Durable–Goods Monopoly with Varying Demand

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Motivation

• Back to school sales
  – New influx of demand → reduce prices in September.
  – But causes people to delay purchase in August.
  – How much should reduce price?

• Pricing with varying demand
  – What happens if new demand falls over time?
  – What happens if new demand is uncertain?

• Objective: Derive optimal pricing strategy for durable–goods monopolist facing fluctuating demand from new cohorts.
Durable Goods Monopoly

- No new entry of consumers (Stokey, 1979)
  - Consumers enter market in period 1.
  - Firm choose prices \( \{p_1, \ldots, p_T\} \).
  - Agents choose when to buy.
  - Solution: charge static monopoly price forever.

- Identical entry each period (Conlisk et al, 1984)
  - Solution: charge static monopoly price forever.

Varying Demand

- What if new demand varies over time?
  - Theory of dynamic pricing.
  - Scope for intertemporal price discrimination.

- Technique
  - Method to solve dynamic mechanism design problems.
  - Simple marginal revenue interpretation.

- Fast rises and slow falls
  - Demand growing \( \implies \) price increases quickly.
  - Demand dying \( \implies \) price decreases slowly.

- Application: propagation of demand cycles.
  - Prices exceed the average–demand price.
  - The lowest price is at last period of the “slump”.
**The Model**

- Time is discrete, \( t \in \{1, \ldots, T\} \), where allow \( T = \infty \).
  - Consumers’ and firm’s information represented by filtered space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)\).
  - Common time-\( t \) discount rate, \( \delta_t \in (\epsilon, 1 - \epsilon) \), is \( \mathcal{F}_t \)-adapted.
  - Total discount factor \( \Delta_t := \prod_{i=1}^{t} \delta_s \).
- Consider consumer with value \( \theta \in [\underline{\theta}, \overline{\theta}] \)
  - If buy at time \( t \) and price \( p_t \) get \((\theta - p_t)\Delta_t\).
  - If do not purchase get zero.
- Each period consumers of measure \( f_t(\theta) \) enter market
  - Distribution function \( F_t(\theta) \), survival function \( F_t(\theta) \).
  - Total measure \( F_t(\theta) \).
  - New demand, \( f_t(\theta) \), is \( \mathcal{F}_t \)-adapted.

**Payoffs**

- Consumer’s Problem
  - Consider consumer \((\theta, t)\) with value \( \theta \) who enters at time \( t \).
  - Given sequence of \( \mathcal{F}_t \)-adapted prices \( \{p_1, \ldots, p_T\} \).
  - Choose purchasing time \( \tau(\theta, t) \) to maximise expected utility.
    \[
    u_t(\theta) = \mathcal{E}[(\theta - p_{\tau})\Delta_{\tau}]
    \]
- Firm’s Problem
  - Assume marginal cost is zero.
  - Choose \( \mathcal{F}_t \)-adapted prices \( \{p_t\} \) to maximise expected profit
    \[
    \Pi = \mathcal{E} \left[ \sum_{t=1}^{T} \int_{\theta}^{\overline{\theta}} \Delta_{\tau^*(\theta, t)} p_{\tau^*(\theta, t)} \, dF_t \right]
    \]
    where \( \tau^*(\theta, t) \) maximises the consumer’s utility, \( u_t(\theta) \).
- Notable assumptions: No resale. Firm commits to prices.
Consumer Surplus and Welfare

- Purchase time optimal so use envelope theorem,

\[
u_t(\theta) = \mathcal{E} \left[ \int_{\theta}^{\bar{\theta}} \Delta_{\tau^*(\theta,t)} dx + u(\theta, t) \right]
\]

using Milgrom–Segal (2002) since space of stopping times complex.

- Consumer surplus from generation \( t \),

\[
\int_{\theta}^{\bar{\theta}} u_t(\theta) dF_t = \mathcal{E} \left[ \int_{\theta}^{\bar{\theta}} \Delta_{\tau^*(\theta,t)} F_t(\theta) d\theta \right]
\]

- Welfare from generation \( t \),

\[
W_t = \mathcal{E} \left[ \int_{\theta}^{\bar{\theta}} \theta \Delta_{\tau^*(\theta,t)} dF_t \right]
\]

Costs are zero so the welfare is maximised by setting \( p_t = 0 \).

Firm’s Problem

- Define marginal revenue with respect to price as

\[
m_t(\theta) := \theta f_t(\theta) - F_t(\theta)
\]

- Expected profit is welfare minus consumer surplus,

\[
\Pi = \mathcal{E} \left[ \sum_{t=1}^{T} \int_{\theta}^{\bar{\theta}} \Delta_{\tau^*(\theta,t)} m_t(\theta) d\theta \right]
\]

- Profit is discounted sum of marginal revenues.
- Marginal revenue sticks to each agent \((\theta, t)\).
- The firm’s problem is to chooses prices \( \{p_1, \ldots, p_T\} \) to maximise \( \Pi \) subject to \( \tau^*(\theta,t) \) maximising \( u_t(\theta) \).
**Consumer’s Problem and Cutoffs**

**Lemma 1.** The earliest purchasing rule, $\tau^*(\theta, t)$, obeys:

- **[existence]** $\tau^*(\theta, t)$ exists.
- **[\theta–monotonicity]** $\tau^*(\theta, t)$ is decreasing in $\theta$.
- **[non–discrimination]** $\tau^*(\theta, t_L) \geq t_H \implies \tau^*(\theta, t_L) = \tau^*(\theta, t_H)$, for $t_H \geq t_L$.

- Characterise $\tau^*(\theta, t)$ by $\mathcal{F}_t$–adapted cutoffs
  
  \[ \theta^*_t := \inf \{ \theta : \tau^*(\theta, t) = t \} \]

- Back out prices from cutoffs:
  \[ (\theta^*_t - p^*_t)\Delta t = \max_{\tau \geq t+1} \mathcal{E} [(\theta^*_t - p^*_\tau)\Delta \tau] \]

**General Solution**

**Definition.** Cumulative marginal revenue equals $M_1(\theta) := m_1(\theta)$ and $M_t(\theta) := m_t(\theta) + \min\{M_{t-1}(\theta), 0\}$.

**Assumption (MON).** $M_t(\theta)$ is quasi–increasing ($\forall t$).

**Theorem 1.** Under (MON) the optimal cutoffs are $\theta^*_t = M_t^{-1}(0)$.

- Period $t = 1$
  - Sell to agent $\theta$ iff $m_1(\theta) \geq 0$

- Period $t = 2$
  - Form cumulative MR, $M_2(\theta) = m_2(\theta) + \min\{M_{t-1}(\theta), 0\}$
  - Sell to agent $\theta$ iff $M_2(\theta) \geq 0$

- Cutoffs are determined by past demand.
- Prices are determined by future cutoffs.
(1) Monotone Deterministic Demand

- Suppose demand deterministic.

**Proposition 2a.** Suppose demand is increasing, $m_{t+1}(0) \geq m_t(0)$. Then $\theta_t^* = m_t^{-1}(0)$ and prices are $p_t^* = m_t^{-1}(0)$.

**Proposition 2b.** Suppose demand is decreasing, $m_{t+1}(0) \leq m_t(0)$. Then $\theta_t^* = m_{t}^{-1}(0)$ and prices are

$$p_t^* = \sum_{s=t}^{T} \mathcal{E} \left[ \frac{\Delta_{s} - \Delta_{s+1}}{\Delta_t} \right] m_s^{-1}(0) \mid \mathcal{F}_t$$

- Myopic price: $p_t^M := m_t^{-1}(0)$.
- Average–Demand price: $p_t^A := m_{\leq t}(0)$

(2) Deterministic Cycles

- Suppose demand follows $K$ repetitions of $\{f_1(\theta), \ldots, f_T(\theta)\}$

**Proposition 4.** For $k \geq 2$, cycles are stationary.

**Proposition 5.** For $k \geq 2$, optimal prices always lie above the average–demand price, $\overline{m}_{\leq T}^{-1}(0)$.

- Price discrimination bad for all customers.

**Proposition 6.** For $k \geq 2$, if cycles are simple the price is lowest at the end of the slump.
(3) IID Demand

• Demand drawn from \( \{m_i(\theta)\} \) with prob \( \{q_i\} \).
  – Average marginal revenue \( m^A(\theta) = \sum_i q_im_i(\theta) \).
  – Average-demand price \( p^A := [m^A]^{-1}(0) \).

Proposition 7. The SLLN implies \( \lim_{t \to \infty} \theta^*_t \geq p^A \) and \( \lim_{t \to \infty} p^*_t \geq p^A \) a.s..

• Stochastic equivalent of Proposition 5 (i.e. with deterministic cycles, prices exceed the average-demand price).

Applications

Summary

• Derived optimal pricing strategy for durable-goods monopolist facing varying demand.

• Award good to agents with positive cumulative MR.

• Prices rise quickly and fall slowly.

• Asymmetry pushes prices upwards.

The End