Bidding into the Red: Web Appendix

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A Wealth Effects and Bidding Behaviour

This Section considers symmetric changes in wealth which will also allow us to assess the effect of minimum wealth requirements, entry fees and default punishments. Proposition 1 analyses the affect of wealth on the bidding functions, while Proposition 2 considers the affect of wealth on utilities and solvency rates.

Proposition 1. *Suppose (HR) holds. Symmetric changes in wealth affect bidding in the FPA and SPA as follows:*
(a) Bids decrease in wealth.
(b) Bids are convex in wealth.
(c) As $w \to \infty$, the bidding function converges pointwise to the unlimited liability bidding function.
(d) In the SPA, as $w \to 0$, the bidding function converges pointwise to $v(\theta_i, \overline{s})$.

Proof. See Appendix A.2. □

Part (a) was first established by Waehrer (1995), albeit with a different proof. It says when bidders become wealthier they have more to lose from going bankrupt and so bid more conservatively. Part (b) says any change in wealth has a smaller effect when bidders are richer. Part (c) shows that when agents are wealthy enough, they act as if they have unlimited liability. Part (d) shows that in a SPA, very poor bidders have nothing to lose and so bid as if they are the luckiest person in the world. In fact, when $w = 0$ there are a continuum of equilibria: it is weakly dominant to bid anything above $v(\theta_i, \overline{s})$.

Proposition 2. *Suppose (HR) holds. Symmetric changes in wealth affect utilities in the FPA and SPA as follows:*
(a) Increasing all bidders' wealth by Δw increases interim utilities by more than Δw.

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(b) Interim utilities are concave in wealth.
(c) The ex–post probability of solvency increases in wealth.
(d) Suppose $(1 - F(\theta_i))/f(\theta_i) \to 0$ as $\theta_i \to \overline{\theta}$. As $N \to \infty$ and $w \to 0$, so the probability the winner declares bankruptcy converges to one.

Proof. See Appendix A.3. □

Part (c) says that a symmetric increase in wealth leads bids to fall and solvency rates to rise. Part (a), which states that utility from the auction increases in wealth, then follows from the Insolvency Effect. Part (b) means that an extra dollar in wealth is more important when the bidders are poor. Finally, part (d) says that lots of poor bidders will drive themselves into bankruptcy.[1]

A.1 Asymmetric Changes in Bidders' Wealth

Suppose wealth levels differ among bidders and the seller holds a second–price auction. Since poorer bidders are more aggressive, the auction may be inefficient and not award the object to the agent with the highest valuation.

Proposition 3. *Suppose wealth varies among bidders (and may be unknown or common knowledge). Then in the SPA:*
(a) Increasing bidder i's wealth by Δw increases i's interim utility by less than Δw.
(b) An increase in everyone's wealth by Δw increases the valuation of the winner.

Proof. See Appendix A.4 □

Part (a) says that an increase in bidder i's wealth makes i less aggressive, puts them at a competitive disadvantage and means they get less utility from the auction. Part (b) says that when everyones' wealth increases the allocation is more efficient. That is, inefficient allocations are more of a problem when all the bidders are poor, than when they are wealthy.

Wealth asymmetries have two effects. First, the allocation may be inefficient. Second, if there is inefficiency, the winning bidder is more likely to go bankrupt than the efficient bidder since both their wealth and valuation are lower. If wealth is observable then, under small–recovery, both effects lower revenue (4.3). The seller may then react in several ways. They may handicap poor bidders to correct for the asymmetry. Since the seller is bankruptcy–averse, the handicap should be large enough so that allocation is biased *towards* wealthy bidders. The seller may use an extreme form of handicap and specify a minimum wealth requirement. This

[1] Results analogous to those in Propositions 1 and 2 also apply to the all–pay auction: In a symmetric equilibrium, an increase in the wealth of all bidders by Δw decreases bids, increases the probability of solvency, and increases utilities by more than Δw. As $w \to \infty$, bids converge to those under an unlimited liability all–pay auction. As $w \to 0$, bids converge to those of a FPA where bidders have no wealth. See Board (2005) for details.

is particulary useful where the wealth level is endogenous. The seller may also wish to reduce bankruptcy by giving the lowest type some surplus. In the extreme, the seller may even avoid auctions all together and negotiate with the a single bidder (e.g. Manelli and Vincent (1995) and Zheng (2001)).

A.2 Proof of Proposition 1

Second–Price Auction. (a) Differentiating (3.1) with respect to w and using the envelope theorem,

$$[1 - B_w(\theta_i, w)][1 - G(B(\theta_i, w \,|\, \theta_i))] = 1 \tag{A.1}$$

Hence $B(\theta_i, w)$ is decreasing in w.

(b) Differentiating (A.1) with respect to w,

$$B_{ww}(\theta_i, w) E_s[\sigma_i^*] = (1 - B_w(\theta_i, w))^2 g(B(\theta_i, w) - w|\theta_i)$$

Hence $B(\theta_i, w)$ is convex in w.

(c) If we assume $v(\theta_i, \overline{s}) < \infty$, then $B(\theta_i, w) = E_s[v(\theta_i, s)]$ when $w \geq v(\theta_i, \overline{s})$. When valuations are unbounded the proof is slightly harder. Observe that $B(\theta_i, w)$ is decreasing as $w \to \infty$, so must converge. Using (3.1),

$$\begin{aligned} B(\theta_i, w) &= E_s[v(\theta_i, s)] + E_s[v(\theta_i, s) + w - B(\theta_i, s) \wedge 0] \\ &\leq E_s[v(\theta_i, s)] + E_s[(v(\theta_i, s) - B(\theta_i, s))\mathbf{1}_{v(\theta_i,s) - B(\theta_i,s) \leq -w}] \end{aligned}$$

$B(\theta_i, w)$ is finite and $v(\theta_i, s)$ is integrable, so the second term converges to zero as $w \to \infty$ (see Williams (1991, Lemma 13.1)).

(d) $B(\theta_i, w)$ is increasing as $w \to 0$, so must converge. For $w > 0$, $B(\theta_i, w) \leq v(\theta_i, \overline{s})$. Equation (3.1) implies $E_s[v(\theta_i, s) - B(\theta_i, w)] \leq w$, so $\lim_{w \to 0} E_s[v(\theta_i, s) - B(\theta_i, w)] = 0$ and $\lim_{w \to 0} B(\theta_i, w) \geq v(\theta_i, \overline{s})$. Putting this together, $\lim_{w \to 0} B(\theta_i, w) = v(\theta_i, \overline{s})$.

First–Price Auction. (a) Let $w_H \geq w_L$. For any θ_i suppose $U_H(\theta_i) - w_H = U_L(\theta_i) - w_L$, i.e.

$$E_s[(v(\theta_i, s) + w_H - b(\theta_i, w_H)) \vee 0] - w_H = E_s[(v(\theta_i, s) + w_L - b(\theta_i, w_L)) \vee 0] - w_L$$

so $b(\theta_i, w_H) \leq b(\theta_i, w_L)$. Higher wealth and lower bids imply solvency is greater under w_H so,

$$U_H'(\theta_i) = E_s[v_{\theta_i}(\theta_i, s)\sigma_H^*] F^{N-1}(\theta_i) \geq E_s[v_{\theta_i}(\theta_i, s)\sigma_L^*] F^{N-1}(\theta_i) = U_L'(\theta_i)$$

Since $U_H(\underline{\theta}) - w_H = U_L(\underline{\theta}) - w_L = 0$, we thus have $U_H(\theta_i) - w_H \geq U_L(\theta_i) - w_L$ $(\forall \theta_i)$. We also have $b(\theta_i, w_H) \leq b(\theta_i, w_L)$ and $\sigma_H^* \geq \sigma_L^*$.

(b) Consider a varying wealth $\tilde{w}$ and a constant wealth w such that $E_w[\tilde{w}] = w$. Denote the bids under $\tilde{w}$ (respectively w) by the random variable $b(\theta_i, \tilde{w})$ (and the constant $b(\theta_i, w)$). Suppose $E_w[\tilde{U}(\theta_i) - \tilde{w}] = U(\theta_i) - w$, which implies that $E_w[u(\theta_i, \tilde{w} - b(\theta_i, \tilde{w}))] = u(\theta_i, w - b(\theta_i, w))$. From the envelope theorem,

$$\begin{aligned} E_w[\tilde{U}'(\theta_i)] &= E_w[u_{\theta_i}(\theta_i, \tilde{w} - b(\theta_i, \tilde{w}))]F^{N-1}(\theta_i) \\ U'(\theta_i) &= u_{\theta_i}(\theta_i, w - b(\theta_i, w))F^{N-1}(\theta_i) \end{aligned}$$

For the lowest type $E_w[\tilde{U}(\underline{\theta})] = U(\underline{\theta})$, and Lemma 1 of Maskin and Riley (1984) and (HR) imply $E_w[\tilde{U}'(\theta_i)] \leq U'(\theta_i)$ when $E_w[\tilde{U}(\theta_i)] = U(\theta_i)$. Hence $E_w[\tilde{U}(\theta_i)] \leq U(\theta_i)$. Finally, $u(\theta_i, w - b(\theta_i, w))$ is convex and increasing in $w - b(\theta_i, w)$, so Jensen's inequality means $E_w[\tilde{w} - b(\theta_i, \tilde{w})] \leq w - b(\theta_i, w)$ and $b(\theta_i, w) \leq E_w[b(\theta_i, \tilde{w})]$. That is, $b(\theta_i, w)$ is convex in w.

(c) If we assume $v(\theta_i, \overline{s}) < \infty$, then $b(\theta_i, w) = E_s[v(\theta_i, s)]$ when $w \geq v(\theta_i, \overline{s})$. When valuations are unbounded the proof is harder. Observe that $b^{\mathrm{LL}}(\theta_i, w)$ is decreasing as $w \to \infty$, so must converge. The utility under limited liability obeys

$$\begin{aligned} \lim_{w\to\infty} u^{\mathrm{LL}}(\theta_i, w - b^{\mathrm{LL}}(\theta_i, w)) - w &= \lim_{w\to\infty} E_s[(v(\theta_i, s) + w - b^{\mathrm{LL}}(\theta_i, w)) \vee 0] - w \\ &= \lim_{w\to\infty} E_s[v(\theta_i, s) - b^{\mathrm{LL}}(\theta_i, w)] - \lim_{w\to\infty} E_s[(v(\theta_i, s) + w - b^{\mathrm{LL}}(\theta_i, w)) \wedge 0] \\ &= \lim_{w\to\infty} E_s[v(\theta_i, s) - b^{\mathrm{LL}}(\theta_i, w)] \end{aligned} \tag{A.2}$$

where the third line uses the same approach as with the SPA, above. Fix type θ_i and $w > 0$, and observe $\{v(\alpha, s) - b^{\mathrm{LL}}(\alpha, w)\}_{\alpha \leq \theta_i}$ is uniformly integrable, since it is dominated by an integrable function, $|v(\alpha, s) - b^{\mathrm{LL}}(\alpha, w)| \leq |v(\theta_i, s)| + |b^{\mathrm{LL}}(\theta_i, w)|$. The markov inequality (see Williams (1991, Section 6.4)) implies that as bidders become richer, bankruptcy rates converge to zero,

$$\begin{aligned} \lim_{w\to\infty} E_s[1 - \sigma_i^*] &= \lim_{w\to\infty} \Pr[b(\theta_i, w) - v(\theta_i, s) \geq w] \\ &\leq \lim_{w\to\infty} \frac{1}{w} E_s[(b(\theta_i, w) - v(\theta_i, s))\mathbf{1}_{b(\theta_i, w) - v(\theta_i, s) \geq w}] = 0 \end{aligned}$$

where the last line uses uniform integrability (see Williams (1991, Section 13.2)). Thus utility under limited liability satisfies

$$\begin{aligned} \lim_{w\to\infty} U^{\mathrm{LL}}(\theta_i, w) - w &= \lim_{w\to\infty} E_{\theta_{-i}}\left[\int_{\underline{\theta}}^{\theta_i} E_s\left[\frac{\partial}{\partial \theta_i} v(\alpha, s)\sigma_i^*\right] P_i(\alpha, \theta_{-i})\, d\alpha\right] \\ &= E_{\theta_{-i}}\left[\int_{\underline{\theta}}^{\theta_i} E_s\left[\frac{\partial}{\partial \theta_i} v(\alpha, s)\right] P_i(\alpha, \theta_{-i})\, d\alpha\right] \\ &= U^{\mathrm{UL}}(\theta_i, w) - w \end{aligned} \tag{A.3}$$

From equations (A.2) and (A.3), we have $\lim_{w\to\infty} b^{\mathrm{LL}}(\theta_i, w) = b^{\mathrm{UL}}(\theta_i)$, as required.

A.3 Proof of Proposition 2

Second–Price Auction. (a) Utility in the SPA is

$$U(\theta_i) = E_{\theta_{-i}} E_s[(v(\theta_i, s) + w - B(\theta_{(2)}, w)) \vee 0 \,|\, \theta_i = \theta_{(1)}] P_i(\theta) + w(1 - P_i(\theta))$$

A symmetric rise in wealth does not effect the likelihood of winning, $P_i(\theta)$, but it lowers $B(\theta_{(2)}, w)$. By equation (A.1) and the envelope theorem,

$$\begin{aligned} \frac{dU(\theta_i)}{dw} &= \left[1 - B_w(\theta_{(2)}, w)\right] \left[1 - G(B(\theta_{(2)}, w) - w \,|\, \theta_{(1)})\right] P_i(\theta) + (1 - P_i(\theta)) \\ &= \left[\frac{1 - G(B(\theta_{(2)}, w) - w \,|\, \theta_{(1)})}{1 - G(B(\theta_{(2)}, w) - w \,|\, \theta_{(2)})}\right] P_i(\theta) + (1 - P_i(\theta)) \geq 1 \end{aligned} \tag{A.4}$$

as required. (b) Equation (A.1) implies that an increase in wealth reduces $B(\theta_{(2)}, w) - w$. Applying (HR) and Shaked and Shanthikumar (1994, Equation 1.B.2), an increase in wealth reduces the term in square brackets in (A.4). Since wealth has no effect on any other part of equation (A.4), $U(\theta_i)$ is convex in wealth. (c) Solvency increases in wealth since bids fall and allocations are unaffected.

(d) Equation (4.2) implies that for any standard auction,

$$\begin{aligned} \lim_{N\to\infty} \lim_{w\to 0} E_{\theta,s}[(v(\theta_i, s) + w - y_i(\theta)) \vee 0 \,|\, \theta_i = \theta_{(1)}] &\leq \lim_{N\to\infty} \lim_{w\to 0} E_{\theta,s} \left[\frac{\partial v(\theta_{(1)}, \bar{s})}{\partial \theta_i} \frac{1 - F(\theta_{(1)})}{f(\theta_{(1)})}\right] + w \\ &= 0. \end{aligned} \tag{A.5}$$

For any random variable X, $E[\max\{X, 0\}] = 0$ implies $E[\mathbf{1}_{X>0}] = 0$ (e.g. Royden (1988, Lemma 3.8)). Therefore (A.5) implies that the probability of solvency obeys

$$\lim_{N\to\infty} \lim_{w\to 0} E_{\theta,s}[\sigma^*_{(1)}] = \lim_{N\to\infty} \lim_{w\to 0} E_{\theta,s}[\mathbf{1}_{v(\theta_i,s)+w-y_i(\theta)>0} \,|\, \theta_i = \theta_{(1)}] = 0$$

First–Price Auction. (a)–(c) follows from the proof of Proposition 1(a)–(b). Part (d) is the same as for the SPA.

A.4 Proof of Proposition 3

(a) Consider bidder i and denote the largest opposing bid by B_1. Let $F_{B_1}(\cdot)$ be the distribution function of B_1. Agent i's utility is then

$$\int_0^{B(\theta_i, w_i)} \max\{0, v(\theta_i, s) + w_i - B_1\} \, dF_{B_1}(B_1) + \int_{B(\theta_i, w_i)}^{\infty} w_i \, dF_{B_1}(B_1)$$

Differentiating with respect to w_i yields

$$\int_0^{B(\theta_i,w_i)} \mathbf{1}_{v(\theta_i,s)+w_i-B_1\geq 0}\, dF_{B_1}(B_1) + \int_{B(\theta_i,w_i)}^{\infty} dF_{B_1}(B_1) \leq 1$$

(b) Pick any two bidders. If one bidder has a higher signal and lower wealth, they will always beat their opponent. Hence, let us consider two bidders (θ_L, w_L) and (θ_H, w_H), where $\theta_H \geq \theta_L$ and $w_H \geq w_L$. Suppose $B(\theta_H, w_H) - B(\theta_L, w_L) \geq 0$. Then we wish to show that

$$\Delta(x) := B(\theta_H, w_H + x) - B(\theta_L, w_L + x)$$

is positive, for $x \geq 0$. When $\Delta(x) = 0$ then $G(B(\theta_H, w_H + x) - w_H \,|\, \theta_H) \leq G(B(\theta_L, w_L + x) - w_L \,|\, \theta_L)$ and, using (A.1), $B_w(\theta_H, w_H) \geq B_w(\theta_L, w_L)$. Hence $\Delta'(x) \geq 0$. Since $\Delta(0) \geq 0$, we have $\Delta(x) \geq 0$, for $x \geq 0$.

B All–Pay Auctions

B.1 Limited Liability and the All–Pay Auction

Let us focus on the increasing symmetric pure bidding strategy, $\beta^{\mathrm{LL}}(\theta_i)$, assuming it exists. The bidding function will then consist of two components. When $\beta^{\mathrm{LL}}(\theta_i) > w$ bidding is identical to the FPA with no wealth but value $v(\theta_i, s) + w$. When $\beta^{\mathrm{LL}}(\theta_i) \leq w$ bidders pay their bid when they lose, and so will be more cautious.[2]

Proposition 4. *Suppose an increasing symmetric pure bidding strategy exists in the APA and FPA. Then, in the APA, bidding is more aggressive under limited liability than unlimited liability. The APA also yields higher interim utilities and a higher probability of solvency than the FPA. Under small–recovery the seller also prefers the APA.*

Proof. Proposition 2with $z_i(\theta) = y_i(\theta) = \beta(\theta_i)$ implies bidding is more aggressive under limited liability than unlimited liability. Comparing the APA and FPA, when $U_{\mathrm{FP}}(\theta_i) = U_{\mathrm{AP}}(\theta_i)$ it must be the case that $\beta(\theta_i) \leq b(\theta_i)$ and $\sigma^*_{i,\mathrm{AP}} \geq \sigma^*_{i,\mathrm{FP}}$. From (3.4),

$$U'_{\mathrm{FP}}(\theta_i) = E_s[v_{\theta_i}(\theta_i, s)\sigma^*_{i,\mathrm{FP}}]P_i(\theta) \leq E_s[v_{\theta_i}(\theta, s)\sigma^*_{i,\mathrm{AP}}]P_i(\theta) = U'_{\mathrm{AP}}(\theta_i)$$

Since $U_{\mathrm{FP}}(\underline{\theta}) \leq U_{\mathrm{AP}}(\underline{\theta})$, it follows that $U_{\mathrm{FP}}(\theta_i) \leq U_{\mathrm{AP}}(\theta_i)$. Thus $\beta(\theta_i) \leq b(\theta_i)$ and $\sigma^*_{i,\mathrm{AP}} \geq \sigma^*_{i,\mathrm{FP}}$. This implies that, under small recovery, revenue (4.3) is also larger under the APA than the FPA. □

[2] The bidding will be continuous by the standard argument (e.g. Fudenberg and Tirole (1991, p.223)). When $\beta^{\mathrm{LL}}(\theta_i) > w$ existence is guaranteed by (HR), as in the first–price auction. When $\beta^{\mathrm{LL}}(\theta_i) \leq w$, Athey (2001, Theorem 7) guarantees existence in increasing strategies for risk neutral bidders, however the proof does not immediately extend to this case.

Proposition 4 shows that making losing bidders pay reduces bids and reduces bankruptcy. Intuitively, APA is a half–way house between the unlimited liability auctions (where bankruptcy is impossible) and the FPA under limited liability. This is related to the suggestion of Laffont and Robert (1996) to use the APA to overcome budget constraints.

B.2 Wealth Effects and the All–Pay Auction

Proposition 5 mirrors the results obtained for the FPA and SPA in Propositions 1 and 2.

Proposition 5. *Suppose an increasing symmetric pure bidding strategy exists in the APA. Consider a symmetric change in wealth levels.*
(a) Bids decrease in wealth.
(b) As $w \to \infty$, bidding strategies converge pointwise to those under unlimited liability.
(c) As $w \to 0$, bidding strategies converge pointwise to bidding under the FPA with no wealth.
(d) Increasing all bidders' wealth by Δw increases utilities by more than Δw.
(e) The probability of solvency increases in wealth.

Proof. For (a) and (b) repeat the FPA proof from Proposition 1.

For part (c), define the switching point as θ^* such that $\beta^{\mathrm{LL}}(\theta^*, w) = w$. Since $\beta^{\mathrm{LL}}(\theta_i, w) \geq \beta^{\mathrm{UL}}(\theta_i, w)$, if we define θ^{**} by $\beta^{\mathrm{UL}}(\theta^{**}, w) = w$, then $\theta^{**} \geq \theta^*$. Since $\beta^{\mathrm{UL}}(\theta_i, w)$ is independent of w, as $w \to 0$ so $\theta^{**} \to \underline{\theta}$ and $\theta^* \to \underline{\theta}$. Bidding for $\beta(\theta_i, w) \geq w$ is identical to that under a FPA with value $v(\theta_i, s) + w$ and initial condition $\beta^{\mathrm{LL}}(\theta^*, w) = w$. Hence as $w \to 0$, so $\theta^* \to \underline{\theta}$ and $v(\theta_i, s) + w \to v(\theta_i, s)$, so $\beta^{\mathrm{LL}}(\theta_i, w)$ converges pointwise to the first–price bidding function with no wealth. As in Proposition 1, part (a) implies part (e) which implies (d). □

C Entry Fees and Reservation Prices in a First Price Auction

Consider a first price auction with reserve price r and entry fee e. The marginal type, who is indifferent between entering the auction and staying out, denoted θ_0, is defined by

$$E_s[\max\{v(\theta_0, s) + w - e - r, 0\} - w + e]F^{N-1}(\theta_0) = e \tag{C.1}$$

Equation (C.1) implies that the entry fee and reservation price are substitutes: holding θ_0 fixed, a larger entry fee implies a lower reservation price. Assuming bidders use a symmetric, increasing bidding strategy, $b(\theta_i)$, those agents who enter the auction choose $\hat{\theta}_i$ to maximise utility,

$$E_s[\max\{v(\theta_i, s) + w - e - b(\hat{\theta}_i)\} - w + e]F^{N-1}(\hat{\theta}_i) + (w - e) \tag{C.2}$$

Differentiating with respect to $\hat{\theta}_i$ and evaluating at $\hat{\theta}_i = \theta_i$ yields a differentiation equation for the bidding function. The initial condition is given by $b(\theta_0) = r$.

Proposition 6. *Suppose (HR) holds and consider two first–price auctions,* (r_1, e_1) *and* (r_2, e_2)*, which induce the same types of bidders to enter, where* $r_1 \geq r_2$ *and* $e_1 \leq e_2$*. Then utilities and solvency rates are higher under* (r_2, e_2)*. Under small–recovery, the seller also prefers* (r_2, e_2)*.*

Proof. Denote the equilibrium utility of type θ_i under (r_1, e_1) and (r_2, e_2) by $U_r(\theta_i)$ and $U_e(\theta_i)$, respectively. Similarly, denote the bidding function by $b_r(\theta_i)$ and $b_e(\theta_i)$, respectively. Observe that, by assumption, the two auctions have the same marginal type who gains utility $U_r(\theta_0) = U_e(\theta_0) = w$.

Now suppose that, for any $\theta_i \geq \theta_0$, $U_e(\theta_i) = U_r(\theta_i)$. It follows from utility (C.2) that

$$E_s[\max\{v(\theta_i, s) + w - e_2 - b_e(\theta_i), 0\}] \geq E_s[\max\{v(\theta_i, s) + w - e_1 - b_r(\theta_i), 0\}] \tag{C.3}$$

Hence $b_e(\theta_i) + e_2 \geq b_r(\theta_i) + e_1$ and bankruptcy is more likely under (r_1, e_1) than (r_2, e_2), i.e. $\sigma^*_{i,e} \geq \sigma^*_{i,r}$. From (3.4),

$$U'_e(\theta_i) = E_s[v_{\theta_i}(\theta_i, s)\sigma^*_{i,e}]P_i(\theta) \geq E_s[v_{\theta_i}(\theta, s)\sigma^*_{i,r}]P_i(\theta) = U'_r(\theta_i)$$

Thus $U_e(\theta_i) = U_r(\theta_i)$ implies $U'_e(\theta_i) \geq U'_r(\theta_i)$ and, using the fact that $U_e(\theta_0) = U_r(\theta_0)$, we have $U_e(\theta_i) \geq U_r(\theta_i)$ for all $\theta_i \geq \theta_0$. It thus follows from (C.2) that (C.3) holds for all $\theta_i \geq \theta_0$. Hence $b_e(\theta_i) + e_2 \geq b_r(\theta_i) + e_1$ and bankruptcy is more likely under (r_1, e_1) than (r_2, e_2). From the seller's perspective, revenue (4.3) is thus larger under (r_2, e_2) than (r_1, e_1) if the small–recovery assumption holds. □

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