

$$
\begin{align*}
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{0}\right), S(p, q)+v\left(w_{0}\right)\right)}{\partial u_{1}}=0  \tag{4.1}\\
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{0}\right), S(p, q)+v\left(w_{0}\right)\right)}{\partial u_{2}}=0
\end{align*}
$$

$$
\begin{aligned}
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{1}\right), S(p, q)+v\left(w_{1}\right)\right)}{\partial u_{1}} \neq 0 \\
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{1}\right), S(p, q)+v\left(w_{1}\right)\right)}{\partial u_{2}}=0 \\
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{2}\right), S(p, q)+v\left(w_{2}\right)\right)}{\partial u_{1}}=0 \\
& \frac{\partial f_{U}\left(D(p, q)+m\left(I_{2}\right), S(p, q)+v\left(w_{2}\right)\right)}{\partial u_{2}} \neq 0
\end{aligned}
$$

Observational equivalence implies that for any values of $(I, w)$,

$$
\operatorname{rank}\left(\begin{array}{ccc}
\widetilde{D}_{p}(p, q) & \widetilde{S}_{p}(p, q) & \Delta_{p}+s_{1} D_{p}(p, q)+s_{2} S_{p}(p, q) \\
\widetilde{D}_{q}(p, q) & \widetilde{S}_{q}(p, q) & \Delta_{q}+s_{1} D_{q}(p, q)+s_{2} S_{q}(p, q) \\
\widetilde{m}_{I}(I) & 0 & s_{1} m_{I}(I) \\
0 & \widetilde{v}_{W}(w) & s_{2} v_{W}(w)
\end{array}\right)=2
$$

where

$$
s_{1}=\partial \log f_{U}(D(p, q)+m(I), S(p, q)+v(w)) / \partial u_{1}
$$

and

$$
s_{2}=\partial \log f_{U}(D(p, q)+m(I), S(p, q)+v(w)) / \partial u_{2}
$$

Letting $(I, w)=\left(I_{0}, w_{0}\right)$, we get that

$$
\operatorname{rank}\left(\begin{array}{ccc}
\widetilde{D}_{p}(p, q) & \widetilde{S}_{p}(p, q) & \Delta_{p} \\
\widetilde{D}_{q}(p, q) & \widetilde{S}_{q}(p, q) & \Delta_{q} \\
\widetilde{m}_{I}\left(I_{0}\right) & 0 & 0 \\
0 & \widetilde{v}_{W}\left(w_{0}\right) & 0
\end{array}\right)=2
$$

Since, by assumption, the matrix

$$
\left(\begin{array}{cc}
\widetilde{D}_{p}(p, q) & \widetilde{S}_{p}(p, q) \\
\widetilde{D}_{q}(p, q) & \widetilde{S}_{q}(p, q)
\end{array}\right)
$$

is invertible, the third column must be a linear combination of the first two. It follows that for some $\lambda_{1}^{0}=\lambda_{1}\left(p, q, I_{0}, w_{0}\right)$ and $\lambda_{2}^{0}=\lambda_{2}\left(p, q, I_{0}, w_{0}\right)$,

$$
\lambda_{1}^{0} \tilde{m}_{I}\left(I_{0}\right)=0 \quad \text { and } \quad \lambda_{2}^{0} \widetilde{v}_{W}\left(w_{0}\right)=0
$$

Since $\widetilde{m}_{I}\left(I_{0}\right) \neq 0$ and $\widetilde{v}_{W}\left(w_{0}\right) \neq 0$, it must be that $\lambda_{1}^{0}=\lambda_{2}^{0}=0$. Hence, $\Delta_{p}(p, q)=\Delta_{q}(p, q)=0$. Since $\Delta_{p}(p, q)$ and $\Delta_{q}(p, q)$ do not depend on $I, w$, it follows that for all $(I, w)$,

$$
\operatorname{rank}\left(\begin{array}{ccc}
\widetilde{D}_{p}(p, q) & \widetilde{S}_{p}(p, q) & s_{1} D_{p}(p, q)+s_{2} S_{p}(p, q) \\
\widetilde{D}_{q}(p, q) & \widetilde{S}_{q}(p, q) & s_{1} D_{q}(p, q)+s_{2} S_{q}(p, q) \\
\widetilde{m}_{I}(I) & 0 & s_{1} m_{I}(I) \\
0 & \widetilde{v}_{W}(w) & s_{2} v_{W}(w)
\end{array}\right)=2
$$

Letting $(I, w)=\left(I_{1}, w_{1}\right)$, for arbitrary $I_{1}$ and for $w_{1}$ as in (4.1), the matrix becomes

$$
\left(\begin{array}{ccc}
\widetilde{D}_{p}(p, q) & \widetilde{S}_{p}(p, q) & s_{1} D_{p}(p, q) \\
\widetilde{D}_{q}(p, q) & \widetilde{S}_{q}(p, q) & s_{1} D_{q}(p, q) \\
\widetilde{m}_{I}\left(I_{1}\right) & 0 & s_{1} m_{I}\left(I_{1}\right) \\
0 & \widetilde{v}_{W}\left(w_{1}\right) & 0
\end{array}\right)
$$

Again, linear independence of the first two columns and the matrix having rank 2 implies that the third column is a linear combination of the first two. The zeroes in the fourth row imply that the coefficient of the second column is zero. Hence, for some $\lambda_{1}^{1}=\lambda_{1}\left(p, q, I_{1}, w_{1}\right)$,

$$
\begin{aligned}
& \lambda_{1}^{1}\left(p, q, I_{1}, w_{1}\right) \widetilde{D}_{p}(p, q)=D_{p}(p, q) \\
& \lambda_{1}^{1}\left(p, q, I_{1}, w_{1}\right) \widetilde{D}_{q}(p, q)=D_{q}(p, q) \\
& \lambda_{1}^{1}\left(p, q, I_{1}, w_{1}\right) \widetilde{m}_{I}\left(I_{1}\right)=m_{I}\left(I_{1}\right)
\end{aligned}
$$

Since $I_{1}$ was arbitrary and $\widetilde{D}$ and $D$ are not functions of $I, w$, the first two equations imply that $\lambda_{1}^{1}$ is not a function of $I, w$. Likewise, reaching these equations by fixing $I_{1}$, varying ( $p, q$ ) arbitrarely, and letting $w_{1}$ satisfy (4.1), the third equation implies that $\lambda_{1}^{1}$ is not a function of $p, q$. Hence, $\lambda_{1}^{1}$ is a constant. It follows that the derivatives $D_{p}, D_{q}$, and $m_{I}$ are identified up to scale.

An analogous argument can be used to show that $S_{p}, S_{q}$, and $v_{W}$ are also identified up to scale.

## 5. IDENTIFICATION IN NONPARAMETRIC NONSEPARABLE MODELS

We next apply our results to two standard nonparametric models with nonadditive unobservable random terms. We first consider the single equation model, with $G=1$, considered in Matzkin (1999, 2003):

$$
y=m(x, u)
$$

We show below that in this model, with $m$ strictly increasing in $u$, application of our theorems implies the well known result that for all $u$,

$$
\frac{\partial m(x, u)}{\partial x}
$$

the partial derivative of $m$ with respect to $x$, for any fixed value of $x$ and $u$, is identified.
In Section 5.2, we consider the triangular model with nonadditive unobservable random terms considered in Chesher (2003) and Imbens and Newey (2003):

$$
\begin{aligned}
& y_{1}=m_{1}\left(y_{2}, u_{1}\right), \\
& y_{2}=m_{2}\left(x, u_{2}\right) .
\end{aligned}
$$

Assuming that $X$ is distributed independently of ( $u_{1}, u_{2}$ ), and that $m_{1}$ and $m_{2}$ are strictly increasing, respectively, in $u_{1}$ and $u_{2}$, we derive the well-known result that for all $u_{1}, y_{2}$,

$$
\frac{\partial m_{1}\left(y_{2}, u_{1}\right)}{\partial y_{2}}
$$

is identified. ${ }^{4}$

### 5.1. Single Equation Model

Consider the model

$$
y=m(x, u)
$$

with $y, u \in R, u$ and $x$ independently distributed, $f_{U} \in \Phi$, and the inverse of $m$ belonging to $\Gamma$. Letting $r$ denote the inverse of $m$ with respect to $u$, we have the model

$$
u=r(y, x)
$$

with $\partial r(y, x) / \partial y>0$. Let $\widetilde{r} \in \Gamma$ be an alternative function, so that $\widetilde{u}=\widetilde{r}(y, x)$. The condition for observational equivalence requires that the matrix

$$
\left[\begin{array}{ll}
\widetilde{r}_{y} & s r_{y}+\Delta_{y} \\
\widetilde{r}_{x} & s r_{x}+\Delta_{x}
\end{array}\right]
$$

has rank 1, for all $y, x$, where $s=\partial \log f_{U}(r(y, x)) / \partial u$. Hence, for all $y, x$,

$$
s r_{x} \widetilde{r}_{y}+\Delta_{x} \widetilde{x}_{y}=s r_{y} \widetilde{r}_{x}+\Delta_{y} \widetilde{r}_{x}
$$

[^3]or
$$
s\left(r_{y} \widetilde{r}_{x}-r_{x} \widetilde{r}_{y}\right)=\Delta_{x} \widetilde{r}_{y}-\Delta_{y} \widetilde{r}_{x} .
$$

Note that

$$
\frac{\partial}{\partial y}\left(\frac{r_{x}}{r_{y}}\right)=\frac{r_{y x}}{r_{y}}-\frac{r_{y y}}{r_{y}} \frac{r_{x}}{r_{y}}
$$

Hence,

$$
\begin{aligned}
\Delta_{x} & =\frac{r_{y x}}{r_{y}}-\frac{\widetilde{r}_{y x}}{\widetilde{r}_{y}} \\
& =\frac{r_{y y}}{r_{y}} \frac{r_{x}}{r_{y}}-\frac{\widetilde{r}_{y y}}{\widetilde{r}_{y}} \frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}+\frac{\partial}{\partial y}\left(\frac{r_{x}}{r_{y}}-\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}\right)
\end{aligned}
$$

and, since

$$
\begin{aligned}
& \Delta_{y}=\frac{r_{y y}}{r_{y}}-\frac{\widetilde{r}_{y y}}{\widetilde{r}_{y}} \\
& \frac{-\Delta_{x} \widetilde{r}_{y}+\Delta_{y} \widetilde{r}_{x}}{\widetilde{r}_{y}}=\frac{r_{y y}}{r_{y}}\left(\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}-\frac{r_{x}}{r_{y}}\right)+\frac{\partial}{\partial y}\left(\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}-\frac{r_{x}}{r_{y}}\right)
\end{aligned}
$$

Hence, the rank condition implies that

$$
s r_{y}\left(\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}-\frac{r_{x}}{r_{y}}\right)+\frac{r_{y y}}{r_{y}}\left(\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}-\frac{r_{x}}{r_{y}}\right)+\frac{\partial}{\partial y}\left(\frac{\widetilde{r}_{x}}{\widetilde{r}_{y}}-\frac{r_{x}}{r_{y}}\right)=0 .
$$

Writing explicitly the arguments of all functions and multiplying both sides of the equality by $f_{U}(r(y, x)) r_{y}(y, x)$ gives

$$
\frac{\partial}{\partial y}\left[f_{U}(r(y, x)) r_{y}(y, x)\left(\frac{\widetilde{r}_{x}(y, x)}{\widetilde{r}_{y}(y, x)}-\frac{r_{x}(y, x)}{r_{y}(y, x)}\right)\right]=0 .
$$

Observational equivalence then implies that the function $v$ defined by

$$
v(y, x)=\left[f_{U}(r(y, x)) r_{y}(y, x)\left(\frac{\widetilde{r}_{x}(y, x)}{\widetilde{r}_{y}(y, x)}-\frac{r_{x}(y, x)}{r_{y}(y, x)}\right)\right]
$$

is a constant function of $y$. Since for any $x$, the range of $r(\cdot, x)$ is $R$ and $f_{U} r_{y} \rightarrow$ 0 as $|y| \rightarrow \infty$, as long as the ratios $\widetilde{r}_{x} / \widetilde{r}_{y}$ and $r_{x} / r_{y}$ are uniformly bounded, it must be that for any $y, x, v(y, x)=0$. Since $r_{y}>0$, it follows from these conditions that for all $y, x$ at which $f_{U}(r(y, x))>0$,

$$
\begin{equation*}
\frac{\widetilde{r}_{x}(y, x)}{\widetilde{r}_{y}(y, x)}=\frac{r_{x}(y, x)}{r_{y}(y, x)} \tag{5.1}
\end{equation*}
$$

Hence, observational equivalence implies that the ratio of the derivatives of $r$ is identified. Since

$$
u=r(m(x, u), x),
$$

it follows that

$$
\frac{\partial m(x, u)}{\partial x}=-\frac{r_{x}(y, x)}{r_{y}(y, x)}
$$

is identified.

### 5.2. A Triangular Model

Consider now the model

$$
\begin{aligned}
& y_{1}=m_{1}\left(y_{2}, u_{1}\right), \\
& y_{2}=m_{2}\left(x, u_{2}\right)
\end{aligned}
$$

with $y_{1}, y_{2}, u_{1}, u_{2} \in R, m_{1}$ strictly increasing in $u_{1}$, and $m_{2}$ strictly increasing in $u_{2}$. Assume that $x$ is distributed independently of $\left(u_{1}, u_{2}\right)$ and that the density of $u$ belongs to $\Phi$. Let $r^{1}$ denote the inverse of $m_{1}$ with respect to $u_{1}$ and let $r^{2}$ denote the inverse of $m_{2}$ with respect to $u_{2}$. Hence

$$
\begin{aligned}
& u_{1}=r^{1}\left(y_{1}, y_{2}\right), \\
& u_{2}=r^{2}\left(y_{2}, x\right)
\end{aligned}
$$

Consider the alternative model

$$
\begin{aligned}
& \widetilde{u}_{1}=\widetilde{r}^{1}\left(y_{1}, y_{2}\right), \\
& \widetilde{u}_{2}=\widetilde{r}^{2}\left(y_{2}, x\right)
\end{aligned}
$$

Assume that $r=\left(r^{1}, r^{2}\right) \in \Gamma$ and $\widetilde{r}=\left(\widetilde{r}^{1}, \widetilde{r}^{2}\right) \in \Gamma$.
Observational equivalence implies that the rank of the matrix

$$
\left[\begin{array}{ccc}
\widetilde{r}_{y_{1}}^{1} & 0 & s_{1} r_{y_{1}}^{1}+\Delta_{y_{1}} \\
\widetilde{r}_{y_{2}}^{1} & \widetilde{r}_{y_{2}}^{2} & s_{1} r_{y_{2}}^{1}+s_{2} r_{y_{2}}^{2}+\Delta_{y_{2}} \\
0 & \widetilde{r}_{x}^{2} & s_{2} r_{x}^{2}+\Delta_{x}
\end{array}\right]
$$

is 2 , where

$$
s_{1}=\partial \log f_{U_{1}, U_{2}}\left(r^{1}\left(y_{1}, y_{2}\right), r^{2}\left(y_{2}, x\right)\right) / \partial u_{1}
$$

and

$$
s_{2}=\partial \log f_{U_{1}, U_{2}}\left(r^{1}\left(y_{1}, y_{2}\right), r^{2}\left(y_{2}, x\right)\right) / \partial u_{2} .
$$

Since the first two columns are linearly independent, the third column must be a linear combination of the first two. Hence, for some $\lambda_{1}, \lambda_{2}$,

$$
\begin{aligned}
& \lambda_{1} \widetilde{r}_{y_{1}}^{1}=s_{1} r_{y_{1}}^{1}+\frac{r_{y_{1} y_{1}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{1} y_{1}}^{1}}{\widetilde{r}_{y_{1}}^{1}}, \\
& \lambda_{1} \widetilde{r}_{y_{2}}^{1}+\lambda_{2} \widetilde{r}_{y_{2}}^{2}=s_{1} r_{y_{2}}^{1}+s_{2} r_{y_{2}}^{2}+\frac{r_{y_{1} y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{1} y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}+\frac{r_{y_{2} y_{2}}^{2}}{r_{y_{2}}^{2}}-\frac{\widetilde{r}_{y_{2} y_{2}}^{2}}{\widetilde{r}_{y_{2}}^{2}} \\
& \lambda_{2} \widetilde{r}_{x}^{2}=s_{2} r_{x}^{2}+\frac{r_{y_{2} x}^{2}}{r_{y_{2}}^{2}}-\frac{\widetilde{r}_{y_{2} x}^{2}}{\widetilde{r}_{y_{2}}^{2}}
\end{aligned}
$$

Solving for $\lambda_{1}$ and $\lambda_{2}$ from the first two equations, and substituting them into the third, one gets, after rearranging terms, the expression

$$
\begin{aligned}
& s_{1} r_{y_{1}}^{1}\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right)-s_{2} r_{y_{2}}^{2} \frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}}\left(\frac{r_{x}^{2}}{r_{y_{2}}^{2}}-\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}\right) \\
& \quad-\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} \frac{\partial}{\partial y_{2}}\left(\frac{r_{x}^{2}}{r_{y_{2}}^{2}}-\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}\right)-\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} \frac{r_{y_{2} y_{2}}^{2}}{r_{y_{2}}^{2}}\left(\frac{r_{x}^{2}}{r_{y_{2}}^{2}}-\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}\right) \\
& \quad-\frac{\partial}{\partial y_{1}}\left(\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}-\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}\right)+\frac{r_{y_{1} y_{1}}^{1}}{r_{y_{1}}^{1}}\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right) \\
& =0 .
\end{aligned}
$$

Multiplying both sides of the equality by $f_{U} r_{y_{1}}^{1} r_{y_{2}}^{2}$ gives

$$
\begin{aligned}
& r_{y_{2}}^{2}\left(\left.\frac{\partial f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right)}{\partial y_{1}}\right|_{u_{2}=r^{2}\left(y_{2}, x\right)}\right) r_{y_{1}}^{1}\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right) \\
& \quad+r_{y_{2}}^{2} f_{U}\left(\frac{\partial r_{y_{1}}^{1}}{\partial y_{1}}\right)\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right)+r_{y_{2}}^{2} f_{U} r_{y_{1}}^{1} \frac{\partial\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right)}{\partial y_{1}} \\
& \quad+\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} r_{y_{1}}^{1}\left(\left.\frac{\partial f_{U}\left(u_{1}, r^{2}\left(y_{2}, x\right)\right)}{\partial y_{2}}\right|_{u_{1}=r^{1}\left(y_{1}, y_{2}\right)}\right) r_{y_{2}}^{2}\left(\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}-\frac{r_{x}^{2}}{r_{y_{2}}^{2}}\right) \\
& \quad+\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} r_{y_{1}}^{1} f_{U}\left(\frac{\partial r_{y_{2}}^{2}}{\partial y_{2}}\right)\left(\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}-\frac{r_{x}^{2}}{r_{y_{2}}^{2}}\right)+\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} f_{U} r_{y_{1}}^{1} r_{y_{2}}^{2} \frac{\partial\left(\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}-\frac{r_{x}^{2}}{r_{y_{2}}^{2}}\right)}{\partial y_{2}} \\
& =0
\end{aligned}
$$

or, after gathering terms,

$$
\begin{aligned}
& r_{y_{2}}^{2} \frac{\partial}{\partial y_{1}}\left[\left.f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right)\right|_{u_{2}=r^{2}\left(y_{2}, x\right)} r_{y_{1}}^{1}\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{{\widetilde{r_{2}}}_{1}^{1}}{\widetilde{r}_{y_{1}}^{1}}\right)\right] \\
& \quad+\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} r_{y_{1}}^{1} \frac{\partial}{\partial y_{2}}\left[\left(\left.f_{U}\left(u_{1}, r^{2}\left(y_{2}, x\right)\right)\right|_{u_{1}=r^{1}\left(y_{1}, y_{2}\right)}\right) r_{y_{2}}^{2}\left(\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}-\frac{r_{x}^{2}}{r_{y_{2}}^{2}}\right)\right] \\
& \quad=0
\end{aligned}
$$

Hence, after dividing by $r_{y_{2}}^{2} r_{y_{1}}^{1}$, it follows that observational equivalence implies that

$$
\begin{align*}
& \frac{1}{r_{y_{1}}^{1}} \frac{\partial}{\partial y_{1}}\left[\left.f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right)\right|_{u_{2}=r^{2}\left(y_{2}, x\right)} r_{y_{1}}^{1}\left(\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}-\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{y}_{y_{1}}^{1}}\right)\right]  \tag{5.2}\\
& \quad+\frac{\widetilde{r}_{y_{2}}^{2}}{\widetilde{r}_{x}^{2}} \frac{1}{r_{y_{2}}^{2}} \frac{\partial}{\partial y_{2}}\left[\left(\left.f_{U}\left(u_{1}, r^{2}\left(y_{2}, x\right)\right)\right|_{u_{1}=r^{1}\left(y_{1}, y_{2}\right)}\right) r_{y_{2}}^{2}\left(\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}-\frac{r_{x}^{2}}{r_{y_{2}}^{2}}\right)\right] \\
& =0
\end{align*}
$$

Note that the first term does not depend on $x$, other than through $u_{2}$, and the second term does not depend on $y_{1}$, other than through $u_{1}$. Since the independence between $x$ and $\left(u_{1}, u_{2}\right)$ implies independence between $x$ and $u_{2}$, the result, (5.1), derived in the single equation model, applied to $r^{2}$, can be used to prove that

$$
\frac{\widetilde{r}_{x}^{2}}{\widetilde{r}_{y_{2}}^{2}}=\frac{r_{x}^{2}}{r_{y_{2}}^{2}} .
$$

This means that the ratio of the derivatives of the structural function $r^{2}$ can be identified. This also implies that the second term in (5.2) equals zero. So (5.2) becomes

$$
\frac{\partial}{\partial y_{1}}\left[\left.f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right)\right|_{u_{2}=r^{2}\left(y_{2}, x\right)} r_{y_{1}}^{1}\left(\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}-\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}\right)\right]=0 .
$$

In other words, the function $v$ defined by

$$
v\left(y_{1}, y_{2}, u_{2}\right)=\left[\left.f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right)\right|_{u_{2}=r^{2}\left(y_{2}, x\right)} r_{y_{1}}^{1}\left(\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}-\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}}\right)\right]
$$

must be constant in $y_{1}$. Since for any $y_{2}$ and $u_{2}, f_{U}\left(r^{1}\left(y_{1}, y_{2}\right), u_{2}\right) r_{y_{1}}^{1} \rightarrow 0$ as $\left|y_{1}\right| \rightarrow \infty$, as long as the ratios of the derivatives of $r^{1}$ and $\widetilde{r}^{1}$ are uniformly
bounded, we can conclude that for all $\left(y_{1}, y_{2}, u_{2}\right), v\left(y_{1}, y_{2}, u_{2}\right)=0$. Hence, under these conditions, observational equivalence implies that

$$
\frac{\widetilde{r}_{y_{2}}^{1}}{\widetilde{r}_{y_{1}}^{1}}=\frac{r_{y_{2}}^{1}}{r_{y_{1}}^{1}} .
$$

This shows that the ratio of the derivatives of the structural function $r^{1}$ can be identified.
Since

$$
u_{1}=r^{1}\left(y_{1}, y_{2}\right)=r^{1}\left(m_{1}\left(y_{2}, u_{1}\right), y_{2}\right)
$$

and $r_{y_{1}}^{1} \neq 0$, the implicit function theorem implies that

$$
\frac{\partial m_{1}\left(y_{2}, u_{1}\right)}{\partial y_{2}}=-\frac{\frac{\partial r^{1}\left(m_{1}\left(y_{2}, u_{1}\right), y_{2}\right)}{\partial y_{2}}}{\frac{\partial r^{1}\left(m_{1}\left(y_{2}, u_{1}\right), y_{2}\right)}{\partial y_{1}}}=-\left.\frac{\frac{\partial r^{1}\left(y_{1}, y_{2}\right)}{\partial y_{2}}}{\frac{\partial r^{1}\left(y_{1}, y_{2}\right)}{\partial y_{1}}}\right|_{y_{1}=m_{1}\left(y_{2}, u_{1}\right)} .
$$

Hence, the partial derivative of $m_{1}$ with respect to $y_{2}$ is identified.

## 6. OBSERVATIONAL EQUIVALENCE OF TRANSFORMATIONS OF STRUCTURAL FUNCTIONS

A stylized way of analyzing observational equivalence can be derived by considering directly the mapping from the vectors of observable and unobservable explanatory variables, $X$ and $U$, to an alternative vector of unobservable explanatory variables, $\widetilde{U}$, generated by an alternative function, $\widetilde{r} .^{5}$ To define such a relationship, we note that given the function $r$ and an alternative function $\widetilde{r} \in \Gamma$, we can express $\widetilde{r}$ as a transformation $\tilde{g}$ of $U$ and $X$ by defining $\widetilde{g}$ for all $x, u$ as

$$
\begin{equation*}
\widetilde{g}(u, x)=\widetilde{r}(h(x, u), x), \tag{6.1}
\end{equation*}
$$

where $h(x, u)$ is the reduced form function derived from the structural function $r$. By our assumptions on $r$ and $\widetilde{r}$, it follows that $\widetilde{g}$ is invertible in $u$ and that

$$
\begin{equation*}
\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|=\left|\frac{\partial \widetilde{r}(h(x, u), x)}{\partial y}\right|\left|\frac{\partial h(x, u)}{\partial u}\right|>0 . \tag{6.2}
\end{equation*}
$$

[^4]The representation of $\widetilde{r}$ in terms of the transformation $g$ implies that for all $y, x$,

$$
\begin{equation*}
\widetilde{r}(y, x)=\widetilde{g}(r(y, x), x) . \tag{6.3}
\end{equation*}
$$

Recall that, for any given $x$, ( $\left.\widetilde{r}, f_{\tilde{U} \mid X=x}\right)$ generates the same distribution of $Y$ given $X=x$, as $\left(r, f_{U}\right)$ does, if and only if for all $y$,

$$
\begin{equation*}
f_{\widetilde{U} \mid X=x}(\widetilde{r}(y, x))\left|\frac{\partial \widetilde{r}(y, x)}{\partial y}\right|=f_{U}(r(y, x))\left|\frac{\partial r(y, x)}{\partial y}\right| \tag{6.4}
\end{equation*}
$$

Hence, using (6.1)-(6.3), we can state that the transformation $\widetilde{g}(U, X)$ generates the same distribution of $Y$ given $X=x$ as $\left(r, f_{U}\right)$ generates if and only if for all $u$,

$$
\begin{equation*}
f_{\tilde{U} \mid X=x}(\widetilde{g}(u, x))\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|=f_{U}(u) . \tag{6.5}
\end{equation*}
$$

The analogous results to Theorems 3.1 and 3.3 are Theorems 6.1 and 6.2 below.

THEOREM 6.1: Suppose that $\left(r, f_{U}\right) \in(\Gamma \times \Phi)$ and that $\widetilde{r} \in \Gamma$. Define the transformation $\widetilde{g}$ by (6.1) and let $\widetilde{U}=\widetilde{g}(U, X)$ be such that for all $x, f_{\widetilde{U} \mid X=x} \in \Phi$. Then $\partial \tilde{U}_{\tilde{U} \mid X=x}(\widetilde{u}) / \partial x=0$ for all $x$ and $\widetilde{u}$ if and only if for all $u$ and $x$,

$$
\begin{align*}
& {\left[-\frac{\partial \log \left(f_{U}(u)\right)}{\partial u}+\frac{\partial}{\partial u} \log \left(\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|\right)\right]\left[\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{-1} \frac{\partial \widetilde{g}(u, x)}{\partial x}\right]}  \tag{6.6}\\
& \quad=\frac{\partial}{\partial x} \log \left(\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|\right)
\end{align*}
$$

THEOREM 6.2: Suppose that $\left(r, f_{U}\right) \in(\Gamma \times \Phi)$ and that $\widetilde{r} \in \Gamma$. Define the transformation $\tilde{g}$ by (6.1) and let $\widetilde{U}=\widetilde{g}(U, X)$. Suppose further that for all $x$, $f_{\tilde{U} \mid X=x} \in \Phi$. Then $\left(f_{\tilde{U}}, \widetilde{g}(r(y, x), x)\right)$ is observationally equivalent to $\left(r, f_{U}\right)$ if and only if for all $u, x$, the rank of the matrix

$$
\left(\begin{array}{cc}
\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{\prime} & \frac{\partial \log f_{U}(u)}{\partial u}-\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial u} \\
\left(\frac{\partial \widetilde{g}(u, x)}{\partial x}\right)^{\prime} & -\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial x}
\end{array}\right)
$$

is $G$.

The proofs of Theorems 6.1 and 6.2 use arguments similar to the ones used to derive Theorems 3.1 and 3.3, and are given in the Appendix. To provide an example of the usefulness of these results, suppose that $X \in R$ and $G=2$, and consider evaluating the implications of observational equivalence when the relationship between $\tilde{r}$ and $r$ is given by

$$
\widetilde{r}(y, x)=\widetilde{B}(x) r(y, x)
$$

where $\widetilde{B}(x)$ is a $2 \times 2$ matrix of functions. One such example for $\widetilde{B}(x)$ could be ${ }^{6}$
(6.7) $\quad \widetilde{B}(x)=\left(\begin{array}{cc}\cos (x) & \sin (x) \\ -\sin (x) & \cos (x)\end{array}\right)$.

Application of Theorem 6.2 yields the result that observational equivalence implies that for all $x, u$, the matrix

$$
\left(\begin{array}{ccc}
\cos (x) & -\sin (x) & \frac{\partial \log f_{U}(u)}{\partial u_{1}} \\
\sin (x) & \cos (x) & \frac{\partial \log f_{U}(u)}{\partial u_{2}} \\
-u_{1} \sin (x)+u_{2} \cos (x) & -u_{1} \cos (x)-u_{2} \sin (x) & 0
\end{array}\right)
$$

must have rank 2 . This holds if and only if for all $u_{1}, u_{2}$,

$$
\frac{\partial f_{U}\left(u_{1}, u_{2}\right) / \partial u_{1}}{\partial f_{U}\left(u_{1}, u_{2}\right) / \partial u_{2}}=\frac{u_{1}}{u_{2}}
$$

Note that this condition is satisfied by the bivariate independent standard normal density. Hence, if $U$ is distributed $N(0, I), \widetilde{B}(x)$ is as specified above, and

$$
\widetilde{u}=\widetilde{r}(y, x)=\widetilde{g}(r(y, x), x)=\widetilde{B}(x) r(y, x),
$$

it follows by Theorem 6.2 that $\left(\tilde{r}, f_{\tilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$.

## 7. CONCLUSIONS

We have developed several characterizations of observational equivalence for nonparametric simultaneous equations models with nonadditive unobservable variables.

The models that we considered can be described as

$$
U=r(Y, X)
$$

${ }^{6}$ This is the example in Benkard and Berry (2006, p. 1433, footnote 4).
where $U \in R^{G}$ is a vector of unobservable exogenous variables, distributed independently of $X, X \in R^{K}$ is a vector of observable exogenous variables, $Y$ is a vector of observable endogenous variables, and $r$ is a function such that, conditional on $X, r$ is one-to-one.

Our characterizations were developed by considering an alternative function, $\widetilde{r}$, and analyzing the density of $\widetilde{U}=\widetilde{r}(Y, X)$. We asked what restrictions $\stackrel{\text { on }}{\widetilde{U}} \widetilde{r}, r$, and the density, $f_{U}$, of $U$ are necessary and sufficient to guarantee that $\widetilde{U}$ is distributed independently of $X$. We showed that these restrictions characterize observational equivalence and we provided an expression for them in terms of a restriction on the rank of a matrix.

The use of the new results was exemplified by deriving known results about identification in nonadditive single equation models and triangular equations models. An example of a separable demand and supply model provided insight into the power of separability restrictions.

We also developed a simplified approach to characterize observational equivalence, which is useful when the alternative function, $\widetilde{r}$, is defined as a transformation of the function $r$.

## APPENDIX

PROOF OF THEOREM 6.1: Define the function $g: R^{G+K} \rightarrow R^{G}$ by

$$
u=g(\widetilde{u}, x),
$$

where $\widetilde{u}$ is defined, as in Section 6, by

$$
\widetilde{u}=\widetilde{g}(u, x)
$$

Then, since

$$
u=g(\widetilde{g}(u, x), x)
$$

we get, by differentiating this expression with respect to $u$ and with respect to $x$, that
(A.1) $\quad I=\frac{\partial g(\widetilde{g}(u, x), x)}{\partial \widetilde{u}} \frac{\partial \widetilde{g}(u, x)}{\partial u}$
and
(A.2) $0=\frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}} \frac{\partial \widetilde{g}(u, x)}{\partial x}+\frac{\partial g(\widetilde{u}, x)}{\partial x}$.

To derive (6.6), we rewrite (6.5) as
(A.3) $\quad f_{\widetilde{U} \mid X=x}(\widetilde{u})=f_{U}(g(\widetilde{u}, x))\left|\frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}}\right|$.

Independence between $\widetilde{U}$ and $X$ is equivalent to requiring that for all $\widetilde{u}, x$,

$$
\frac{\partial \log f_{\tilde{U}_{\mid X=x}}(\widetilde{u})}{\partial x}=0
$$

Taking logs on both sides of (A.3) and differentiating the resulting expressions with respect to $x$, we get
(A.4) $\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{u})}{\partial x}$

$$
=\frac{\partial \log f_{U}(g(\widetilde{u}, x))}{\partial u} \frac{\partial g(\widetilde{u}, x)}{\partial x}+\frac{\partial \log \left|\frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}}\right|}{\partial x}
$$

We will get expressions for the terms in the right-hand side of (A.4) in terms of $\tilde{g}, u$, and $x$. By (A.1) and (A.2),

$$
\frac{\partial g(\widetilde{u}, x)}{\partial x}=-\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{-1}\left(\frac{\partial \widetilde{g}(u, x)}{\partial x}\right)
$$

Differentiating with respect to $\widetilde{u}$ the expression

$$
\tilde{u}=\tilde{g}(g(\widetilde{u}, x), x)
$$

one gets

$$
I=\frac{\partial \widetilde{g}(g(\widetilde{u}, x), x)}{\partial u} \frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}}
$$

Hence,
(A.5) $\quad 1=\left|\frac{\partial g(\widetilde{g}(u, x), x)}{\partial \widetilde{u}}\right|\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|$.

Taking logs and differentiating both sides of (A.5) with respect to $x$ and with respect to $u$ we get

$$
\begin{aligned}
0= & \frac{\partial \log \left|\frac{\partial g(\widetilde{g}(u, x), x)}{\partial \widetilde{u}}\right|}{\partial \widetilde{u}} \frac{\partial \widetilde{g}(u, x)}{\partial x}+\frac{\partial \log \left|\frac{\partial g(\widetilde{g}(u, x), x)}{\partial \widetilde{u}}\right|}{\partial x} \\
& +\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial x}
\end{aligned}
$$

and

$$
0=\frac{\partial \log \left|\frac{\partial g(\widetilde{g}(u, x), x)}{\partial \widetilde{u}}\right|}{\partial \widetilde{u}} \frac{\partial \widetilde{g}(u, x)}{\partial u}+\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial u}
$$

Hence,

$$
\begin{aligned}
& \frac{\partial \log \left|\frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}}\right|}{\partial x} \\
& =\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial u}\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{-1} \frac{\partial \widetilde{g}(u, x)}{\partial x}-\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial x} .
\end{aligned}
$$

Substituting into (A.4), we get

$$
\begin{aligned}
& \frac{\partial \log f_{\tilde{U} \mid X=x}(\widetilde{u})}{\partial x} \\
& =\frac{\partial \log f_{U}(g(\widetilde{u}, x))}{\partial u} \frac{\partial g(\widetilde{u}, x)}{\partial x}+\frac{\partial \log \left|\frac{\partial g(\widetilde{u}, x)}{\partial \widetilde{u}}\right|}{\partial x} \\
& =-\frac{\partial \log f_{U}(g(\widetilde{u}, x))}{\partial u}\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{-1}\left(\frac{\partial \widetilde{g}(u, x)}{\partial x}\right) \\
& \quad+\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial u}\left(\frac{\partial \widetilde{g}(u, x)}{\partial u}\right)^{-1} \frac{\partial \widetilde{g}(u, x)}{\partial x} \\
& \quad-\frac{\partial \log \left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|}{\partial x} .
\end{aligned}
$$

Substituting $g(\widetilde{u}, x)$ by $u$, and setting $\partial \log f_{\tilde{U} \mid X=x}(\widetilde{u}) / \partial x=0$, we get (6.6).
Q.E.D.

Proof of Theorem 6.2: Let

$$
\begin{aligned}
\widetilde{s}_{u} & =\left(\left.\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{u})}{\partial \widetilde{u}}\right|_{\tilde{u}=\widetilde{g}(u, x)}\right), \quad \widetilde{s}_{x}=\left(\left.\frac{\partial \log f_{\widetilde{U} \mid X=x}(\widetilde{u})}{\partial x}\right|_{\tilde{u}=\widetilde{g}(u, x)}\right), \\
s_{u} & =\frac{\partial \log f_{U}(u)}{\partial u}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{g}_{u} & =\frac{\partial \widetilde{g}(u, x)}{\partial u}, \quad \widetilde{g}_{x}=\frac{\partial \widetilde{g}(u, x)}{\partial x} \\
\Delta_{u} & =\frac{\partial \log \left(\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|\right)}{\partial u}, \quad \Delta_{x}=\frac{\partial \log \left(\left|\frac{\partial \widetilde{g}(u, x)}{\partial u}\right|\right)}{\partial x} .
\end{aligned}
$$

Taking logs of both sides of (6.5) and differentiating with respect to $u$ and $x$, we get

$$
\begin{aligned}
& \widetilde{s}_{u}^{\prime} \widetilde{g}_{u}+\Delta_{u}^{\prime}=s_{u}^{\prime}, \\
& \widetilde{s}_{x}^{\prime}+\widetilde{s}_{u}^{\prime} \widetilde{g}_{x}+\Delta_{x}^{\prime}=0
\end{aligned}
$$

or, after transposing,
(А.6) $\tilde{g}_{u}^{\prime} \widetilde{s}_{u}+\Delta_{u}=s_{u}$,
(A.7) $\quad \tilde{s}_{x}+\tilde{g}_{x}^{\prime} \tilde{s}_{u}+\Delta_{x}=0$.

Using the first equality to solve for $\widetilde{s}_{u}$ and substituting the result into the second equality, one gets

$$
\widetilde{s}_{x}=-\widetilde{g}_{x}^{\prime}\left(\widetilde{g}_{u}^{\prime}\right)^{-1} s_{u}+\widetilde{g}_{x}^{\prime}\left(\widetilde{g}_{u}^{\prime}\right)^{-1} \Delta_{u}-\Delta_{x} .
$$

By Theorem 6.1, $\tilde{U}$ is independent of $X$ if and only if $\widetilde{s}_{x}=0$. Consider the matrix
(A.8) $\quad\left(\begin{array}{cc}\widetilde{g}_{u}^{\prime} & s_{u}-\Delta_{u} \\ \widetilde{g}_{x}^{\prime} & -\Delta_{x}\end{array}\right)$.

Observational equivalence implies independence between $\tilde{U}$ and $X$, and, by Theorem 6.1, that $\widetilde{s}_{x}=0$. When $\widetilde{s}_{x}=0$, equations (A.6) and (A.7) imply that $s_{u}-\Delta_{u}$ is a linear combination of the columns of $\widetilde{g}_{u}^{\prime}$, and that $-\Delta_{x}$ is that same linear combination, but of the columns of $\widetilde{g}_{x}^{\prime}$. Since $\widetilde{g}_{u}$ is invertible, this implies that the rank of the matrix must be $G$. Hence, observational equivalence implies that the rank of (A.8) is $G$.

Conversely, suppose that the rank of the matrix in (A.8) is $G$. It follows by the invertibility of $\widetilde{g}_{u}$ that there exists a unique $\lambda \in R^{G}$ such that

$$
\widetilde{g}_{u}^{\prime} \lambda=s_{u}-\Delta_{u}
$$

By (A.6),

$$
\widetilde{g}_{u}^{\prime} \widetilde{s}_{u}=s_{u}-\Delta_{u}
$$

Hence,

$$
\lambda=\widetilde{s}_{u}
$$

Since the matrix in (A.8) has rank $G$, it must be that

$$
-\Delta_{x}=\widetilde{g}_{x}^{\prime} \lambda
$$

By solving for $\lambda$ from the first equation, we get that

$$
\lambda=\left(\widetilde{g}_{u}^{\prime}\right)^{-1}\left(s_{u}-\Delta_{u}\right)
$$

Then it follows that

$$
\begin{aligned}
-\Delta_{x} & =\widetilde{g}_{x}^{\prime} \lambda \\
& =\widetilde{g}_{x}^{\prime}\left(\widetilde{g}_{u}^{\prime}\right)^{-1}\left(s_{u}-\Delta_{u}\right)
\end{aligned}
$$

This implies that

$$
\tilde{s}_{x}=\widetilde{g}_{x}^{\prime}\left(\widetilde{g}_{u}^{\prime}\right)^{-1}\left(s_{u}-\Delta_{u}\right)+\Delta_{x}=0
$$

By Theorem 6.1, it follows that $\widetilde{U}$ is independent of $X$. By the equivalence between (6.4) and (6.5) and the definition of observational equivalence in (3.1), $\left(\widetilde{g}(r(y, x), x), f_{\widetilde{U}}\right)$ is observationally equivalent to $\left(r, f_{U}\right)$.
Q.E.D.

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[^1]:    ${ }^{2}$ Brown (1983) and Roehrig (1988) used a mapping like this. They analyzed the restrictions that independence between the observable and unobservable explanatory variables imposes on this mapping, deriving different results than the ones we derive in this paper.

[^2]:    ${ }^{3}$ I am grateful to Whitney Newey for detailed comments on this and the example in Section 4.2, which included the following new result with its proof. See Matzkin (2005, 2007a, 2007b, 2008) for other sets of conditions.

[^3]:    ${ }^{4}$ See Matzkin (2008) for identification of $\partial m_{1}\left(y_{2}, u_{1}\right) / \partial y_{2}$ when $y_{2}=m_{2}\left(y_{1}, x, u_{2}\right)$.

[^4]:    ${ }^{5}$ See Brown (1983) for an earlier development of this approach.

