Liquidity Shocks and the Business Cycle: What next? *

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Abstract

This paper studies the properties of an economy subject to random liquidity shocks as proposed by Kiyotaki and Moore [2008]. Liquidity shocks affect the ease with which equity can be used as to finance the down-payment of new investment projects. If these shocks are sufficiently large, they have the effects of investment specific technological shocks. We argue that, as with investment shocks, liquidity shocks face several challenges to explain business cycle fluctuations and asset prices. We explain where the weaknesses of the model lie and propose several amendments.

Keywords: Business Cycle, Investment, Liquidity.

JEL Classification: E22, E32, E44.

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1 Overview

Motivation: This paper studies liquidity shocks in a real business cycle framework. Liquidity shocks constrain the fraction of assets that may be traded in a given period. These shocks were introduced to the literature by Kiyotaki and Moore [2008] (henceforth KM). KM show that in conjunction with other financing constraints, these shocks could be a potential sources of business cycle fluctuations. This possibility has led to substantial amount of research that associates liquidity shocks to financial disturbances that cause economic fluctuations.\(^1\) However, these studies have failed to find substantial effects of liquidity shocks unless they incorporate additional frictions.\(^2\) However, in doing so, the transparency of smaller scale models is often lost. The reader is left without knowing what is really driving results and why these additional frictions are needed.

This paper complements this research agenda by taking a step backwards. It studies in detail the most stripped-down version of the KM model. In particular, the paper characterizes equilibria and the effects of liquidity shocks via global non-linear analysis. The mechanisms are described by compartmentalizing the model into pieces to build intuition. In doing so, the paper explains why these shocks may fail to explain quantitatively business cycles and asset prices unless additional frictions are introduced. We attempt to contribute to the literature by conveying some insights on which, where and how strong should additional features be incorporated in order to improve the performance of the model. The final section of the paper explores some of these extensions. The paper also contributes to the literature by providing an algorithm that allows for the computation of global solutions with occasionally binding financial constraints.

The meaning of liquidity. It is convenient to define liquidity here. In this paper, liquidity is interpreted as a property of an asset. An asset is liquid if gains from trade are sufficient to guarantee trade. Liquidity shocks are shocks to the fractions of assets that are liquid. The amount of liquidity is the fraction of liquid assets.\(^3\)

The propagation mechanics of liquidity shocks. In the model, investment has two characteristics that cause liquidity to become a source of business cycle fluctuations. First, access to investment projects is limited to a fraction of the population. This characteristic causes a need to reallocate resources from those agents that do not posses these opportunities to those who do. Second, investment is subject to moral-hazard. This second characteristic requires investment to be partially financed internally (investment requires a down-payment). Hence, resources

\(^1\)Citations go here.

\(^2\)New Keynesian model.

\(^3\)Some authors use liquidity as a synonym of volume of trade. In the context of the model, the distinction between liquid assets and traded assets is important. For example, there could be states in which equity is entirely liquid, but it is not traded at all.
cannot be deployed efficiently against a future promise of repayment. Thus, the combination of these two characteristics creates gains from trading existing assets. Selling part of their existing assets obtains internal funds to relax external financing constraints. Liquidity shocks interrupt the amount of trade despite gains from trade. When these shocks are sufficiently strong, they will drive aggregate investment below efficient levels as entrepreneurs will not be able to access enough external financing. These shocks also cause a wedge between the price of capital and its replacement cost, a measure of inefficient investment.

KM provided the intuition that liquidity shocks play a potential role as a direct source and amplification mechanism of business cycle fluctuations. Can they be quantitatively relevant? The answer is no, not on their own. This paper explains why and what should we do next. We do so performing several tasks.

Task 1. Analyzing the model. To answer these questions, our first task is to study the full stochastic version of the KM model. This provides further insights on how liquidity shocks operate. Whereas most of the intuition in KM still will hold in a stochastic environment, this version uncovers some characteristics that are not deduced immediately from the analysis around a non-stochastic steady-state. In particular, we find that the state-space has two regions separated by a liquidity frontier. Each region has the properties that KM find for two distinct classes deterministic steady states. One region is entirely governed by the dynamics of real business cycle model. In the other region liquidity shocks play the role of shocks to the efficiency of investment providing an additional source of fluctuations. This region is characterized by binding enforcement constraints that render the allocation of resources to investment inefficient compared to frictionless benchmark. Moreover, because only in this region enforcement constraints are binding, investment wedges show up only in this region. We interpret this wedge as Tobin’s q (henceforth, we refer to this wedge simply a q).

The non-linear analysis describes the geometry of the liquidity frontier that separates these two regions. In particular, the liquidity frontier is increasing in the return to capital: the higher the return to capital, the lower the liquidity shocks have to be to activate the constraints. This implies that output is more sensible to liquidity shocks in periods where TFP is high or the capital stock is low. This uncovers a strong non-linear impact of liquidity shock on output and imposes some restrictions on asset price movements and quantities.

Task 2: Quantifying the model. The main quantitative implication of the analysis is that liquidity shocks on their own may not explain strong recessions. Our calibration exercise is purposely designed in such a way that the effects of liquidity shocks have the strongest possible effects. Despite this advantage, when we study the impulse response to an extreme liquidity shock...
dry-up episode, we find that the response of output is a drop in 0.7% on impact. The response of output to liquidity shocks is weak because liquidity shocks resemble investment shocks. Investment shocks have weak effects in neoclassical environments because output is a function of the capital stock that, in turn, moves very slowly in comparison to investment. In fact, we find a theoretical upper-bound to the maximal impact when the labor supply schedule remains constant. For a standard calibration, this bound equals $\sqrt[12]{0.9} = 0.87\%$. This is the reason why the bulk of business cycle studies focus on total factor productivity shocks.\footnote{This result is known in the literature, at least, since Barro and King [1984] and is also recently discussed in Justiniano et al. [2010].}

The non-linear nature of liquidity shocks also imply that their effects on output are negligible if they are not close to a full market shutdown. This means that models that linearize the system close or above to the liquidity frontier will have even more problems in obtaining a substantial effect on output. On the asset-pricing dimension, we find that to obtain a reasonable mean and variance of the risk-free rate, liquidity shocks must fluctuate close to the liquidity frontier. However, if liquidity shocks fall too often into the unconstrained region, the variation in $q$ is too low. However, if liquidity shocks fall deep inside the constrained region too often, risk-free rates become excessively volatile.

This finding suggest that one needs to introduce additional frictions. In particular, we argue that these must affect the labor market (or other variable inputs) in a way that interacts with liquidity shocks to explain sizeable recessions that do not compromise asset prices.\footnote{A similar conclusion is found in Greenwood et al. [2000] or Justiniano et al. [2010] in their studies of random investment specific shocks.}

Task 3: Exploring additional frictions. The final task of this paper is to propose additional features as possible amendments to the model. One simplistic way to reproduce stronger effects is by introducing variable capital utilization as in Greenwood et al. [1988] into the model. With variable capital utilization, entrepreneurs face a trade-off between incrementing the utilization of capital and depreciating capital. When liquidity shocks hit, they cause a reduction in utilization by increasing the opportunity cost in the use of capital. This feature illustrates the type of mechanics that one needs to obtain greater quantitative effects. Though this avenue is successful in causing a much sharper reduction in output, this friction is counterfactual in two dimensions. First, it delivers a counterfactual effect on the marginal product of labor, something that did not occur during the last recession. Second, it generates a counterfactual increase in consumption. We then discuss how to introduce sector capacity constraints and specificity. We discuss a model with two features: (1) capital is specific to consumption and investment sectors and (2) there is limit to the amount of labor that a given sector can absorb in a given period. This model is capable of delivering a strong reduction in output (and unemployment) without a counterfactual increase in consumption. We then discuss how working capital constraints in the spirit of the early work Christiano and Eichenbaum [1992] and the recent work by Jermann...
and Quadrini [2010] and Bigio [2009]. This extension delivers also a sharp reduction in output together with an increase in the labor wedge. We then discuss how the KM mechanism operates when combined with new-Keynesian features as is the case of del Negro et al. [2010] and Ajello [2012].

Relationship with the literature. Our paper is close to the works that follow up on the KM model. del Negro et al. [2010] introduces nominal rigidities into the KM model. Nominal rigidities are an example of an amplification mechanism that our model is looking for in order to explain an important part of the business cycles. That paper corroborates our finding that without such an amplification mechanism, liquidity shocks on their own cannot have important implications on output. Our papers are complementary as theirs tries to explain the recent financial crises as caused by a strong liquidity shock. The focus here is on studying liquidity shocks in an RBC. Other than that, our paper differs from theirs because it studies the behavior of the model globally whereas theirs is restricted to a log-linearized version of the model. We believe that the findings in both papers complement each other. Ajello [2012] is similar to del Negro et al. [2010], but he models interruptions in investment through a wedge between the purchase and sales price of capital. Again, the new-Keynesian layer in that model allows to explain large movements in output. Perhaps Shi [2012] is the closest to ours. Like us, the goal of that paper is to provide a quantitative analysis of the KM model. Both our papers agree in that the model need additional features to explain asset prices. However, that paper concludes that liquidity shocks can have a strong effect on output because they indirectly affect the supply of labor. All of the suggestions in our paper operate by affecting the demand for hours, so the mechanisms are quite different.

In terms of the techniques, our paper is similar to He and Krishnamurthy [2008] and Brunnermeier and Sannikov [2009] in that we also stress the importance of global methods in understanding the non-linear dynamics of these financial frictions. Like us, these papers deliver regions of the state-space where constraints prevent efficient investment. This paper complements that work as a step towards understanding how changes in liquidity (rather than wealth) affect the allocation of resources to investment.\footnote{These papers study environments in which agents are heterogenous because some are limited in their access to saving instruments (investment opportunities). Investment opportunities are also illiquid assets because of hidden action (gains from trade don’t guarantee trade or intermediation). Thus, these papers focus on how a low relative wealth of agents with access to these opportunities distorts the allocation of other entrepreneurs from investing efficiently.}

Another related paper is Lorenzoni and Walentin [2009]. In that paper, a wedge between the cost of capital and the price of equity shows up as combination of limited access to investment opportunities (gains from trade) and limited enforcement (the inefficiency that prevents trade). Our papers share in common that investment is inefficient due to lack of commitment. In Lorenzoni and Walentin [2009] entrepreneurs may potentially default on debt, whereas in KM,
agents can default on equity generated by new projects. Thus, as in He and Krishnamurthy [2008] and Brunnermeier and Sannikov [2009], shocks propagate by affecting the relative wealth of agents carrying out investment opportunities, which is something we extract from. Our paper is related to theirs because both stress that the relation between q and investment is governed by two forces: shocks that increase the demand for investment (e.g., an increase in productivity) will induce a positive correlation between q and investment. The correlation moves in the opposite direction when the enforcement constraints are more binding tighter (e.g., with a fall in liquidity).

Finally, Buera and Moll [2012] present an environment in which collateral constraints affect the reallocation of capital. The shocks studied by Kiyotaki and Moore are different because they affect the amount of capital that can be sold as opposed to be borrowed. However, these authors study how capital may be misallocated to firms of lower productivity tighter constraints are faced. This misallocation shows up as a negative shock to measured total factor productivity. Our analysis of course, abstracts from heterogeneity, but it is worth mentioning that liquidity shocks could have additional kicks through this channel. However, these effects require a low persistence in firm productivity shocks. Shourideh and Zetlin-Jones [2012] provide a quantitative version of this model and show that with intermediate goods and imperfect substitution across inputs, the model can deliver important fluctuations in output. This is yet another source of frictions that affect the demand for labor.

Organization. The first part of the paper describes KM’s model. The following sections correspond to tasks 1 through 3. The final section concludes.

2 Kiyotaki and Moore’s Model

The model is formulated in discrete time with an infinite horizon. There are two populations with unit measure, entrepreneurs and workers. Workers provide labor elastically and don’t save. Entrepreneurs don’t work but invest in physical capital that they use in privately owned firms. Each period, entrepreneurs are randomly assigned one of two types, investors and savers. We use superscripts $i$ and $s$ to refer to either type.

There are two aggregate shocks: a productivity shock $A \in A$ and where a liquidity shock $\phi \in \Phi \subset [0, 1]$, and where $A$ and $\Phi$ are discrete. The nature of these shocks will affect the ability to sell equity and will be clear soon. These shocks form Markov process that evolves according to stationary transition probability $\Pi : (A \times \Phi) \times (A \times \Phi) \rightarrow [0, 1]$.

It will be shown that the aggregate state for this economy is given by the aggregate capital stock, $K \in K$ in addition to $A$ and $\phi$. $K$ is shown to be compact later on. The aggregate state is summarized by a single vector $s = \{A, \phi, K\}$ and $s \in S \equiv A \times \Phi \times K$. 
2.1 Preferences of Entrepreneurs

We follow the exposition of Angeletos [2007] for the description of preferences which are of the class introduced by Epstein and Zin [1989]. Although the real-side effects of liquidity shocks are roughly the same under log-utility specifications, I use Epstein-Zin preferences to give the model a fair shot when assessing its asset-pricing properties. Thus, preferences of entrepreneur of type $j$ are given recursively by:

$$V(s) = U(c^j) + \beta \cdot U(\mathcal{CE}(U^{-1}(V^j(s'))))$$

where

$$\mathcal{CE} = \Upsilon^{-1}(E \Upsilon(\cdot))$$

where the expectation is taken over time $t$ information.\(^9\)

The term $\mathcal{CE}$ refers to the certainty equivalent utility with respect to the CRRA $\Upsilon$ transformation.\(^10\) $\Upsilon$ and $U$ are given by,

$$\Upsilon(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad U(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

$\gamma$ captures the risk-aversion of the agent whereas his elasticity of intertemporal substitution is captured by $\sigma$.

2.2 Production

At the beginning of each period, entrepreneurs take their capital endowment, $k \in [0, \infty]$. They then choose labor inputs, $l$, optimally to maximize profits. Production is carried out according to a Cobb-Douglas production function $F(k,l) \equiv k^\alpha l^{(1-\alpha)}$, where $\alpha$ is the capital intensity. Because the production function is homogeneous, maximization of profits requires maximizing over the labor to capital ratio:

$$r_t = \max_{l/k} \left( [A_t F(1,l/k)] - w_t l/k \right)$$

where $w_t$ is the wage and $A_t$ is the aggregate productivity shock in period $t$. The return to capital $r_t$ is the solution to this problem.

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9Utility in Epstein and Zin [1989] is defined (equation 3.5) differently. This representation is just a monotone transformation of the specification in that paper. The specification here is obtained by applying $U^{-1}$ to that equation.

10The transformation $\Upsilon(c)$ characterizes the relative risk aversion through $\gamma > 0$. The function $U(c)$ captures intertemporal substitution through $\sigma > 0$. When $\sigma \gamma < 1$, the second term in the utility function is convex in $V^j(s')$ and concave when the inequality is reversed. When $\gamma = \frac{1}{\sigma}$, one obtains the standard expected discounted. If these terms are further equalized to $1$, the specification yields the log-utility representation.
2.3 Entrepreneur Types

Entrepreneurs are able to invest only upon the arrival of random investment opportunities. Investment opportunities are distributed i.i.d. across time and agents. An investment opportunity is available with probability $\pi$. Hence, each period, entrepreneurs are segmented into two groups, investors and savers, with masses $\pi$ and $1 - \pi$ respectively. The entrepreneur’s budget constraint is:

$$c_t + i^d_t + q_t \Delta e^+_{t+1} = r_t n_t + q_t \Delta e^-_{t+1}$$  \hspace{1cm} (1)

The right hand side of (1) corresponds to the resources available to the entrepreneur. The first term is the return to equity holdings where $r_t$ is the return on equity (and capital), and $n_t$ is the amount of equity held by the entrepreneur. The second term in the right is the value of sales of equity, $\Delta e^+_{t+1}$. This term is the difference between the next period’s stock of equity $e^-_{t+1}$ and the non-depreciated fraction of equity owned in the current period $\lambda e^-_{t}$. The entrepreneur uses these funds to consume $c_t$, to finance down payment for investment projects, $i^d_t$, and to purchase outside equity $\Delta e^+_{t+1}$. Each unit of $e^-_{t}$ entitles other entrepreneurs to rights over the revenues generated by the entrepreneurs capital and $e^+_{t}$ entitles the entrepreneur to revenues generated by other entrepreneurs. The net equity for each entrepreneur is therefore:

$$n_t = k_t + e^+_{t} - e^-_{t}$$ \hspace{1cm} (2)

The difference between saving and investing entrepreneurs is that the former are not able to invest directly. Thus, they are constrained to set $i^d_t = 0$. Outside equity and issued equity evolve according to

$$e^+_{t+1} = \lambda e^+_{t} + \Delta e^+_{t+1}$$ \hspace{1cm} (3)

and

$$e^-_{t+1} = \lambda e^-_{t} + \Delta e^-_{t+1} + i^s_t,$$ \hspace{1cm} (4)

respectively. Notice that the stock of equity is augmented by sales of equity $\Delta e^+_{t+1}$ and an amount $i^s_t$ which is specified by the investment contract in the next section. Finally, the timing protocol is such that investment decisions are taken at the beginning of each period. That is, entrepreneurs choose consumption and a corporate structure before observing future shocks.
2.4 Investment, optimal financing and the effects of liquidity shocks

*Investment opportunities and financing.* When an investment opportunity is available, entrepreneurs choose a scale for an investment project, $i_t$. Projects increment the firm’s capital stock one for one with the size of the project. Each project is funded by a combination of internal funding, $i^d_t$, and external funds $i^f$. External funds are obtained by selling equity that entitles other entrepreneurs to the proceeds of the new project. Thus, $i_t = i^d_t + i^f_t$. In general, the ownership of capital created by this project may differ from the sources of funding. In particular, investing entrepreneurs are entitled to the proceeds of a fraction $i^i_t$ of total investment, and the rest, $i^s_t$, entitles other entrepreneurs to those proceeds. Again, $i_t = i^i_t + i^s_t$.

Because the market for equity is competitive and equity is homogeneous, the rights to $i^s_t$ are sold at the market price of equity $q_t$. Therefore, external financing satisfies $i^f = q_t i^s_t$. Notice that at the end of the period, the investing entrepreneur increases his equity in $i^i_t = i_t - i^s_t$ while he has contributed only $i_t - q_t i^s_t$. In addition, investment is subject to moral-hazard because entrepreneurs may divert funds from the project. By diverting funds, they are able to increment their equity stock up to a fraction $1 - \theta$ of the total investment without honoring the fraction of equity sold. There are no enforcement or commitment technologies available. The assumption that issues of new equity are subject to moral hazard as opposed to equity in place tries to capture the idea that financial transactions on assets that are already in place are more easily enforced than those on assets that do not exist yet. In essence, we assume that funds from existing physical capital may not be diverted.

Utility will be shown to be an increasing function of equity only. The incentive compatibility condition for external financing is equivalent to:

$$\begin{align*}
(1 - \theta) i_t & \leq i^i_t, \text{ or } i^s_t \leq \theta i_t \\
\end{align*}$$

(5)

This condition states that the lender’s stake in the project may not be higher than $\theta$. Therefore, taking $i^d_t$ as given, the entrepreneur solves the following problem when it decides how much to invest:

**Problem 1** (Optimal Financing I).

$$\max_{i^i_t > 0} i^i_t$$

\[\text{11\textsuperscript{1}}\] The distinction between inside and outside equity makes this a $q$-theory of investment. The wedge occurs as a combination of two things. First, only a fraction $\pi$ of agents have access to investment opportunities, which generates a demand for outside equity. Limited enforcement causes the supply of outside equity to be limited by the incentive compatibility constraints. The value of $q$ must adjust to equate demand with supply and this price may differ from 1.
taking $i_t^d > 0$ as given and subject to:

$$
i_t^d + i_t^f = i_t, \quad i_t^i + i_t^s = i_t, \quad q_t i_t^s = i_t^f$$

$$i_t^s \leq \theta i_t$$

Substituting out all the constraints, the problem may be rewritten in terms of $i_t^d$ and $i_t^s$ only.

**Problem 2** (Optimal Financing II).

$$\max_{i_t^d} i_t^d + (q_t - 1) i_t^s$$

taking $i_t^d$ as given and subject to:

$$i_t^s \leq \theta (i_t^d + q_t i_t^s)$$

The interpretation of this objective is clear. For every project, the investing entrepreneur increases his stock of equity $i_t^d + (q_t - 1) i_t^s$, which is the sum of the down payment plus the gains from selling equity corresponding to the new project, $i_t^s$. The constraint says that the amount of outside funding is limited by the incentive compatibility constraint.

If $q_t \in (1, \frac{1}{\theta})$, the problem is maximized at the points where the incentive compatibility constraint binds. Therefore, within this price range, for every unit of investment $i_t$, the investing entrepreneur finances the amount $(1 - \theta q_t)$ units of consumption and owns the fraction $(1 - \theta)$. This defines the replacement cost of equity,

$$q_t^R = \left[ \frac{(1 - \theta q_t)}{(1 - \theta)} \right]$$

$q_t^R$ is less than 1, when $q_t > 1$ and equal to 1 when $q_t = 1$. When $q_t = 1$, the entrepreneur is indifferent on the scale of the project, so $i_t^s$ is indeterminate within $[0, \theta i_t]$. The difference between $q_t^R$ and $q_t$ is a wedge between the cost of purchasing outside equity and the cost of generating inside equity. The physical capital run by the entrepreneur evolves according to

$$k_{t+1} = \lambda k_t + i_t$$

so using the definition of equity (2):

$$n_{t+1} = \lambda n_t + i_t - i_t^s + (\Delta e_t^+ - \Delta e_t^-)$$

(6)

**Resellability Constraints.** In addition to the borrowing constraint imposed by moral hazard, there is a constraint on the sales of equity created in previous periods. Resellability constraints impose a limit on sales of equity that may be sold at every period. These constraints depend on the liquidity shock $\phi_t$: 

10
\[ \Delta e_{t+1}^+ - \Delta e_{t+1}^- \leq \lambda \phi_t n_t \quad (7) \]

Kiyotaki and Moore motivate these constraints by adverse selection in the equity market. Bigio [2009] and Kurlat [2009] show that such a constraint will follow from adverse selection in the quality of assets. There are multiple alternative explanations to why liquidity may vary over the business cycle. We discuss some alternative explanations in the concluding section. Here, what matters is that liquidity shocks, \( \phi_t \), prevent equity markets from materializing gains from trade.

**Agent problems.** When \( q_t > 1 \), the cost of increasing equity by purchasing outside equity is larger than putting the same amount as down-payment and co-financing the rest \( (q_t > q^R) \). Moreover, when \( q_t > 1 \), by selling equity and using the amount as down-payment, the agent increases his equity by \( q_t (q_t^R)^{-1} > 1 \). Since when \( q_t > 1 \), investing entrepreneurs must set \( \Delta e_{t+1}^+ = 0 \) and (7) binds. Using these facts, the investing entrepreneur’s budget constraint (1) may be rewritten in a convenient way by substituting (6) and \( i^d_t = i_t - q_t \phi_t \). The budget constraint is reduced to:

\[
c_t + q_t^R n_{t+1} = (r_t + q_t^i \lambda) n_t
\]

where \( q_t^i = q_t \phi_t + q^R (1 - \phi_t) \). When \( q_t = 1 \), this constraint is identical to the saving entrepreneur’s budget constraint when the evolution of its equity is replaced into (1). Thus without loss of generality, the entrepreneur’s problem simplifies to:

**Problem 3 (Saver’s ).**

\[
V^s_t (w_t) = \max_{c_t, n_{t+1}} U (c_t) + \beta \cdot U \left( \mathcal{CE}_t \left( U^{-1} (V_{t+1} (w_{t+1})) \right) \right)
\]

subject to the following budget constraint:

\[
c_t + q_t n_{t+1} = (r_t + \lambda q_t) n_t \equiv w^s_t
\]

\( V_{t+1} \) represents the entrepreneur’s future value. \( V_{t+1} \) does not include a type superscript since types are random and the expectation term is also over the type space also. \( w_{t+1} \) is the entrepreneur’s virtual wealth which also depends on the type. The investing entrepreneur solves the following problem:

**Problem 4 (Investor’s).**

\[
V^I_t (w_t) = \max_{i_t, c_t, n_{t+1}} U (c_t) + \beta \cdot U \left( \mathcal{CE}_t \left( U^{-1} (V_{t+1} (w_{t+1})) \right) \right)
\]
subject to following budget constraint:

\[ c_t + q_t^R n_{t+1} = (r_t + q_t^i \lambda) n_t \equiv w_t^{i+1} \]

Finally, workers provide labor \( l_t \) to production in exchange for consumption goods \( c_t \) in order to maximize their static utility.

**Problem 5 (Worker’s).**

\[
U_t = \max_{c,l} \frac{1}{\left(1 - \frac{1}{\psi}\right)} \left[ c - \frac{\omega}{(1 + \nu)} (l)^{1+\nu}\right]^{1-\frac{1}{\psi}}
\]

and subject to the following budget constraint:

\[ c = \omega_l l \]

Along the discussion we have indirectly shown the following Lemma:

**Lemma 1.** In any equilibrium \( q \in [1, \frac{1}{\theta}] \). In addition, when Tobin’s \( q \) is \( q > 1 \), the liquidity constraint (7) binds for all investing entrepreneurs. When \( q = 1 \), policies for saving and investing entrepreneurs are identical and aggregate investment is obtained by market clearing in the equity market.

\( q_t \) is never below 1 since capital is reversible. Models with adjustment costs would not have this property. The following assumption is imposed so that liquidity plays some role in the model:

**Assumption 1.** \( \theta < 1 - \pi \)

If this condition is not met, then the economy does not require liquidity at all in order to exploit investment opportunities efficiently. Before proceeding to the calibration of the model, we provide two digressions: one about the intuition behind the amplification mechanism behind liquidity shocks and the other about a broader interpretation of the financial contracts in the model.

### 2.5 Intuition behind the effects

A brief digression on the role of \( \phi_t \) is useful before studying the equilibrium in more detail. Notice that the problem of optimal financing subject to the enforcement constraints is essentially static. Hence, we can use a static analysis to understand the effects of liquidity shocks.

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\(^{12}\)In Sargent [1980], investment is irreversible, so \( q \) may be below one in equilibrium. This happens at \( q=1 \) when the demand for consumption goods generates negative investment. \( q \) must adjust to below one such that the demand for investment is 0.
Figure 1: **Liquidity constraints** The left panel shows how borrowing constraints impose a cap on the amount of equity that can be sold to finance a down payment. The middle panel shows how liquidity shocks affect the amount of resources available as a down payment. The right panel shows the price effect of the shock, which may or may not reinforce the liquidity shock.

Whenever $q_t > 1$, external financing allows investing entrepreneurs to arbitrage. In such instances, the entrepreneur wants to finance investment projects as much as possible. We have already shown that when constraints are binding, the entrepreneur owns the fraction $(1 - \theta)$ of investment while he finances only $(1 - q_t \theta)$. If he uses less external financing he misses the opportunity to obtain more equity.

To gain intuition, let’s abstract from the consumption decision and assume that the entrepreneur uses $x_t \equiv q_t \phi_t \lambda n_t$ to finance the down-payment. Thus, $i_t^d = x_t$. The constraints impose a restriction on the amount external equity that may be issued, $i_t^s \leq \theta i_t$. External financing is obtained by selling $i_t^s$ equity at a price $q_t$ so the amount of external funds for the project is $i_t^f = q_t i_t^s$. Since $i_t = i_t^f + i_t^d$, external financing satisfies,

$$i_t^f \leq q_t \theta (i_t^f + x_t).$$  \hspace{1cm} (8)

Figure 1 describes the simple intuition behind the liquidity channel. The figure explains how changes in the amount $x_t$ of sales of equity corresponding to older projects affects investment by restricting the amount of external financing for new projects. Panel (a) in Figure 1 plots the right and left hand sides of restriction on outside funding given by the inequality (8). Outside funding, $i_t^f$, is restricted to lie in the area where the affine function is above the 45-degree line. Since $q_t > 1$, the left panel shows that the liquidity constraint imposes a cap on the capacity to raise funds to finance the project. Panel (b) shows the effects of a decreases in $x_t$ without considering any price effects. The fall in the down-payment reduces the intercept of the function.
defined by the right hand side of Figure 1. External funding, therefore, falls together with the whole scale of the project. Since investment falls, the price of $q$ must rise such that the demand for saving instruments fall to match the fall in supply. The increase in the price of equity implies that the amount financed externally is higher. The effect on the price increases the slope and intercept, which partially counterbalances the original effect. This effect is captured in the panel to the right of Figure 1.

3 Equilibrium

The right-hand side of the entrepreneurs’ budget constraints defines their corresponding wealth: $w^s_t = (r_t + q_t \lambda) n_t$ and $w^i_t = (r_t + q^i_t \lambda) n_t$. A stationary recursive equilibrium is defined by considering the distribution of this wealth vector.

**Definition 1** (Stationary Recursive Competitive Equilibrium). A recursive competitive equilibrium is a set of price functions, $q, \omega, r : S \to \mathbb{R}_+$ allocation functions, $n^{i,j}, c^j, l^j, \tilde{v}^j : S \to \mathbb{R}_+$ for $j = i, s, w$, a sequence of distributions $\Lambda_t$ of equity holdings, and a transition function, $\Xi$, for the aggregate state such that the following hold: (1) (Optimality of Policies) Given, $q$ and $\omega$, solve the problems 3, 4 and 5. (2) Goods market clear. (3) Labor markets clear at price $w$. (4) Equity market clear at price $q$. (5) Firms maximize profits. (6) Aggregate capital evolves according to $K'_t(s) = I(s) + \lambda K(s)$. (7) $\Lambda_t$ and $\Xi$ are consistent with the policy functions obtained from problems 3, 4 and 5.

The equilibrium concept defined above is recursive for the aggregate state variables. What we mean by this is that the sequence of wealth distributions, $\Lambda_t$, is a relevant state variable that determine individual quantities but not aggregate quantities.

**Optimal Policies:** Note that the problem for both entrepreneurs is similar. Both types choose consumption and equity holdings for the next period but differ in the effective cost of equity that each of them faces. The following propositions describe the policy functions and show that these are linear functions of the wealth vector.

**Proposition 1** (Savers policies). Policy functions for saving entrepreneurs are given by:

$$c^s_t = (1 - \varsigma^s_t) w^s_t \quad q_t n^s_{t+1} = \varsigma^s_t w^s_t$$

The corresponding policies for investing entrepreneurs are,

**Proposition 2** (Investors policies). Policy functions for investing entrepreneurs satisfy:
(i) For all \( s_t \) such that \( q_t > 1 \), optimal policies are:

\[
\begin{align*}
\varsigma^i_t &= (1 - \varsigma^i_t) w^i_t \\
q^R n^i_{t+1} &= \varsigma^i_t w^i_t \\
i_{t+1} &= \frac{n_{t+1} + (\phi_t - 1) \lambda n_t}{(1 - \theta)}
\end{align*}
\]

and

(ii) For all \( s_t \) such that \( q_t = 1 \), policies are identical to the saving entrepreneurs and \( i_t \) is indeterminate at the individual level.

The marginal propensities to save of the previous proposition will depend only on the conditional expectation of conditional returns:

\[
R_{ss}^s \equiv \left( r_t + q_{t+1} \lambda \right) \left( 1 - \varsigma^s_{t+1} \right), \quad R_{si}^i \equiv \left( r_t + q^i_{t+1} \lambda \right) q^i_t, \quad R_{is}^i \equiv \left( r_t + q^i_{t+1} \lambda \right) q^i_t
\]

The following proposition characterizes these marginal propensities.

**Proposition 3 (Recursion).** The functions \( \varsigma^i_t \) and \( \varsigma^s_t \), in Propositions 1 and 2 solve the following:

\[
\begin{align*}
(1 - \varsigma^i_t)^{-1} &= 1 + \beta^\sigma \Omega^i_t \left( (1 - \varsigma^i_{t+1})^{\frac{1}{\gamma}} , (1 - \varsigma^s_{t+1})^{\frac{1}{\gamma}} \right)^{\sigma - 1} \\
(1 - \varsigma^s_t)^{-1} &= 1 + \beta^\sigma \Omega^s_t \left( (1 - \varsigma^i_{t+1})^{\frac{1}{\gamma}} , (1 - \varsigma^s_{t+1})^{\frac{1}{\gamma}} \right)^{\sigma - 1}
\end{align*}
\]

and

\[
\Omega^s_t \left( a^s_{t+1}, a^i_{t+1} \right) = \Upsilon^{-1} E_t \left[ (1 - \pi) \Upsilon \left( a^s_{t+1} R_{ss}^s \right) + \pi \Upsilon \left( a^i_{t+1} R_{si}^i \right) \right]
\]

and

\[
\Omega^i_t \left( a^s_{t+1}, a^i_{t+1} \right) = \Upsilon^{-1} E_t \left[ (1 - \pi) \Upsilon \left( a^s_{t+1} R_{is}^i \right) + \pi \Upsilon \left( a^i_{t+1} R_{ii}^i \right) \right]
\]

When \( (\sigma, \gamma) = (1, 1) \) then \( \varsigma^s_t = \varsigma^i_t = \beta. \)

A proof for the three previous propositions is provided in the appendix. Because policy functions are linear functions of wealth, the economy admits aggregation (this is the well-known Gorman aggregation result). The rest of the equilibrium conditions are summarized in the appendix.

\[13\]We show in the appendix that (10) and (11) conform a contraction for preference parameters such that \( 1 - \gamma / \sigma - 1 \) and returns satisfying \( \beta^\sigma E_t \left[ R^j \right], j \in \{i, s\} \geq 1 \). However, as long as the entrepreneurs problem is defined uniquely, equilibria can be computed by checking that the iterations on (10) and (11). Upon convergence an equilibrium is found.
Aggregate output is the sum of the two shares:

$$Y_t = \left[ (1 - \alpha) \right] \frac{1 - \beta}{\alpha + \nu} A_t^{\xi+1} K_t^{\xi+1}$$  \hspace{1cm} (12)$$

Since, $0 < \xi + 1 < 1$, this ensures that this economy has decreasing returns to scale at the aggregate level. This ensures that $K$ is bounded. By consistency $K_t = \int n_t(w)\Lambda dw$. Since investment opportunities are i.i.d, the fraction of equity owned by investing and saving entrepreneurs are $\pi K_t$ and $(1 - \pi) K_t$ respectively. The aggregate consumption, $C_t^s$, and equity holdings, $N_{t+1}^s$, of saving entrepreneurs are:

$$C_t^s = (1 - \varsigma_t) (r_t + q_t \lambda) (1 - \pi) K_t$$

$$q_{t+1} N_{t+1}^s = \varsigma_t (r_t + q_t \lambda) (1 - \pi) K_t.$$  \hspace{1cm} (13)

When $q > 1$, these aggregate variables corresponding to investing entrepreneurs, $C_t^i$ and $N_{t+1}^i$, are:

$$C_t^i = (1 - \varsigma_t^i) (r_t + \lambda q_t^i) \pi K_t$$

$$q_t^R N_{t+1}^i = \varsigma_t^i (r_t + \lambda q_t^i) \pi K_t.$$  \hspace{1cm} (14)

The evolution of marginal propensities to save and portfolio weights are given by the solution to the fixed point problem in Proposition 3. In equilibrium, the end of the period fraction of aggregate investment owned by investing entrepreneurs $I_t^i (s_t)$ must satisfy the aggregate version of (7):

$$I_t^i (s_t) \leq \left( \frac{\varsigma_t^i (r_t + \lambda q_t^i)}{q_t^R} - (1 - \phi_t) \lambda \right) \pi K_t.$$  \hspace{1cm} (15)

$(1 - \phi_t) \pi \lambda K_t$ is the lowest possible amount of equity that remains in hands of investing entrepreneurs after they sell equity of older projects. $\frac{\varsigma_t^i (r_t + \lambda q_t^i) \pi K_t}{q_t^R}$ is the equilibrium aggregate amount of equity holdings. The difference between these two quantities is the highest possible amount of equity they may hold, which corresponds to new investment projects. Similarly, for saving entrepreneurs, the amount of equity corresponding to new projects is,

$$I_t^s (s_t) \geq \left( \frac{\varsigma_t^s (r_t + \lambda q_t) (1 - \pi)}{q_t} - \phi_t \pi \lambda - (1 - \pi) \lambda \right) K_t.$$  \hspace{1cm} (14)

In equilibrium, the incentive compatibility constraints (5) that hold at the individual level also hold at the aggregate level. Thus,

$$I_t^i (s_t) \leq \left( \frac{1 - \theta}{\theta} \right) I_t^s (s_t)$$  \hspace{1cm} (15)
The above condition is characterized by a quadratic equation as a function of \( q_t \). The following proposition is used to characterize market clearing in the equity market.

**Proposition 4** (Market Clearing). For \((\sigma, \gamma)\) sufficiently close to \((1, 1)\), there exists a unique \( q_t \) that clears out the equity market and it is given by:

\[
q_t = \begin{cases} 
1 & \text{if } 1 > x_2 > x_1 \\
x_2 & \text{if } x_2 > 1 > x_1 \\
1 & \text{if otherwise}
\end{cases}
\]

where the terms \( x_2 \) and \( x_1 \) are continuous functions of \((\phi_t, \zeta_t^i, \zeta_t^s, \iota_t, r_t)\) and the parameters.

The explicit solution for \((x_2, x_1)\) is provided in the appendix. The solution accounts for the incentive constraint. When \( q > 1 \), the constraints are binding for all entrepreneurs. When, \( q = 1 \), investment at the individual level is not determined. At the aggregate level, without loss of generality, one can set the constraints above at equality when \( q_t = 1 \). By adding \( I_i^t(s_t) \) and \( I_s^t(s_t) \), one obtains aggregate investment. Because, \( q \) is continuous in \( \zeta^s \) and \( \zeta^i \), then clearly \( \zeta^s_t, \zeta^i_t, I_t, \) and \( q_t \) are continuous also.

4 Task 1: State-Space Analysis

4.1 The Liquidity Frontier

Substituting (13) and (14) into (15) at equality for \( q_t = 1 \), we obtain the minimum level of liquidity required such that constraints are not binding. Formally, we have:

**Proposition 5** (Liquidity Frontier). \( \exists! \), \( \phi^*: \mathbb{A} \times \mathbb{K} \to \mathbb{R} \) defined by:

\[
\phi^*(A, K) = \frac{[(1 - \pi)(1 - \theta) - \theta \pi]}{\lambda \pi} [\zeta_t^s(r(A, K) + \lambda) - \lambda]
\]

liquidity constraints (7) bind iff \( \phi_t < \phi^* \).

Here, \( r(A, K) \) is defined by equation (24). The proposition states that for any \((K, A)\) selection of the state-space, \( S \), there is a threshold value \( \phi^* \) such that if liquidity shocks fall below that value, the liquidity constraints bind. We call the function \( \phi^* \) the liquidity frontier, as it separates the state space into two regions. \( S^n \) is the set of points where the liquidity constraint binds so \( \phi^* = \partial S^n \). The interpretation of the liquidity frontier is simple. \( [\zeta_t^s(r(A, K) + \lambda) - \lambda] \) is the amount of entrepreneurs want to hold at \( q = 1 \). Since both types are identical when \( q = 1 \), then \( \zeta_t^s \) characterizes the demand for equity by both groups. By propositions 1 and 2, we know that \( \zeta_t^s(r(A, K) + \lambda) \) is the demand for equity stock per unit of wealth and \( \lambda \) the
remaining stock of equity per unit of wealth. The difference between these quantities is the per-unit-of-capital demand for investment.

\[(1 - \pi)(1 - \theta) - \theta \pi\] is the degree of enforcement constraint in this economy. As either the fraction of savers, \((1 - \pi)\), or the private benefit of diverting resources, \((1 - \theta)\), increase, the economy requires more liquid assets to finance a larger amount of down-payment in order to carry out investment efficiently. \(\lambda \pi \phi\) is maximal supply of equity. Therefore, the liquidity frontier is the degree of liquidity that allows the largest supply of assets to equal the demand for investment project times the degree of enforcement exactly at the point where \(q = 1\). There are several lessons obtained from this analysis.

**Lessons:** Proposition 5 shows that liquidity shocks have stronger effects as the return to capital \(r(A, K)\) is larger. This function is increasing in productivity, \(A\), and decreasing in the capital stock, \(K\) (decreasing returns to scale). This is intuitive as the demand for investment is greater when returns are high.

The proposition is also important because it gives us a good idea of the magnitude that liquidity shocks must have in order to cause a disruption on efficient investment. Since \(\varsigma_s \approx 1\), (because it is a marginal propensity to save wealth at a high frequency), then \([\varsigma_s (r(A, K) + \lambda) - \lambda] \approx r(A, K)\). In addition, if \(\lambda \approx 1\), then,

\[
\phi^*(A, K) \approx (1 - \theta) \left( \frac{(1 - \pi)}{\pi} - \frac{\theta}{(1 - \theta)} \right) r(A, K)
\]

By assumption, \(\theta \in (0, (1 - \pi))\). This implies that \(\theta \to 0\), \(\phi^*(A, K)\) will be close to \(\frac{(1 - \pi)}{\pi}\), that the economy requires a very large amount of liquidity to run efficiently. On the other hand, as \(\theta \to (1 - \pi)\), then \(\phi^*(A, K)\) will be close to zero implying that the economy does not require liquidity at all. Moreover, we can show that for two values of \(\theta\) there is almost-observational equivalence result.

**Proposition 6 (Observational Equivalence).** Let \(\rho\) be a recursive competitive equilibrium defined for parameters \(x \equiv (\theta, \Pi, \Phi)\). Then there exists another recursive competitive equilibrium \(\rho'\) for parameters \(x' \equiv (\theta', \Pi', \Phi')\) which determines the same stochastic process for prices and allocations as in \(\rho\) iff:

\[
\frac{(1 - \theta')(1 - \pi)}{\theta' \pi \lambda} \left( \frac{\varsigma_t^s (r_t + \lambda q_t)}{q_t} - \lambda \right) \in [0, 1] \tag{16}
\]

for every \(q_t, r_t, \varsigma_t^s \in \rho\) when \(q > 1\) and \(\phi^*(A, K; \theta') \in [0, 1]\) for \(q = 1\). If (16) holds then, the new triplet of parameters \(\Pi', \Phi'\) is constructed in the following way: for every \(\phi \in \Phi, \phi \geq \phi^*(A, K; \theta)\), assign to \(\Phi'\) any value \(\phi' \in (\phi^*(A, K; \theta'), 1)\). For every \(\phi < \phi^*(A, K; \theta)\), assign the value given by (16). This procedure defines a map from \(\Phi\) to \(\Phi'\). Finally, \(\Pi'\) should be consistent with \(\Pi\).
We provide a sketch of the proof here. Condition (16) is obtained by substituting (13) and (14) into (15) at equality. The left hand side of the condition is the solution to a $\phi'$ such that for $\theta'$, (15) also holds at equality. If this is the case, then the same q and investment allocations clear out the equity market. Therefore, if the condition required by the proposition holds, $\phi'$ as constructed, will yield the same allocations and prices as the original equilibrium. Since prices are the same, transition functions will be the same and so will the policy functions. If the $\phi^* (A, K; x') \not\in [0, 1]$ then liquidity under the new parameters is insufficient. This shows the if part.

By contradiction, assume that the two equilibria are observationally equivalent. If (16) is violated for some $s \in S$, then, it is impossible to find a liquidity shock between 0 and 1, such that (15) is solved at equality. For these states, $q_t > 1$ in the original equilibrium, but $q_t = 1$ in the equilibrium with the alternative parameters.

This observational equivalence implies that $(\theta, \Phi)$ are not identified from aggregate outcomes. An estimation scheme should use micro-level data to identify these parameters.

A final by-product of Proposition 5 is its use to compute an estimate of the amount of outside liquidity needed to run the economy efficiently. In order to guarantee efficient investment, an outside source of liquidity must be provided by fraction $(\phi^* - \phi_t)\lambda\pi$ of total capital stock every period. In this case, $\phi^*$ should be evaluated at $q_t = 1$ at all $s_t$. $(\phi^* - \phi_t)$ is a random variable. The stationary distribution of the state-space then can be used to compute the expected liquidity deficit for this economy. Several policy exercises can be computed using this analysis. For example, one can compute the amount of government subsidy required to run the economy efficiently.

Figure 2, plots the liquidity frontier as a function of returns $r (A, K)$, for three different values of $\beta$. Simulations show that the marginal propensities to save for values of $\theta = \{0.9, 0.7, 0.5\}$ are close to $\beta = \{0.99, 975, 0.945\}$ respectively. The frontier is increasing in the returns showing that weaker shocks will activate the constraints for higher returns. As the elasticity of intertemporal substitution increases, weaker shocks activate the liquidity constraints.

### 4.2 Asset prices

Consider the price of riskless bond in zero net supply. Note that saving entrepreneurs determine the price of this asset. This follows since saving and investing entrepreneurs cannot have their euler equations satisfied with equality at the same time when $q > 1$. Since the return to private equity for investing entrepreneurs is higher, only saving entrepreneurs will purchase a bond. Thus, the euler equation of the latter should determine pricing kernels.

The first order condition yields a pricing formula for any synthetic asset with return $r^{s,n}_{t+1}$. 
Figure 2: Liquidity Frontier. Numerical Examples.
\[ 1 = \beta \left[ \frac{s_t^p}{(1 - s_t^p)} \right]^{-\frac{1}{\gamma}} E_t \left[ CE_t \left( (1 - s_{t+1}^j \right)^{1-\sigma} R_{t+1}^i \right) \right]^{\gamma - \frac{1}{\gamma}} \left( 1 - s_{t+1}^j \left( (1-\gamma) \left( R_{t+1}^i \right)^{-\gamma} r_{t+1}^{s,n} \right) \right] \]

where \( R^s \) is the gross return of the portfolio of saving entrepreneurs. The stochastic discount factor here is expressed as a function of marginal propensities to consume and the return to the average portfolio. Hence the stochastic discount factor is given by:

\[ \mu_t (s_{t+1}) = \beta \left[ \frac{s_t^p}{(1 - s_t^p)} \right]^{-\frac{1}{\gamma}} \left[ CE_t \left( (1 - s_{t+1}^j \right)^{1-\sigma} R_{t+1}^i \right) \right]^{\gamma - \frac{1}{\gamma}} \left( 1 - s_{t+1}^j \left( (1-\gamma) \left( R_{t+1}^i \right)^{-\gamma} \right) \right] \]

where

\[ E_t \left[ (1 - s_{t+1}^j \left( (1-\gamma) \left( R_{t+1}^i \right)^{-\gamma} \right) \right] = \left[ \pi \left( 1 - s_{t+1}^j \right) \left( (1-\gamma) \left( R_{t+1}^i \right)^{-\gamma} \right) (1 - \pi) \left( 1 - s_{t+1}^j \right) \left( (1-\gamma) \left( R_{t+1}^i \right)^{-\gamma} \right) \right] \]

The return to a riskless bond in zero net supply is given by \( R^b = E [\mu_t]^{-1} \). On the other hand, the return to equity for the saving entrepreneur defined in equation (9) can be decomposed in the following way:

\[ R_{t+1}^s = \pi \frac{r_t + \lambda q_{t+1}^R}{q_t} + (1 - \pi) \frac{r_t + \lambda q_{t+1}}{q_t} \]

\[ = \frac{r_t + \lambda q_{t+1}}{q_t} - \lambda \pi \left[ (1 - \phi_{t+1}) \left( q_{t+1} - q_{t+1}^R \right) \right] \]

This expression is convenient because it allows us to decompose the return on equity. The first term is the standard return on equity considering the depreciation. The second term reduces the return on equity by a liquidity component. With probability \( \pi \) a fraction \( (1 - \phi_{t+1}) \) of the remaining equity \( \lambda \), becomes illiquid. This fraction reduces the return by the wedge \( (q_{t+1} - q_{t+1}^R) \). Notice that this term disappears when \( \phi_{t+1} = 1 \).

Combining the euler equations corresponding to equity and a riskless bond the following condition is obtained:

\[ E \left[ \mu (s') \left( R^s (s') - R^b (s') \right) \right] = 0 \tag{17} \]

The excess returns on equity is decomposed in the following way:
\[
E[R^s(s')] - R^b(s') = \frac{-cov(\mu(s'), R^s(s'))}{E(\mu(s'))} \\
= \frac{-cov(\mu(s'), (r(s') + q(s') \lambda))}{q(s) E(\mu(s'))} \\
\text{Pure Risk Premium} \\
+ \pi \lambda \frac{cov(\mu(s'), (1 - \phi_{t+1}) (q(s') - qR(s')))}{q(s) E(\mu(s'))} \\
\text{Liquidity Premium}
\]

Hence, the private equity premium is obtained by the composition of two terms. These are, first a standard risk adjustment component: \(\frac{cov(\mu(s'), (r(s') + q(s') \lambda))}{q(s) E(\mu(s'))}\) and second, a liquidity component that is given by the effects covariance between the wedge \((q_t - q_{t+1})\) scaled by \((1 - \phi_{t+1})\) and the stochastic discount factor. When liquidity constraints do not bind, the second term vanishes. The lower the probability that constraints bind in the future, the liquidity component will be smaller.

5 Task 2: Quantitative Analysis

5.1 Calibration

The nature of this exercise is to make liquidity shocks have the largest effect possible in terms of output. Thus, we pick parameters within the limits allowed by the literature and see how far we can get with this model.

The model period is a quarter. The calibration of technology parameters is standard to business cycle theory. Following the standard in the RBC literature, technology shocks are modeled such that their natural logarithm follows an AR(1) process. The persistence of the process, \(\rho_A\), is set to 0.95 and the standard deviation of the innovations of this process, \(\sigma_A\), is set to 0.016. The weight of capital in production \(\alpha\) is set to 0.36\(^{14}\). The depreciation of capital is set to \(\lambda = 0.974\) so that annual depreciation is 10%\(^{15}\).

The probability of investment opportunities, \(\pi\), is set to 0.1 so that it matches the plant level evidence of Cooper et al. [1999]. That data suggests that around 40% to 20% plants augment a considerable part of their physical capital stock. By setting \(\pi\) to 0.1, the arrival of investment opportunities is such that close to 30% of firms invest by the end of a year.

\(^{14}\)Acemoglu and Guerrieri [2008] have shown that the capital share of aggregate output has been roughly constant over the last 60 years.

\(^{15}\)It is common to find calibrations where annual depreciation is 5%. This parameter is crucial in determining the magnitude of the response of output to liquidity shocks. In particular, this response is greater the larger the depreciation rate. We discuss this aspect later on.
There is less consensus about parameters governing preferences. The key parameters for determining the allocation between consumption and investment are the time discount factor, \( \beta \), and the elasticity of intertemporal substitution, \( \sigma \). The discount factor, \( \beta \), is set to 0.9762 based on a complete markets benchmark, as in Campanale et al. [2010]. For our base scenario, we set \( \sigma \) and \( \gamma \) to 1 to recover a log-preference structure, which is a convenient benchmark as consumption is linear in wealth. Marginal propensities to save \( \varsigma^i \) and \( \varsigma^s \) are roughly constant over the state-space under alternative calibrations, implying that this choice has no impact on aggregate quantities. \( \sigma \) and \( \gamma \) are then picked in line with the asset pricing literature. The Frisch-elasticity is determined by the inverse of the parameter \( \nu \), which is set to 2.

Two sets of parameters remain to be calibrated. The first is \( \theta \), the parameter that governs limited enforcement \( \Phi \) and \( \Pi \). The limited enforcement parameter satisfies \( 0 \leq \theta < 1 - \pi = 0.9 \). For comparison reasons only, we set \( \theta = 0.15 \), as this is the value chosen by del Negro et al. [2010] and Ajello [2012]. The lower and upper bounds of \( \Phi \) are calibrated by setting \( (\phi_L, \phi_H) = (0.0, 0.3) \). This choice renders an unconditional mean of liquidity of 0.15 as in del Negro et al. [2010].

In the alternative calibration, we choose \( \Phi \) using as a target a mean risk-free rate of roughly 1% and volatility of 1.7%, which are roughly the values for the U.S. economy.\(^{16}\) We argued previously that \( \theta \) and \( \pi \) determine the liquidity frontier. \( \Phi \) and \( \Pi \) determine how often and how deeply will liquidity shocks enter the liquidity constrained region. Lower values \( \theta \) and \( \phi \) (on average) cause \( q \) to increase. These parameters interplay to determine the volatility of the price of equity and the stochastic discount factor since they influence the effect on consumption after the entrepreneur switches from one type to another (equations (20) and (21)). Based on this consideration, we set \( (\phi_L, \phi_H) = (0.17, 0.25) \) in the alternative scenario.

Finally, we describe the calibration strategy for the transition matrix \( \Pi \). There are three desirable features that the transition probability \( \Pi \) should have: (1) That it induces \( (A_t, \phi_t) \) to be correlated. Bigio [2009] and Kurlat [2009] study the relation between liquidity and aggregate productivity when liquidity is determined by the solution to a problem of asymmetric information. Both papers provide theoretical results in which the relation between productivity and liquidity is positive. (2) That \( A_t \) and \( \phi_t \) are independent of \( \phi_{t-1} \). In principle, at least in quarterly frequency, liquidity should not explain aggregate productivity. We relax this assumption in the appendix where we discuss the effects of rare and persistent liquidity crises. (3) \( A_t \) has positive time correlation, as in the RBC-literature. We calibrate the transition to obtain these properties. In the computer \( A \) is discrete. We let \( \Pi_A \) be the discrete approximation to an AR(1) process with typical element \( \pi_{a',a} = \Pr \{ A_{t+1} = a' | A_t = a \} \). To construct the joint process for \( A_t \) and \( \phi_t \) we first construct a monotone map from \( \Phi \) to \( A \) and denote it by \( A(\phi) \). This map is simply the order of elements in \( \Phi \). Let \( \Pi_u \) be a uniform transition matrix and let

\(^{16}\)We discuss the asset pricing properties of the model in the appendix.
Table 1: Calibration

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Preferences</th>
<th>Technology</th>
<th>Aggregate Shocks</th>
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<tr>
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<tr>
<td></td>
<td>μ_A = 0.95</td>
<td>σ_A = 0.016</td>
<td>R = 0.95</td>
</tr>
</tbody>
</table>

$\mathbf{\mathcal{N}} \in [0, 1]$. We fill in $\Pi$ by letting its typical element $\pi_{(a',\phi') \times (a,\phi)}$ be defined in the following way:

$$
\pi_{(a',\phi') \times (a,\phi)} \equiv \Pr \{ A_{t+1} = a', \phi_{t+1} = \phi' | A_t = a, \phi_t = \phi \}
$$

$$
= \pi_{a',a} \times \left( \mathbf{\mathcal{N}} \cdot \pi_{A(\phi'),a} + (1 - \mathbf{\mathcal{N}}) \cdot \Pi_a \right)
$$

Approximated in this way, $\Pi$ has all of the 3 properties we desire. The parametrization $\mathbf{\mathcal{N}}$ governs the correlation between $A_t$ and $\phi_t$. By construction $(A, \phi)$ are independent of $\phi$, but $A$ follows an AR(1) process. Under this assumption, the model requires 5 parameters to calibrate $\Pi$: $(\rho_A, \sigma_A)$ the standard parameters of the $AR(1)$ process for technology, $(\phi_L, \phi_H)$ the lower and upper bounds of $\Phi$ and $\mathbf{\mathcal{N}}$ the degree of correlation between liquidity shocks and technology shocks.

When we study the asset pricing properties of the model, we find that the standard deviation of the risk-free rate approximates better the values found in the data when $\mathbf{\mathcal{N}}$ is close to 1. It is worth noting that when liquidity shocks are endogenous, as in Bigio [2009] or Kurlat [2009], a high autocorrelation, $\mathbf{\mathcal{N}}$ and $(\phi_L, \phi_H)$ are delivered as outcomes of the model. Also, the equity premium is higher for higher values of this correlation. For these reason, we set $\mathbf{\mathcal{N}} = 0.95$.

The same risk-free rate may be obtained by different combinations of $\Phi$ and new parameter $\mathbf{\mathcal{N}}$. However, this choice comes at the expense of affecting its own volatility and the mean of the equity premium.

5.2 Equilibrium throughout the State-Space

In this section, we describe the model obtained by computing the recursive competitive equilibria.\(^{17}\) These equilibria are computed using a grid size of 6 for $A$ and $\Phi$. The grid size for the capital stock is 120. No important changes in the moments of the data generated by the model are obtained by increasing the grid size. Figure 3 plots $q$, $q^i$ and $q^R$ over the state-space. Each panel corresponds to a combination of an aggregate productivity shock and a liquidity shock.

\(^{17}\)Fortran\textcopyright{} and Matlab\textcopyright{} codes are available upon request. The codes run in approximately 25 minutes and 2 hours respectively. For log-preferences, the computation time is reduced more than 100 times.
The aggregate capital state is plotted in the x-axis of each panel. Some plots show a threshold capital level for which $q$ is above one and equal to 1 for capital above that level. Those points of the state-space coincide with points where the liquidity frontier is crossed. The figure helps explain how there are two forces in determining $q$ in the model. By comparing the panels in the left from those in the right, we observe how $q$ is lower in states where liquidity is higher. This happens because more liquidity relaxes the enforcement constraint at the aggregate level. To clear out the equity market when liquidity is tighter, $q$ must increase to deter saving entrepreneurs from investing in projects that don’t satisfy the incentive compatibility constraint. However, by comparing the upper row with the one on the bottom, we observe the effects of higher aggregate productivity. $q$ increases when productivity is higher. This happens because agents demand more investment at these states since returns are predicted to be higher in the future. Inside the area below the liquidity frontier, the enforcement constraint is binding at an aggregate level. In these states, $q$ must increase to make investing entrepreneurs invest more and deter saving entrepreneurs from financing projects externally.

The corresponding plots for aggregate investment are depicted in Figure 4. Each panel plots 5 curves. These curves are the aggregate investment, investment in new projects owned by saving and investing entrepreneurs (the aggregate levels of $i^s$ and $i^i$) and the net change in the equity position of both groups. The plots show that investment is close to being a linear function of capital, but this relation changes when the liquidity frontier is crossed. Why? Notice that there is a kink in aggregate investment when $q$ rises above 1, which corresponds to points inside the liquidity frontier. Outside the liquidity constrained region, the lower the capital stock, the larger the demand to replenish the capital stock (returns to equity are higher). Inside the liquidity constrained region, the lower the capital stock, the lower the amount of liquid assets for the same liquidity shock. Because liquidity is lower, investing entrepreneurs have a smaller amount to use as down-payment: investment must be lower. This effect is explained in Section 2.5. In addition, there is an important wealth transfer from investing to saving entrepreneurs in the liquidity constrained region. Thus, the overall effect on investment is also influenced by a wealth effect.

The plots show that investment is greater as aggregate productivity increases but falls when there is less liquidity. Inside each plot, we observe that the fraction of investment owned by investing entrepreneurs is always around $(1 - \theta)$ which is 85% in our base calibration. The net increase in equity is much less though. Inside the model, investing entrepreneurs are financing new projects by selling older ones. This implies that they are selling a significant fraction of their old equity stock. Outside the liquidity constrained region, the increase in equity for both groups is exactly proportional to their relative populations. Below the liquidity frontier, the increase in equity by investing entrepreneurs increases because investing entrepreneurs sell older projects at a price $q$ and their cost of generating equity is $(q^R)^{-1}$. Arbitrage opportunities translate into a
wealth transfer from savers to investing entrepreneurs. Clearly, inside the liquidity constrained region, the volume of trade in old projects falls as liquidity shocks approach to the liquidity frontier.

Marginal propensities to save are constant and equal to $\beta$ for log-utility. When $\sigma < 1$, they satisfy the following characteristics: (1) in the constrained region, investing entrepreneurs have a lower marginal propensity to save, which is strongly governed by a wealth effect; (2) the marginal propensity to save of saving entrepreneurs is decreasing in $q$ and the converse is true about investing entrepreneurs. These relations are reverted when $\sigma > 1$. Nevertheless, simulations shows that for different choices of $\sigma$, marginal propensities to save are not constant but don’t move very much.

As in He and Krishnamurthy [2008] and Brunnermeier and Sannikov [2009], this model also highlights the importance of global methods in understanding the interaction between liquidity shocks and the constraints. Is linearizing the model a bad idea? Yes and no. No because the behavior of inside either region seems to be very linear for our choice of parameters. This, in fact, is not true for other parameters. However, linearization may become problematic. The reason is that it may be highly likely that the liquidity frontier is crossed often because in stochastic environments, the capital stock is often above the desired level of capital. These states necessarily lie in the unconstrained region. Bigio [2009] shows that if illiquidity is caused by asymmetric information, liquidity shocks will fall close to the liquidity frontier. We later argue that in order to replicate a reasonable risk-free rate, liquidity shocks must fall close to the liquidity frontier. Because it is given in an analytical form (at least for log-preferences), the liquidity frontier may be used to asses how close the deterministic steady-states of an approximated model is to the liquidity frontier.

Figure 7 depicts the marginal uncondition distribution of the capital stock as well as the marginal distribution of conditional on constraints being binding. The next section describes the main quantitative properties of the model.

5.3 Business Cycle Moments

Statistics: Table 2 reports the standard business cycle moments delivered by the model. The table is broken into the moments corresponding to the unconditional distribution and conditioned on the event that the economy is in the liquidity constrained region. All of the results are computed with respect to the model’s stationary distribution of the aggregate state. The bottom row of the table reports the computed occupation time within the region below the liquidity frontier $S_b$ (the region where liquidity constraints bind). For the baseline calibration, the occupation time within the liquidity constrained region is roughly 87%. The first row presents the average value of $q$, which is close to 1.5, meaning that the wedge between the cost
of capital and the transformation rate is about 50%. In the constrained region, the wedge is on average 16% larger than its unconditional mean and 58% larger than in the unconstrained region. However, q is not very volatile. The standard deviation is about 0.5%. The second row reports the average unconditional investment/output ratio which is around 20%. This rate is slightly smaller in the constrained region because investment is inefficiently low in the constrained region. The average within the unconstrained region is considerably larger and equal to 30%. The unconditional volatility of investment over the volatility of output is almost 4 times larger. This relative volatility is much lower within the unconstrained region. This figure is not surprising once one considers that workers are not smoothing consumption (on their own will). The fourth and fifth rows describe the average marginal propensities to consume both entrepreneurs. The seventh column reports the relative size of output within each region. Output is on average 2.5% lower within the constrained region and 15% larger outside this region. Two things explain this large differences: (1) lower output is associated with a lower capital stock, which in turn, increases returns and the probability of binding constraints. (2) low liquidity is associated with low aggregate productivity shock. Similarly, worked hours are considerably larger within the unconstrained region.

The unconditional correlation between q and investment is negative and close to -95%. This happens because, most of the time, the economy is in the liquidity constrained region, and the relation between these two variables is negative in this region (but theoretically undeterminate). The fact that financing constraints can break the relation between q and investment is also found in prior work by Lorenzoni and Walentin [2009]. In our model, the relation between q and investment breaks is explained by Figure 4. The reason is that these correlations are driven by two forces that work in opposite directions: ceteris paribus liquidity shocks reduce these correlations and aggregate productivity shocks increase them. When liquidity shocks hit, the economy is more constrained, so q must rise to clear the market for new equity. This occurs together with a fall in investment. On the other hand, positive productivity shocks drive q as larger returns to equity are expected. q grows to clear out the equity market while investment is larger. The correlation computed under the baseline calibration is considerably lower than in the data. This suggests that the liquidity shocks calibrated here are extremely large, a feature that we openly acknowledge as we want to get as much as we can from the shocks.

Another interesting feature of the model is that it generates right skewness in the stationary distribution of the capital stock compared to a frictionless economy. The reason is that when the capital stock is lower, the liquidity constraints tend to be tighter. This feature is reflected into less investment than otherwise, which, in turn makes the capital stock recover at a slower pace. However, the average capital stock is lower in this economy than in a frictionless economy. One way to see this is that under log-preferences, the marginal propensities to save are constant, so all

\[ 18 \text{In the baseline scenario, these are constants and equal to } \beta. \]
of the effect on investment is through a substitution effect. In the case of saving entrepreneurs, this effect is negative and dominates the increase on investment by investing entrepreneurs.

**Results under the alternative calibration:** Table 3 reports the model’s statistics for the alternative calibration. The calibration is such that the model delivers a reasonable risk-free rate, a feature that is discussed in the appendix. The important thing to note here is that the model requires liquidity shocks to be calibrated so that they fall in the region close to the liquidity frontier to obtain reasonable risk-free rates. Under this alternative calibration, the occupation time of liquidity constraints is 75% which is not very far from the occupation time in the baseline calibration. In contrast, the magnitude of liquidity shocks is much lower. This reflects itself in the average conditional and unconditional values of $q$, which is much lower and implies a lower wedge. The share of investment over output remains roughly at 20%. The volatility of investment over output falls considerably, which reflects the smaller distortions on investment. In addition, the differences in output and hours are also considerably smaller. More interestingly, as opposed to the baseline scenario, average output and hours are larger in the liquidity constrained region. The reason is that liquidity constraints are driven much more with positive aggregate productivity (see discussion on the liquidity frontier), than by strong liquidity shocks. This aspect of the alternative calibration is also reflected in a larger and positive correlation between $q$ and investment. In this scenario, the effects of technology shocks are more powerful than those corresponding to liquidity shocks, so the force towards a positive correlation dominates.

The analysis shows that the average marginal propensities to save for both entrepreneurs are close to each other. They vary in less than 3 decimal points along the whole state space and the standard deviation has an order of magnitude less than 5 decimal points. This suggests, as explained before, that the log utility is not a bad approximation if the discount factor is adjusted accordingly.\(^{19}\) Finally, the computations show that the unconditional distribution of capital is tighter, shifted to the right and less skewed.

Finally, the fact that asset prices and the relation between $q$ and investment are closer to reality in the alternative calibration suggests that the size of liquidity shocks in the model are overstated. This reinforces the conclusions we obtain from the impulse response analysis: without labor frictions, liquidity shocks will not have an important effect on output.

**Impulse Responses:** Figure 5 plots the impulse response of several key variables after a drop in liquidity from the average level of 15% to 0%. The drop occurs only for one quarter after which liquidity follows its stochastic process. The responses are computed setting the economy at its unconditional average state. The dynamics implied by liquidity shocks resemble investment shocks. The plots show that investment falls contemporaneously with the drop in liquidity. The liquidity shock drives up $q$ because there is less amount of liquidity to meet

\(^{19}\)For example, in this scenario, one can choose a value of 0.971 for $\beta$ and the results will be roughly similar.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unconditional</td>
</tr>
<tr>
<td>q</td>
<td>1.5</td>
</tr>
<tr>
<td>STD(q)</td>
<td>0.499</td>
</tr>
<tr>
<td>i/y</td>
<td>0.186</td>
</tr>
<tr>
<td>STD(i)/STD(y)</td>
<td>3.84</td>
</tr>
<tr>
<td>y/E[y]</td>
<td>1</td>
</tr>
<tr>
<td>hours/E[hours]</td>
<td>1</td>
</tr>
<tr>
<td>corr(q,i)</td>
<td>-0.945</td>
</tr>
<tr>
<td>Occupation Time</td>
<td>0.871</td>
</tr>
</tbody>
</table>

Table 2: Key statistics under baseline calibration.

the supply of sources of external financing. An increase in $q$ benefits investing entrepreneurs but at the aggregate level, induces as substitution effect towards consumption. Because the capital in place is fixed, output only responds 1 period after the shock is realized. All of the subsequent effect on output is driven by the fall in the capital stock which is caused by the fall in investment. Because output is constant during the period of the shock, overall consumption increases. Hours and wages co-move with the capital stock so both fall together in the periods after the liquidity shock is gone. The probability of binding constraints jumps in the period where the shock occurs, and remains slightly above the unconditional average as the returns to capital are larger. After the liquidity shock is gone, all of the variables in the model remain at lower levels as the capital stock slowly converges to its stationary level.

The calibration is purposely constructed in such a way that the overall effect on output is the largest possible. The peak drop on output is close to 0.7% of the unconditional mean of output, suggesting that the effects of an extreme fall in liquidity are not very large. We devote the next section to explain why. The dynamics under the alternative calibration are very similar. However, the effects of liquidity shocks are non-linear: a fall in liquidity from 15% to 10% delivers an overall effect 0.08% which is 10 times smaller than when liquidity falls from 15% to 0%.

5.4 Discussion: Why are the effects so small?

There is an intuitive explanation for why the liquidity shocks have a minor impact on output. For any reasonable calibration the ratio of investment to capital ratio will be small. Liquidity shocks introduce a friction on investment (when it is positive). The strongest possible effect of a liquidity shock is to prevent the economy from investing at all. Thus, at most, capital

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20Recall that the liquidity frontier is higher when returns are larger.
can fall by $1 - \lambda$. Without considering labor effects, any disruption acting on investment will drop output by no more than $(1 - \lambda^\alpha)\%$. This is a theoretical upper bound we refer to in the introduction. In our calibration, this number is about 0.89%. In the model, the labor supply reacts positively to a fall in the capital stock so the effect is partially mitigated. On the other hand, liquidity shocks will not drive investment to 0 because investing entrepreneurs can invest by using internal funds. Thus, the fall in 0.7% in output should be expected.

Liquidity shocks also generate the wrong contemporaneous co-movement between output and consumption. This happens because, in the model, liquidity shocks don’t affect any of the factors of production contemporaneously. Since output remains unchanged in the period of the shock, liquidity shocks distort allocations towards consumption. This result is originally pointed out in Barro and King [1984].

The finding that liquidity shocks have a muted impact is related to previous results found by Kocherlakota [2000] and Cordoba and Ripoll [2004] but for different reasons. Kocherlakota [2000] and Cordoba and Ripoll [2004] focus on borrowing limits that depend on asset values as an amplification mechanism based on the model of Kiyotaki and Moore [1997]. Both papers agree that borrowing limits do not have much power as amplification mechanisms of productivity or monetary shocks under standard preferences. According to Liu et al. [2010], the reason behind the muted response of output is that asset prices don’t react much to the shock considered.

The problem with collateral constraints is that even if one finds a shocks that reacts strongly, the effects on output comes from a large reallocation of a production input (capital or land). Unfortunately, this explanation is unsatisfactory. We don’t see large reallocations of capital or land during recessions (Eisfeldt and Rampini [2006]). If there is some factor being reallocated

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unconditional</th>
<th>Constrained</th>
<th>Unconstrained</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>1.02</td>
<td>1.04</td>
<td>1</td>
</tr>
<tr>
<td>STD(q)</td>
<td>0.0298</td>
<td>0.0297</td>
<td>0.0297</td>
</tr>
<tr>
<td>i/y</td>
<td>0.199</td>
<td>0.201</td>
<td>0.191</td>
</tr>
<tr>
<td>STD(i) / STD(y)</td>
<td>1.48</td>
<td>1.45</td>
<td>1.51</td>
</tr>
<tr>
<td>$\varsigma_i$</td>
<td>0.971</td>
<td>0.971</td>
<td>0.971</td>
</tr>
<tr>
<td>STD($\varsigma_i$)</td>
<td>8.75e-005</td>
<td>7.95e-005</td>
<td>5.67e-005</td>
</tr>
<tr>
<td>$\varsigma_s$</td>
<td>0.971</td>
<td>0.971</td>
<td>0.971</td>
</tr>
<tr>
<td>STD($\varsigma_s$)</td>
<td>6.76e-005</td>
<td>7.21e-005</td>
<td>5.67e-005</td>
</tr>
<tr>
<td>y/E[y]</td>
<td>1</td>
<td>1.01</td>
<td>0.982</td>
</tr>
<tr>
<td>hours/E[hours]</td>
<td>1</td>
<td>1</td>
<td>0.997</td>
</tr>
<tr>
<td>corr(q,i)</td>
<td>0.135</td>
<td>0.039</td>
<td>0</td>
</tr>
<tr>
<td>Occupation Time</td>
<td>0.649</td>
<td>0.351</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Key statistics under alternative calibration.
during a crises, it is more likely to be labor.

In summary, liquidity shocks must be combined with other mechanisms that act by affecting the labor input to have strong effects on output. The investment wedge in this paper cannot be directly mapped into the investment wedge in Chari et al. [2007] because the model presented here does not have a representative agent (it has two). Nevertheless, the findings here are consistent with theirs in suggesting that a labor wedges are more likely to be important in explaining the business cycle. The next section introduces variable capital utilization. This will introduce a wedge in labor demand that is a function of $q$. This provides an example of a mechanism that disrupts investment and labor decisions together.

5.5 Asset Prices

It is known from the asset pricing literature that in order to obtain reasonable asset prices, the neoclassical growth model needs significant variation in $q$ to generate a non-negligible equity premium. Liquidity shocks induce such movements. Liquidity shocks cause variation in $q$ without the need of adjustment costs (Jermann [1998]). They also behave as a source of uninsurable idiosyncratic risk. From Constantinides and Duffie [1996], we know these source of variation may explain part of the equity premium. Tallarini [2000] shows that preferences that distinguish the intertemporal elasticity of substitution from risk aversion help explain the equity premium puzzle. Since we extend the framework in KM beyond log utility the model is placed in good footing. In fact, recent asset pricing models combine production adjustment costs with EZ preferences and are capable of explaining asset prices (see for example Campanale et al. [2010] and Croce [2009]). Since liquidity shocks have the potential to explain some asset pricing features, we use asset prices to infer more about liquidity shocks from this quantitative analysis.

For our calibration, we assume, though not modeled explicitly, that the economy experiences exogenous growth in labor productivity. For this reason, asset pricing formulas are computed using a different value for $\beta$ equal 0.991. A similar calibration is used in Campanale et al. [2010]. Tables ?? and 5 describe the asset pricing properties of the model corresponding to the unconditional distribution and the distribution conditional on binding constraints. The first column plots the expected returns of a risk-free bond in zero net supply. The baseline scenario predicts returns above 1.1%. This figure is close to the average in the U.S.. In the constrained region, the risk-free rate is slightly lower but also close to 1%. These figures suggest that this rate does not vary much across regions. The volatility of the risk-free rate is close to 2%, which is slightly above the average for the U.S. economy. However, the equity premium is 0.22%, far below the premium found in the data, 7%. Finally, we find that the liquidity premium of equity is about 10% of the equity premium.
<table>
<thead>
<tr>
<th>Calibration</th>
<th>Risk-Free Rate</th>
<th>Equity Premium</th>
<th>Liquidity Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>1.1069</td>
<td>0.22419</td>
<td>0.022063</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>2.14</td>
<td>0.542</td>
<td>0.625</td>
</tr>
</tbody>
</table>

Table 4: Key asset prices computed from unconditional distribution.

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Risk-Free Rate</th>
<th>Equity Premium</th>
<th>Liquidity Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>1.02</td>
<td>0.23</td>
<td>0.0226</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>2.74</td>
<td>0.142</td>
<td>0.0147</td>
</tr>
</tbody>
</table>

Table 5: Key asset prices computed from distribution conditional on liquidity constrained region.

There is a natural question at this point. Why do other models with EZ-preferences and production explain a substantial equity premium (e.g., Campanale et al. [2010]) whereas this one cannot? There is a simple explanation. Notice that it is the saving entrepreneur who is determining the price of equity. With liquidity shocks, his consumption is going up together with the price. This is also true in Campanale et al. [2010]. A proof of this is that the liquidity premium is positive. This means that the variation \( q \) is helping deliver a higher equity premium. Moreover, when liquidity shocks are realized, the agent expects higher dividends because the aggregate capital stock will be lower, and this will press wages downwards. Recall that the return to capital is linear at the individual level. Thus, liquidity shocks are giving substantial variation on the return to equity. This feature explains why \( \kappa \) needs to be high. With a low correlation between aggregate productivity and liquidity, the return to equity will fluctuate due to changes in prices. By increasing \( \kappa \) we are inducing a comovement in prices and dividends. Thus, we are already giving the model the best chance in delivering good asset pricing properties.

Hence, the problem must lie on the discount factor. The problem is that there is a substantial amount of insurance across entrepreneurs. Once a liquidity shock hits, the consumption of s-entrepreneur’s will increase, but the opposite occurs with i-entrepreneurs. This opposing forces reduce the covariance between the discount factor and returns.

There is yet another problem. Suppose we are able to obtain a substantial equity premium through liquidity shocks. The model will deliver a counterfactual behavior in terms of the business cycle. Conditioning on a level of \( A_t \), both the equity premium and \( q_t \). What does this failure in asset pricing suggest for incorporating other frictions? The analysis suggests that if liquidity shocks need to affect the labor market. If they affect the labor market and they are expected to be persistent, liquidity shocks could reduce the consumption of both...
entrepreneurs simultaneously. This potentially would increase the covariance term. Moreover, they could potentially resolve the negative comovement in the equity price, the equity premium and output by affecting the dividend rate. We tackle these issues in the following section. First, we digress on why liquidity shocks have such a small impact on output.

6 Task 3: Suggested Extensions

In the previous section we conclude that to generate stronger effects on output and the right co-movement with consumption and labor productivity, liquidity shocks must induce contractions in hours. For this we need a contraction in either the labor demand or the labor. Liquidity shocks could deliver changes in the supply of labor by distorting the labor-leisure trade-off of workers. However, these movements would also deliver the wrong output comovements with wages and the labor wedge. Here, we propose four mechanisms by which labor demand is affected by liquidity shocks. The first two mechanisms operate indirectly. These are utilization in capital and discuss the role of new-Keynesian mechanisms. The last two mechanisms operate by affecting labor demand directly. We explore an extension that includes working capital and then another one that includes specificity in capital goods and capacity constraints.

6.1 Extension-I: Variable Capital Utilization

We now modify the problem to introduce variable capital utilization in the spirit of Greenwood et al. [2000]. The purpose of this extension is not to take capital utilization literally. Instead, we want to illustrate a channel by which $q$ alters labor demand. With variable capital utilization, $q$ has a negative effect on employment. With this, liquidity shocks also distort production. One can envision a type of financial frictions that could generate similar results. For example, if capital becomes more costly to replace in a model with a production lag, the firm then may want to save resources by hiring less workers in order to finance more capital purchases in the future. This is the case in a model where labor and investment compete for the firm’s liquidity. However, to keep the analysis simple, we take the route of modeling variable capital utilization without further mention to financing constraints.

Suppose now that, in addition to choosing labor inputs, entrepreneurs also decide on the utilization of their capital stock. Here, $h \in [0, 1]$ is the capital utilization. Production is carried out according to $F(hk, l) \equiv (hk)^\alpha l^{(1-\alpha)}$. Utilization introduces a trade-off. Using more capital in production comes at the cost of faster depreciation. Thus, depreciation is a convex function of utilization $\delta(h) \equiv (1-\lambda) - \frac{(hk)^{(1+d)}}{(1+d)}$. Here, $d > 0$ governs the convexity of this cost function, and $b$ is a normalization constant. In optimizing its problem, the firm now takes into

\[33\]
consideration the value of capital losses $q\delta(h)$ induced by the utilization. Under decentralized production, by arbitrage, the rental rate of capital must incorporate the loss in value of more utilization. Since production is carried out efficiently, the allocations must equal decentralized production. Therefore, the objective of the firm is:  

$$\max_{h,l/k} \left( AF(h, l/k) - w\frac{l}{k} - q\delta(h) \right) k.$$ 

This expression shows that when choosing utilization, the entrepreneur internalizes the effect in the depreciation of capital. Notice that depreciation is adjusted by $q$, which is increasing in the liquidity shock.

**Main implications:** We perform an impulse response analysis with variable capital utilization. We set $d$ to 2 which is close to the values chosen by Greenwood et al. [2000]. $b$ is chosen to normalize utilization to be equal to 1 at the peak of the business cycle. $\lambda$ is calibrated to keep the annual average depreciation rate at 10%. The rest of the parameters are kept. The average labor share of output is unaffected, so there is no need to modify $\alpha$.

What are the effects of liquidity shocks now? We focus on the response of output reported in Figure 6. The most striking finding is the fall in output, which is almost 10 times larger now. The mechanics work in the following way: when liquidity is tighter, $q$ will increase. This increases the cost of capital utilization because the opportunity cost of depreciating capital is larger. Under the calibration, utilization falls in about 12% according to the dashed curve in the bottom right panel of the figure. Since, capital utilization is a complement of labor, the productivity of labor falls. Correspondingly, real wages and hours fall also. In particular, the fall in worked hours is close to 2.5%, which is significantly larger than the 0.3% found in the benchmark model. The reduction in two variable inputs of production, utilization and hours, reduces output contemporaneously. The magnitude of these effects crucially depends on the parameter $d$, which measures the elasticity of the response of utilization. Smaller values of this parameter increases the overall effect on output, but for a close set of parameters, the size of the effect on output does not vary much.

The response of consumption shows the same positive jump at the time of the liquidity shock. A counterfactual result. A decomposition of consumption by agent shows that it falls substantially for workers and investing entrepreneurs. As before, this aggregate increases for saving entrepreneurs due to a strong substitution between consumption and investment. For this calibration, the latter effect dominates. Following the period after the shock vanishes, the capital stock is lower, and the rest of the variables recover slowly together lead by the evolution of the capital stock.

$^{22}$The detail of equilibrium conditions for all of the extensions to the model is available by the author upon request.
The main lesson from this exercise is that a model that delivers a negative effect from $q$ to employment demand has the potential of explaining much larger output fluctuations. However, this avenue doesn’t resolve the problem of counter-cyclical consumption patterns or the fact that marginal product of labor is negatively affected by $\phi$.

6.2 Extension-II: New-Keynesian Channels

Versions of the KM have been extended to incorporate new-Keynesian features. The standard new-Keynesian features are (a) monopolistic competition and (b) nominal price rigidities. These features deliver strong movements in output for the following reason. Assume that the labor supply is fixed as in the KM model. After the occurrence of a liquidity shock, agents adjust the aggregate demand for consumption goods and investment (due to the negative wealth effect). The combination of nominal price rigidities and imperfect substitutability lead to production distortions. These distortions manifest in a contraction in the labor demand and consequently in total output. These are the same mechanisms that explain why in the model of Justiniano et al. [2010], investment shocks are magnified and generate the right co-movement between output and consumption. del Negro et al. [2010] and Ajello [2012] show that a similar effect operates with liquidity shocks. In comparison with the impulse responses in del Negro et al. [2010], the same dramatic liquidity shock produces a similar effect as the model with variable capital utilization we described earlier.

6.3 Extension-III: Working Capital Constraints

An alternative way in which liquidity shocks can affect the demand of labor is by requiring the wage bill to be payed upfront (see the early work by Christiano and Eichenbaum [1992]). With a working capital constraint, liquidity shocks operate by reducing the labor demand directly. This follows because they affect the access to working capital of the firm. We assume a symmetric problem for s-entrepreneurs as the one studied above. The working capital constraint in this case is $w_t l_t \leq x_t$. Here $x_t$ is the liquidity obtained by net-equity sales. The problem for s-entrepreneur that faces these constraints is:

**Problem 6 (Working Capital).**

$$V^s_t (w_t) = \max_{c_t, n_{t+1}, \Delta e_{t+1}, \Delta e_{t+1}} \ U (c_t) + \beta \cdot U \left( C E_t \left( U^{-1} (V_{t+1} (w_{t+1})) \right) \right)$$

---

23However, it is not clear in what direction the labor wedge moves within the new-Keynesian model.

24For the version of their model in which the Central Bank does not respond to the liquidity shock.
subject to the following budget constraint:

\[
c_t + q_t n_{t+1} = (r (A_t, x_t/n_t) + \lambda q_t) n_t \equiv w_t^* \\
(\Delta e_{t+1}^- - \Delta e_{t+1}^+) \leq \lambda \phi_t n_t \\
x_t = q_t (\Delta e_{t+1}^- - \Delta e_{t+1}^+). 
\]

The term \( r_t (x) \) represents the return per unit of equity given that \( x \) are liquid funds per of capital. This return is the value of the following problem:

\[
r (A_t, x) = \max_l A_t F_l (1, l) \\
w_t l \leq x. 
\]

Notice that in this formulation there is an implicit assumption. That is that the entrepreneur is returned the return per-unit of equity of a firm with liquidity equal to \( x_t/n_t \). Implicitly, entrepreneurs supply liquidity to each firm in proportion to their equity and agree on the aggregate amounts.

\[
A_t F_l (1, l) - w_t = \tau_t w_t
\]

and

\[
r_x (A_t, x_t/n_t) q_t = \mu_t.
\]

where \( \mu_t \) is the multiplier on the resellability constraint. Then, \( \tau_t \) the labor wedge and \( \tau_t w_t \) is the multiplier on the working capital constraint. If \( \mu_t \geq 0 \), it is clearly because \( \tau_t \) is positive. Moreover, \( \tau_t \) is positive only if \( w_t l \leq q_t \lambda \phi_t n_t \).

Consider now the effects of \( \phi_t \). Again, this version of the model will deliver a liquidity frontier. This frontier depends on the value of \( q_t \) and the equilibrium labor in the frictionless economy. When \( \phi_t \) falls under this frontier, labor demand will contract. This shock will also activate the wedge between the marginal productivity of labor and wages. Thus, \( \phi_t \) will bring down output and resolve the negative comovement of consumption and output. In a related environment, Bigio [2009] shows that this mechanism is capable of delivering sizeable recessions with the right comovement in consumption and the labor wedge. That paper provides an adverse-selection rationale for contractions in liquidity, but other models can deliver similar effects without asymmetric information. Nevertheless, the conclusion of this section also applies to other versions of liquidity shocks. Naturally, the mechanism can be extended to include constraints on other intermediate goods for production and not necessarily labor.
6.4 Extension-IV: Specificity and Capacity Constraints

A neoclassical economy is described through a single production technology for consumption goods and a linear transformation of consumption to investment goods. This description is equivalent to one in which the production of investment goods is carried also by employing labor and consumption. The requirement for this isomorphism is that this production technology is also homogeneous of degree 1 and that capital and labor are perfectly mobile across sectors. Thus, technology constraints are

\[ i_t = A_t F (k^i_t, l^i_t) = r_t k^i_t + w_t l^i_t \]

and

\[ c_t = A_t F (k^c_t, l^c_t) = r_t k^c_t + w_t l^c_t \]

where \( k^i_t + k^c_t = k_t \) and \( l^c_t + l^i_t = l_s^t \).

Liquidity shocks can have stronger effects if we dispose some of these assumptions, of the neoclassical. Assume now that labor and capital are not perfectly mobile in the short run. We can do so by imposing a short run specificity to the capital used for producing investment goods as in the work in Caballero [2007]. Also, assume that the consumption goods sector can employ workers up to a certain proportion of its capital in use. That is, assume there’s a production capacity constraint.

To incorporate liquidity shocks in a similar way as with the KM model, we need to constrain the amount of borrowing for investment. However, if the production of investment goods uses production inputs, we also need to prevent the suppliers of those inputs from lending factors to the s-entrepreneur. Since production function is homogeneous of degree 1, we want the enforcement constraint \( \theta \) to be interpreted as in the KM model. Thus, we need to impose a constraint of the form \( i_t - i^d_t = r_t k^i_t + w_t l^i_t - i^d_t \leq \theta i_t \). This constraint is isomorphic to the working capital constraint we presented above. Rearranging terms and substituting the expression for the equilibrium wage, we find that employment in the capital goods sector solves \( l^i_t \) must solve:

\[ (\alpha - \theta^i) A_t F (k^i_t, l^i_t) + (l^i_t + l^c_t) \nu l^i_t \leq i^d_t \]

Liquidity shocks operate in the following way. As in the KM model, liquidity shocks will cause a reduction in \( i^d_t \). Given the above constraint, this will affect the employment of hours in the investment goods sector, holding \( l^c_t \) constant. Since capital employed in this sector is now fixed and the presence of capacity constraints in the consumption goods sector, the excess in labor caused by the reduction in \( i^d_t \) in the capital goods sector will not be absorbed by the consumption goods sector. Thus, a first direct effect will be a decline in output equivalent to the decline in investment coming from the contraction in the labor demand. Consumption
will not increase. In fact, it will drop. Since the consumption goods sector cannot absorb the excess labor, its output cannot increase. Moreover, the relative price of consumption would have to fall to compensate for the reduction in investment. Thus, this reduction will also cause a contraction in the demand for labor in the consumption goods sector. Hence, we would find an increase in the labor wedge in the capital goods sector and a reduction in hours in the consumption goods sector.

Thus, the summary is that discarding the mobility assumptions, a reduction in aggregate investment cannot be entirely absorbed by the consumption goods sector. The lack of short-run mobility across sectors seems an extreme assumption for the long-run. However, they can be easily obtained from search frictions. To test this theory, we would have to find evidence that the labor wedge in capital goods sectors (or in financially constrained sectors) was larger than in other sectors.

7 Concluding Remarks

This paper develops a framework that extends analysis of the model formulated by Kiyotaki and Moore [2008] into a fully stochastic environment. We show that two regimes coexist under which the economy behaves as in the standard RBC economy and regimes where the economy suffers liquidity shocks that have the effect of negative investment shocks. The paper describes some quantitative drawbacks of this model. The previous section discusses some potential avenues for solving these quantitative issues. We conclude the paper providing several suggestions for further theoretical explorations for models with liquidity shocks.

Where do liquidity shocks come from? Beyond meeting the aforementioned quantitative challenges, the next step towards a better theory of liquidity is to explain what drives these shocks? We conceive three potential directions to explain large fluctuations in liquidity. First, adverse selection in the market for collateral may explain why liquidity constraints may be suddenly tighter. In a standard real business cycle model with idiosyncratic production risk, the distribution of these shocks don’t matter. If entrepreneurs have private information on the quality of their capital, liquidity shocks may arise endogenously as a problem of asymmetric information at the time they sell older projects. Eisfeldt [2004] studies a problem with asymmetric in general equilibrium to understand risk-sharing. Recently, Bigio [2009] and Kurlat [2009] have independently environments where liquidity shocks arise endogenously in response to asymmetric information on the quality of capital in the context of KM.

A second line of research should study the relation between liquidity and noisy information similar to some recent theories on demand shocks such as Angeletos and La’O [2009] or Lorenzoni [2009]. For example, some assets may become illiquid if there are discrepancies in their valuations caused by noisy information. Phenomena such as self-fulfilling prophecies may arise
if assets become illiquid because there is fear of them being illiquid in the future. A third approach should relate liquidity with an explicitly modeled financial system. He and Krishnamurthy [2008], Brunnermeier and Sannikov [2009], Gertler and Karadi [2009], Gertler and Kiyotaki [2010], and Curdia and Woodford [2009] study aspect when financial institutions are also subjected to limited enforcement constraints. Hence, changes in the net-worth of the financial system have effects on their role as intermediaries. Bigio [2012] presents a model where the financial system is key in determining the equilibrium in markets with asymmetric information. When the financial system’s position gets worse, it can bare less risk and thus aggravate liquidity dry-ups. In addition, an explicit model with financial intermediaries may help resolve the problem of a counter-cyclical $q$ because capital may loose value if agents project that the financial sector will be less able to reallocate capital in the future.

*Coexisting liquid assets.* There also remains a technical challenge for introducing money or other assets in this economy in a fully stochastic environment. A second asset in this environment will feature a discontinuity in its excess demand at the liquidity frontier. This aspect may be problematic to compute equilibria of the model. Perhaps this difficulty may be resolved by introducing a second degree of heterogeneity, but there is no immediate answer to do this question.

Introducing other assets may be important for policy and positive analysis. This extension may be a useful to explain the effects of monetary policy and other unconventional policies in different points of the cycle. For example, it would be useful to understand if debt financing of government deficits may cause a crowding-out effect that worsens the lack of liquidity.
References


8 Proof of Optimal Policies (Propositions 1,2,3)

This section provides a proof of the optimal policies described in Section 3. The strategy is guess and verify. For saving entrepreneurs, the guess is: $V(w; t) = U(a_t^s w^s_t)$, $c^s(w; t) = (1 - \zeta_t^s) w$, $n_{t+1}^s = \frac{c_t^s}{q_t}$ and for the investing entrepreneur, the guess is: $V^i(w^i_t) = U(a_t^i w^i_t)$, $c^i(w^i_t; t) = (1 - \zeta_t^i) w^i_t$, $n_{t+1}^i (w^i_t) = \frac{c_t^i}{q_t}$.

8.1 Step 1: First Order Conditions

Using this guess, the first order conditions for $\zeta_t^s$ is:

$$ (\zeta_t^s) : q_t U'(c_t) = \beta \left( E_t \left[ \pi \Upsilon (a_t^s w^s_{t+1}) + (1 - \pi) \Upsilon (a_t^i w^i_{t+1}) \right] \right)^{\gamma - 1/\sigma} \ldots $$

$$ (1 - \pi) \Upsilon' (a_t^s w^s_{t+1}) a_t^s R^s_{t+1} + \pi \Upsilon' (a_t^i w^i_{t+1}) a_t^i R^i_{t+1}$$

The last term follows from independence in investment opportunities. Observe also that wealth is mapped into future wealth by the following identities: $w^s_{t+1} = R^s_{t+1} w^s_t$, $w^i_{t+1} = R^i_{t+1} w^i_t$, $w^i_{t+1} = R^{ii}_{t+1} w^i_t$, and $w^s_{t+1} = R^{is}_{t+1} w^s_t$. Using these definitions $w^i_t$ is factored out from (18) to obtain:

$$ (n^s) : U'(c_t) = \beta \left( \zeta_t^s w^s_t \right)^{-1/\sigma} \left[ (1 - \pi) \Upsilon (a_t^s R^s_{t+1}) + \pi \Upsilon (a_t^i R^i_{t+1}) \right]^{\gamma - 1/\sigma} \ldots $$

$$ E_t \left[ (1 - \pi) \Upsilon' (a_t^s R^s_{t+1}) a_t^s R^s_{t+1} + \pi \Upsilon' (a_t^i R^i_{t+1}) a_t^i R^i_{t+1} \right] $$

By replacing the guess for consumption and clearing out wealth, an Euler equation in terms of the marginal propensities to save is obtained:

$$ (1 - \zeta_t^s)^{-1/\sigma} = \beta \left( \zeta_t^s \right)^{-1/\sigma} \left[ (1 - \pi) \Upsilon (a_t^s R^s_{t+1}) + \pi \Upsilon (a_t^i R^i_{t+1}) \right]^{\gamma - 1/\sigma} \ldots $$

$$ E_t \left[ (1 - \pi) \Upsilon' (a_t^s R^s_{t+1}) a_t^s R^s_{t+1} + \pi \Upsilon' (a_t^i R^i_{t+1}) a_t^i R^i_{t+1} \right] $$

An identical condition may be obtained for the investing entrepreneur by following the same steps:

$$ (1 - \zeta_t^i)^{-1/\sigma} = \beta \left( \zeta_t^i \right)^{-1/\sigma} \left[ (1 - \pi) \Upsilon (a_t^s R^s_{t+1}) + \pi \Upsilon (a_t^i R^i_{t+1}) \right]^{\gamma - 1/\sigma} \ldots $$

$$ E_t \left[ (1 - \pi) \Upsilon' (a_t^s R^s_{t+1}) a_t^s R^s_{t+1} + \pi \Upsilon' (a_t^i R^i_{t+1}) a_t^i R^i_{t+1} \right] $$
8.2 Step 2: Verification of the Value function

Replacing our guess and using the envelope theorem we obtain $U_{t}^\prime ((1-\varsigma_t)w_t) = V_{w}^\prime (a_t^s w_t) a_t^s$. Thus, $(a_t^s)^{1-\frac{1}{\sigma}} = (1-\varsigma_t)^{-\frac{1}{\sigma}}$ and similarly, for the investing entrepreneur $(a_t^i)^{1-\frac{1}{\sigma}} = (1-\varsigma_t)^{-\frac{1}{\sigma}}$. Rewriting both relations gives us or $a_j^t = (1-\varsigma_j^t)^{1-\sigma}$, $j = i, s$. The certainty equivalent of a unit of wealth tomorrow in the saving state is given by the function:

$$\Omega^s(a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1}E_{t}[(1-\pi) \Upsilon (a_{t+1}^s R_{t+1}^{ss}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{si})]$$

which is homogeneous of degree 1. The same is true about:

$$\Omega^i(a_{t+1}^s, a_{t+1}^i) = \Upsilon^{-1}E_{t}[(1-\pi) \Upsilon (a_{t+1}^s R_{t+1}^{is}) + \pi \Upsilon (a_{t+1}^i R_{t+1}^{ii})]$$

Equations (20) and (21) may be written in terms of this and we obtain:

$$(1-\varsigma_t^s)^{-\frac{1}{\sigma}} = \beta (\varsigma_t^s)^{-1/\sigma} \Omega^s(a_{t+1}^s, a_{t+1}^i)^{1-1/\sigma}$$

and

$$(1-\varsigma_t^i)^{-\frac{1}{\sigma}} = \beta (\varsigma_t^i)^{-1/\sigma} \Omega^i(a_{t+1}^s, a_{t+1}^i)^{1-1/\sigma}$$

Clearing out $\varsigma_t^s$ from the right hand side, and adding 1 to both sides yields:

$$(1-\varsigma_t^s)^{-1} = 1 + \beta \Omega^s(a_{t+1}^s, a_{t+1}^i)^{\sigma-1}$$

and

$$(1-\varsigma_t^i)^{-1} = 1 + \beta \Omega^i(a_{t+1}^s, a_{t+1}^i)^{\sigma-1}$$

Note that any value function satisfies the envelope condition so any pair of functions $(\varsigma_t^s, \varsigma_t^i)$ satisfying this recursion guarantee that the functional form guessed for $V^i$ and $V^s$ satisfy the Bellman equation. This operator appears in the statement of Proposition 3. Propositions 1 and 2 follow immediately. Finally, we show conditions under which 3 is a contraction.

8.3 Step 3: Uniqueness of policy functions

Assume that $\beta^\sigma E_{t} \left[ R_{t+1}^{ss} (s)^{(\sigma-1)} \right] < \beta^\sigma E_{t} \left[ R_{t+1}^{ii} (s)^{(\sigma-1)} \right] < 1$, where the expectation is with respect to the future state $s$ and the agents type. To simplify notation let $x(s) = (1-\varsigma(s))^{-1}$ and $y(s) = (1-\varsigma^s)^{-1}$ so that recursions (20) and (21) become:
\[ x(s) = 1 + (\beta)^\sigma \Omega_t^s \left( x(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right)^{\sigma - 1} \]
\[ y(s) = 1 + (\beta)^\sigma \Omega_t^i \left( x(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right)^{\sigma - 1} \]

where \( \Omega_t^s \) and \( \Omega_t^i \) are defined in the previous step. These equations define an operator \( T : S^{[1,\infty)^2} \rightarrow S^{[1,\infty)^2} \). That is, the operator maps continuous bounded functions from the state space \( S \) to \( [1, \infty] \) into itself. Since \( S \) is compact, then \( S^{[1,\infty]} \) is a complete metric space with respect to the sup-norm for the product space. Thus, \( T \) defines a recursion:

\[(x_{t+1}, y_{t+1}) = T(x_t, y_t)\]

We check that the recursion satisfies Blackwell’s sufficient conditions that guarantee that \( T \) is a contraction.

**Monotonicity.** Whenever \( \sigma < 1 \), the power function \( z^{\frac{1}{\sigma - 1}} \) is decreasing. Thus, for any \( x > x' \) we have: \( x^{\frac{1}{\sigma - 1}} < x'^{\frac{1}{\sigma - 1}} \). Therefore, the certainty equivalent with respect to \( \Upsilon \) for \( (x', y') \) is larger than that of the terms \( (x, y) \). Hence, \( \Omega_t^s \left( x(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right) < \Omega_t' \left( x'(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right) \). Since \( \sigma - 1 \) is negative, then \( \Omega_t^s \left( x(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right)^{\sigma - 1} > \Omega_t' \left( x'(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right)^{\sigma - 1} \). If \( \sigma > 1 \), the argument is the same except that everything is monotone increasing. The analysis is identical for \( \Omega_t^i \). This shows the monotonicity of \( T \).

**Discounting 1.** Assume \( \frac{1 - \gamma}{\sigma - 1} \geq 1. \) \( \Omega_t^i \left( x(s)^{\frac{1}{\sigma - 1}}, y(s)^{\frac{1}{\sigma - 1}} \right)^{\sigma - 1} \) can be transformed into a form of a certainty equivalent of a random variable under a convex function. When under a new probability measure. Using the definition of \( \Omega_t^i \), we have that,

\[
\left( \Upsilon^{-1} E_t \left[ (1 - \pi) \Upsilon \left( (x(s) + \psi)^{\frac{1}{\sigma - 1}} R^{ss} (s) \right) + \pi \Upsilon \left( (y(s) + \psi)^{\frac{1}{\sigma - 1}} R^{si} (s) \right) \right] \right)^{\sigma - 1} = \left( \Upsilon^{-1} E_t \left[ (1 - \pi) \Upsilon \left( (x(s) + \psi)^{\frac{1}{\sigma - 1}} R^{ss} (s) \right) + \pi \Upsilon \left( (y(s) + \psi) R^{si} (s) \right) \right] \right)^{\sigma - 1}
\]

By dividing and multiplying by \( E_t \left[ R^{sj} (s)^{1-\gamma} \right] \) we are transforming the term inside the bracket into an expectation under a new probability measure where the probabilities are now weighted by the function \( R^{sj} (s)^{1-\gamma} / E_t \left[ R^{sj} (s)^{1-\gamma} \right] \). Thus, we have transformed the equation above into the certainty equivalent of a random variable under a convex function. When \( \frac{1 - \gamma}{\sigma - 1} > 1 \), then \( z^{\frac{1}{\sigma - 1}} \) is convex. Observe that any random variable \( x \) and constant \( \psi \), the certainty equivalent of
$(x + \psi)$ under a convex function is smaller than $\psi$ plus the certainty equivalent of $x$. Therefore, we have:

$$1 + (\beta)^{\sigma} \Omega_t^s \left( (x(s) + \psi)^{\frac{1}{\sigma-1}}, (y(s) + \psi)^{\frac{1}{\sigma-1}} \right)^{\sigma-1}$$

$$\leq 1 + (\beta)^{\sigma} \Omega_t^s \left( (x(s))^{\frac{1}{\sigma-1}}, (y(s))^{\frac{1}{\sigma-1}} \right)^{\sigma-1} + \psi \beta^\sigma E_t \left[ R^{sj}(s)^{(1-\gamma)(\sigma-1)} \right]^{\frac{1}{\sigma-1}}$$

where the second inequality follows from the certainty equivalent under a concave function. Hence, $\beta^\sigma E_t \left[ R^{sj}(s)^{(\sigma-1)} \right] < 1$ is sufficient to satisfy discounting. In particular it will hold for $\sigma$ sufficiently close to 1 and $R^{sj}$ bounded. The same argument can be shown to hold for the recursion on $y(s)$.

**Final Step.** To complete the proof one needs to modify Theorem 3.3 in Stokey et al. [1989] from single-valued to two-valued functions. This is done by using a new norm by taking the max over the sup of each component. The rest of the proof is identical.

### 8.4 CRRA Preferences

One can follow the same steps as before to derive the operator for constant relative risk aversion preferences (CRRA). The recursion for the CRRA case is:

$$(1 - \varsigma_t^s)^{-1} = 1 + \beta^\sigma E_t \left[ (1 - \pi_t) \left( 1 - \varsigma_{t+1}^s \right)^{\frac{1}{\sigma}} \left( R_t^{ss} \right)^{1-\frac{1}{\sigma}} + \pi \left( 1 - \varsigma_{t+1}^i \right)^{\frac{1}{\sigma}} \left( R_t^{is} \right)^{1-\frac{1}{\sigma}} \right]^{\sigma}$$

and

$$(1 - \varsigma_t^i)^{-1} = 1 + \beta^\sigma E_t \left[ (1 - \pi_t) \left( 1 - \varsigma_{t+1}^s \right)^{\frac{1}{\sigma}} \left( R_t^{si} \right)^{1-\frac{1}{\sigma}} + \pi \left( 1 - \varsigma_{t+1}^i \right)^{\frac{1}{\sigma}} \left( R_t^{ii} \right)^{1-\frac{1}{\sigma}} \right]^{\sigma}$$
8.5 Logarithmic Preferences

Provided that $1/\sigma = \gamma = 1$, preferences are log, as in KM. Replacing the solution for the constrained guy, guessing and verifying: $\varsigma^s = \varsigma^i = \beta$ one may pull out $\varsigma^i$

\[
(1 - \varsigma^s)^{-1} = 1 + \beta^\sigma (1 - \varsigma^s_{t+1})^{-1} \Omega (1, 1)^{\sigma-1}
\]
\[
= \sum_{j=0}^{\infty} [\beta^\sigma \Omega (1, 1)^{\sigma-1}]^j
\]
\[
= \frac{1}{1 - \beta^\sigma \Omega (1, 1)^{\sigma-1}}
\]

which satisfies

$\varsigma^s = \beta$

for $\sigma = 1$. A similar argument guarantees the conjecture for $\varsigma^i$:

$\varsigma^s = \varsigma^i = \beta$

9 Equilibrium Conditions

Labor Demand: Taking the physical capital $k_t$ as given, firms are run efficiently. Using the first order conditions aggregate labor demand is obtained by integrating over the individual capital stock.

\[
L^d_t = \left[ \frac{A_t (1 - \alpha)}{\omega_t} \right]^{\frac{1}{\alpha}} K_t. \tag{22}
\]

Equilibrium Employment: Workers consume their labor income so $c_t = \omega_t L^s_t (A_t, K_t)$. The solution to the workers problem defines a labor supply schedule. Solving for equilibrium employment pins down the average wage, $\omega_t = \bar{\omega}_{\alpha + \beta} [A_t (1 - \alpha)]^{\frac{\alpha}{\alpha + \beta}} K_t^{\frac{\beta \alpha}{\alpha + \beta}}$ and equilibrium employment

\[
L^*_t (A_t, K_t) = \left[ \frac{(1 - \alpha) A_t}{\bar{\omega}} \right]^{\frac{1}{\alpha + \beta}} [K_t]^{\frac{\alpha}{\alpha + \beta}}. \tag{23}
\]

Returns to Equity: The return to capital owned by an entrepreneur is a function $A_t$ and $\omega_t$. From the firm’s profit function and the equilibrium wage, return per unit of capital is:

\[
r_t = \Gamma (\alpha) [A_t]^{\frac{1}{\alpha + \beta}} \bar{\omega}_{\beta} (K_t)^{\xi} \tag{24}
\]

where $\Gamma (\alpha) \equiv [(1 - \alpha)]^{\frac{\alpha + 1}{\alpha + \beta}} \left[ \frac{1}{1 - \alpha} - 1 \right]$ and $\xi \equiv \frac{\nu (\alpha - 1)}{\alpha + \nu} < 0$. $\xi$ governs the elasticity of aggregate returns as a function of aggregate capital. The closer $\alpha$ is to 1, profits are more elastic
to aggregate capital and the return is lower. The dynamics of the model are obtained by aggregating over the individual states using the policy functions described in the body of the paper.

**Aggregate Output:** Total output is the sum of the return to labor and the return to capital. The labor share of income is,

$$w_t L_t^s = \bar{\omega}^\xi [A_t (1 - \alpha)]^{\frac{\xi+1}{\alpha}} K_t^{\xi+1}.$$ 

The return to equity is (24). By integrating with respect to idiosyncratic capital endowment we obtain the capital share of income:

$$r_t (s_t) K_t = \Gamma (\alpha) [A_t^{\frac{\xi+1}{\alpha}} \omega^\xi K_t^{\xi+1}]$$

Aggregate output is the sum of the two shares:

$$Y_t = \left[ \frac{(1 - \alpha)}{\bar{\omega}} \right]^{\frac{1-\alpha}{\alpha+\phi}} A_t^{\frac{\xi+1}{\alpha}} K_t^{\xi+1},$$

which is the expression found in the text.

### 10 Proof of Equity Market Clearing

We use the market clearing condition for out $I^s (s)$ and $I^d (s)$ and substitute in the policy functions (1) and (2). We have that a solution at equality $q$ must satisfy:

$$\frac{(1 - \pi) \left( \zeta^s (r + q\lambda) / qK - \lambda K \right) - \pi \varphi K}{\theta} \leq \frac{\pi \left[ \zeta^i (r + q^i \lambda K) / q^R - (1 - \phi) K \right]}{(1 - \theta)}$$

Clearing out $K$ from both sides and using the definition of $q^i$ this expression becomes,

$$\frac{(1 - \pi) \zeta^s r - ((1 - \pi) (1 - \zeta^s) \lambda + \pi \phi) q}{\theta q} \leq \frac{\pi \left[ \zeta^i (r + q (1 - \phi)) - (1 - \zeta^i) (1 - \phi) q^R \right]}{(1 - \theta) q^R}$$

$$= \frac{\pi \zeta^i (r + q (1 - \phi))}{(1 - \theta q)} - \frac{\pi (1 - \zeta^i)}{(1 - \theta)} (1 - \phi)$$

We get rid of the denominators and rearrange terms to obtain,

$$\frac{(1 - \pi) \zeta^s r (1 - \theta q) - ((1 - \pi) (1 - \zeta^s) \lambda + \pi \phi) q (1 - \theta q)}{\theta q \pi \zeta^i (r + q \phi) - \theta q (1 - \theta q) \pi \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi)}$$

$$\leq \frac{(1 - \pi) \zeta^s r (1 - \theta q) - ((1 - \pi) (1 - \zeta^s) \lambda + \pi \phi) q (1 - \theta q)}{\theta q \pi \zeta^i (r + q \phi) - \theta q (1 - \theta q) \pi \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi)}$$
From this equation we obtain a quadratic expression for $q$

$$(1 - \pi)\zeta^s r - q\theta (1 - \pi)\zeta^s r - ((1 - \pi) (1 - \zeta^s) \lambda + \pi (1 - \phi)) q$$

$$+ \theta q^2 ((1 - \pi) (1 - \zeta^s) \lambda + \pi (1 - \phi))$$

$$\leq q\theta \pi \zeta^s r + q^2 \theta \pi \zeta^i (1 - \phi) - \theta q \frac{(1 - \zeta^i)}{(1 - \theta)} \phi + q^2 \theta^2 \frac{(1 - \zeta^i)}{(1 - \theta) \pi \phi}$$

This inequality is characterized by a quadratic equation in $q^*$,

$$(q^*)^2 A + q^* B + C = 0$$  \hfill (25)$$

with coefficients,

\[ A = -\theta \left( (1 - \pi) (1 - \zeta^s) - \frac{\pi (1 - \zeta^i)}{(1 - \theta) \phi} \right) \lambda + \frac{\pi (1 - \zeta^i)}{(1 - \theta) \phi} \]

\[ B = \theta r ((1 - \pi)\zeta^s + \pi \zeta^i) + ((1 - \pi) (1 - \zeta^s) \lambda + \pi \phi) - \theta \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi) \pi \]

\[ C = - (1 - \pi)\zeta^s r \]

It is clear that $C$ is negative. A pointwise inspection of $A$ shows that this terms is also negative provided that $(1 - \zeta^i)$ is arbitrarily close to $(1 - \zeta^s)$ to each other. By Assumption 1, $(1 - \pi) > \frac{\theta \pi}{(1 - \theta)}$ and by Proposition 3, for $\sigma$ sufficiently close to 1, $(1 - \zeta^i)$ is arbitrarily close to $(1 - \zeta^s)$. This means that the first term in the brackets is positive, so $A$ will be negative. For $\sigma$ sufficiently close to 1 again, $B$ is also positive because,

$$(1 - \pi) (1 - \zeta^s) \lambda - \theta \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi) \pi > (1 - \pi) (1 - \zeta^s) (\lambda - (1 - \phi)) \geq 0$$

Evaluated at 0, (25) is negative. It reaches a maximum at $-\frac{B}{2A} > 0$ and then will diverge to $-\infty$. The discriminant (25) is also positive. Observe that,

$$B = B_1 + B_2$$

where

$$B_1 = ((1 - \pi) (1 - \zeta^s) + \pi \phi) - \theta \frac{(1 - \zeta^i)}{(1 - \theta)} (1 - \phi) \pi$$

and

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$$B_2 = \theta r \left( (1 - \pi)\varsigma^s + \pi \varsigma^i \right)$$

So then, $B_2 > \theta (1 - \pi)\varsigma^s r$ and $B_1 > \left( (1 - \pi) (1 - \varsigma^s) \lambda + \pi (1 - \varsigma^i) \phi - \theta \frac{(1 - \varsigma^i)(1 - \phi)}{(1 - \theta)} \right) \pi$. Thus, $B^2 - 4AC > B^2 - 4B_1B_2 = (B_1 - B_2)^2 > 0$. Thus, the roots $(q_1, q_2)$ of (25) are both positive. (15) is satisfied when (25) is negative. We let $q_2$ be the greatest amongst these roots. There are three cases:

Case 1: If $1 > q_1$, then $q = 1$ the aggregate incentive compatibility constraint is not binding at $q = 1$.

Case 2: If $1 \in (q_1, q_2)$, then, $q = 1$ violates the aggregate incentive compatibility constraint. Since for $q > 1$, constraints hold with equality, is must be the case that $q = q_2$ is the unique value of $q$ satisfying the constraint (15) at equality.

Case 3: If $1 < q_1$, (15)) does not bind and $q = 1$ once more.
11 Figures
Figure 3: **Value of Tobin’s Q along the state-space.** The figure shows the value of $q$ (solid), $q^R$ (--) and $q^i$ (--) as a functions of aggregate capita states for 4 combinations of aggregate productivity and liquidity shocks. The figures show that at wedge exists the higher the aggregate productivity shock or the tighter the liquidity shock. For all four cases, the wedge is greater the lower the capital stock, reflecting that liquidity matters more, when returns are higher.
Figure 4: Investment along the state-space. The figure shows the value of the functions $i_t$ and $i_t^c$ as a functions of aggregate capita states for 4 combinations of aggregate productivity and liquidity shocks. The figures show that liquidity constraints bind the higher the aggregate productivity shock or the tighter the liquidity shock. For all four cases, constraints bind the lower the capital stock, when returns are higher.
Figure 5: **Liquidity Shock Impulse Response**. The figure plots the response of variables to a fall in the liquidity shock from 15% to a full shutdown. The figure is computed by computing 500000 Montecarlo simulations starting from the unconditional expectation of the state.
Figure 6: **Liquidity Shock Impulse Response with Variable Capital Utilization.** The figure plots the response of variables to a fall in the liquidity shock from 15% to 5% for the model with variable capital utilization. Capital utilization is depicted in the dashed curve in the bottom right panel. The figure is computed by computing 500000 Monte Carlo simulations starting from the unconditional expectation of the state.
Figure 7: **Invariant Distributions**. The figure shows the invariant distribution of capital the capital stock. The lighter color corresponds to probability masses under the liquidity frontier.