SUPPLEMENT TO "LEARNING DYNAMICS IN SOCIAL NETWORKS"

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APPENDIX D: ANALYSIS FROM SECTION 3

D.1. *Limit of Equilibria*

PROPOSITION 2 SHOWS that (c_d^*) are the unique cost-cutoffs that are approximately optimal in $\mathcal{G}_I$ for all large I. This is the appropriate equilibrium notion for our "macroeconomic" perspective, whereby the finite agents simplify their problem by treating the economy as infinite, and are vindicated by the fact that their solution to the simplified problem is indeed approximately optimal for large I.

An alternative "microeconomic" solution concept might assume that agents can somehow overcome the complexity of the finite models $\mathcal{G}_I$ and play the exact equilibria $(c_{I,d})$. The uniqueness of the limit equilibrium suggests that $(c_{I,d})$ converge to (c_d^*).[1] Here, we confirm this conjecture. For notational simplicity, we state the proof for a single type θ, so the number of outlinks (or degree) d is an integer rather than a vector. All told, we need to prove that for all d

$$\lim_{I\to\infty} c_{I,d} = c_d^*. \tag{39}$$

As a preliminary step, note that in the equilibrium $c_I = (c_{I,d})$ of $\mathcal{G}_I$, social information $y_{I,d}$ is equi-Lipschitz as a function of t and so, too, are the cutoffs $c_{I,d} = \pi(y_{I,d})$. By the Arzela–Ascoli theorem, the sequence of cutoff vectors $c_I = (c_{I,d})$ has a subsequence which converges to some $c_\infty = (c_{\infty,d})$ (pointwise for all d). We write $x_\infty := X(c_\infty)$ for the adoption probabilities associated with this strategy in the branching process, as defined in (27).

Equation (39) now follows from the claim (proved below) that the limit behavior c_∞ is indeed optimal, given the induced adoption probabilities x_∞, that is,

$$c_{\infty,d} = \pi\big(1-(1-x_\infty)^d\big). \tag{40}$$

Indeed, given (40) we substitute into (27) (for a single type θ) to get

$$\dot{x}_\infty = E\big[\phi\big(1-(1-x_\infty)^D, \pi^{-1}(c_{\infty,D})\big)\big] = E\big[\Phi\big(1-(1-x_\infty)^D\big)\big].$$

That is, x_∞ solves (10) and so $x_\infty = x^*$. Thus, the limit of the (subsequence of) equilibria in $\mathcal{G}_I$ is an equilibrium in the branching process, that is, $(c_{\infty,d}) = (c_d^*)$. Since the solution to (10) and the associated cost cutoffs are unique, the entire sequence c_I (rather than just a subsequence) must converge to c_∞, completing the proof.

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[1] As always, the cost cutoffs also depend on $t \in [0, 1]$, which we omit to simplify notation; when talking about convergence, we refer to the topology of uniform convergence in t.

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PROOF OF (40): By the triangle inequality,

$$\left|c_{\infty,d} - \pi\left(1-(1-x_\infty)^d\right)\right| \le |c_{\infty,d} - c_{I,d}| + \left|c_{I,d} - \pi\left(1-\left(1-X(c_I)\right)^d\right)\right| + \left|\pi\left(1-\left(1-X(c_I)\right)^d\right) - \pi\left(1-\left(1-X(c_\infty)\right)^d\right)\right|.$$

Along the subsequence of I as c_I converges to c_∞, the first term on the RHS vanishes. The third term vanishes since the operator X and the function π are continuous. Turning to the second term, note that the proof of (28) and in particular the upper bound in (29) do not depend on the strategy c^*, and so implies more strongly that

$$\lim_{I\to\infty} \sup_c \left|Y_{I,d}(c) - \left(1-\left(1-X(c)\right)^d\right)\right| = 0. \tag{41}$$

The equilibrium cutoffs $c_I = (c_{I,d})$ additionally satisfy $\pi(Y_{I,d}(c_I)) = c_{I,d}$, and so

$$\lim_{I\to\infty} \left|c_{I,d} - \pi\left(1-\left(1-X(c_I)\right)^d\right)\right| = 0.$$

Q.E.D.

D.2. *Undirected, Multitype Networks*

Here, we introduce heterogeneous types into the undirected networks of Section 3.2. As in Section 3.1, every agent independently draws a finite type θ and then every agent with type θ independently draws a vector of link-stubs $(D_{\theta,\theta'})_{\theta'}$ to agents of type θ'. We additionally impose the accounting identity $\Pr(\theta)E[D_{\theta,\theta'}] = \Pr(\theta')E[D_{\theta',\theta}]$ and an additional independence assumption on $(D_{\theta,\theta'})_{\theta'}$ across θ'. Next, we connect matching link-stubs (i.e., type (θ,θ')-stubs with (θ',θ)-stubs) at random, and finally discard self-links, multilinks, and left-over link-stubs; the accounting identity guarantees that a vanishing proportion of link-stubs are discarded as $I \to \infty$. The additional independence assumption in turn implies that the typical type-θ' neighbor of a type-θ agent i has $D'_{\theta',\theta}$ links to type-θ agents (including i), and $D_{\theta',\theta''}$ links to type-θ'' agents for all $\theta'' \neq \theta$. That is, the friendship paradox only applies to agent i's own type θ.

When agent i enters, write $\bar{x}_{\theta,\theta'}$ for the probability that her neighbor θ' has adopted conditional on not having observed i adopt earlier. By the same logic as in the body of the paper, adoption probabilities in the branching process follow:

$$\dot{\bar{x}}_{\theta,\theta'} = E\left[\phi\left(1-(1-\bar{x}_{\theta',\theta})^{D'_{\theta',\theta}-1}\prod_{\theta''\neq\theta}(1-\bar{x}_{\theta',\theta''})^{D_{\theta',\theta''}},\right.\right.$$
$$\left.\left.1-(1-\bar{x}_{\theta',\theta})^{D'_{\theta',\theta}}\prod_{\theta''\neq\theta}(1-\bar{x}_{\theta',\theta''})^{D_{\theta',\theta''}}\right)\right]. \tag{42}$$

D.3. *Adding Links in Undirected Networks*

Here, we prove the claim from Section 3.2 that additional links contribute to social learning in undirected networks. As in Theorem 1, given link distributions D, $\tilde{D}$, write $y_d^* = 1-(1-\bar{x}^*)^d$ and $\tilde{y}_d^*$ as the corresponding social learning curves. Letting $\succeq_{\mathrm{LR}}$ represent the likelihood ratio order, we then have the following.

THEOREM 1″: *Assume F has a bounded hazard rate*, (6). *Social learning and welfare increase with links*: *If $\tilde{D} \succeq_{\mathrm{LR}} D$,*

(a) *For any degree* d, $\tilde{y}_d^* \geq y_d^*$,
(b) *In expectation over the degree*, $E[\tilde{y}_{\tilde{D}}^*] \geq E[y_D^*]$.

PROOF: First, observe that $\tilde{D} \succeq_{\text{LR}} D$ implies $\tilde{D}' \succeq_{\text{LR}} D'$ since

$$\frac{\Pr(\tilde{D}' = \bar{d})}{\Pr(\tilde{D}' = \underline{d})} = \frac{\bar{d}}{\underline{d}} \cdot \frac{\Pr(\tilde{D} = \bar{d})}{\Pr(\tilde{D} = \underline{d})} \geq \frac{\bar{d}}{\underline{d}} \cdot \frac{\Pr(D = \bar{d})}{\Pr(D = \underline{d})} = \frac{\Pr(D' = \bar{d})}{\Pr(D' = \underline{d})}.$$

Hence $\tilde{D}' \succeq_{\text{FOSD}} D'$. Under assumption (6), $\phi(1-(1-x)^{d-1}, 1-(1-x)^d) = 1 - \frac{1}{1-x}[(1-x)^d(1-F(\pi(1-(1-x)^d))]$ rises in d since the term in square brackets increases in $(1-x)^d$. Thus the RHS of (13) FOSD-increases in D', and so too does its solution $\bar{x}^*$ by the Single-Crossing Lemma. This implies $\tilde{y}_d^* \geq y_d^*$. Part (b) then follows from the fact that $E[y_D] = E[1-(1-\bar{x})^D]$ increases in D. *Q.E.D.*

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