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Selling options

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Abstract

Contracts often take the form of options: oil fields can be abandoned, planning permission may go unused, and acquired firms can be liquidated. We consider a seller who auctions a dynamic option among N agents. After the auction, the economy evolves and the winning bidder chooses both if and when to execute the option. The revenue-maximising auction consists of an up-front bid and a contingent fee, where the latter is chosen in a Pigouvian manner, so the winning agent's choice of exercise time maximises the seller's revenue. This contingent payment is time- and state-invariant, so the seller does not have to observe post-auction information in order to implement the optimal auction. The revenue-maximising mechanism induces a dynamic distortion: the option is exercised later than under the comparable welfare-maximising mechanism.

JEL classification: C72; D44; D86; D92

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1. Introduction

In many economically important environments, a seller auctions the right to use an asset. After the auction, as the economy evolves, the winning bidder has to decide both whether to exercise this right and when to do so. The seller of such an option contract has a lot of flexibility: the price can depend on the bids of the agents, the date the option is exercised (if ever), and the information revealed after the sale. This paper solves for the seller's optimal mechanism, allowing for all these possibilities.

To illustrate, consider the problem faced by the Government of Alberta, owner of 97% of the Alberta oil sands, the world's second largest oil reserve. The Government first auctions a

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given lease among bidders who differ in their expected extraction costs. After the auction, the winner then chooses when to initiate production, depending on the evolution of the oil price. This exercise decision is particularly important for the oil sands since the majority of fields are only marginally profitable at current prices. Using our model, we show that the Alberta Government's revenue-maximising auction is remarkably simple. The mechanism consists of an up-front bid and a positive contingent payment, where the latter is independent of both the time of extraction and the post-auction information. Practically, this means that the Government should not fine bidders who delay extraction. It should also refrain from taxing "windfall profits" derived from increases in the oil price. [1]

This paper makes two main contributions. First, we allow the seller of the good to choose both who the good is awarded to and when this allocation takes place. This intertemporal decision is important whenever the good being sold is durable, and is particularly relevant for commodities such as oil, land and timber. Second, with the aid of Milgrom and Segal's envelope theorem [10], we integrate the theory of optimal stopping with the theory of mechanism design.

1.1. Outline of the paper

Suppose that N agents compete for the option to use an asset, such as a plot of land. Before the auction, the agents have private information about their revenue from the asset. After the auction, the winning bidder's cost changes over time, either due to changes in the state of the world or due to the arrival of new information. As his cost evolves, the winner must then decide both whether to exercise the option and when to do so. [2]

Without loss of generality, we break the payment to the seller into two components: an up-front charge and a contingent fee, paid when the option is executed. This contingent fee (or strike price) depends upon: (1) all the agents' bids; (2) when the option is exercised; and (3) any information revealed after the auction.

The crucial difference between up-front and contingent payments is that the latter introduces a distortion. Up-front payments are sunk and do not affect when the option is executed. In contrast, an increase in the contingent payment lowers the value of the option and delays execution. Welfare is therefore maximised by setting the contingent fee equal to zero and using an up-front scheme, such as an English auction.

The revenue-maximising auction, in contrast, involves a positive contingent payment. This contingent fee is: (1) declining in the winner's bid and independent of losers' bids; (2) independent of when the option is exercised, and (3) independent of post-auction information. This last property means the seller does not have to observe post-auction data in order to implement the optimal mechanism. The positive contingent fee also induces a dynamic distortion: the option is exercised later than under the welfare-maximising auction.

The optimal contingent payment is chosen in a particularly simple fashion that makes the problem very tractable. The winning agent chooses an exercise time to maximise his valuation minus

[1] There are a number of other important applications where a seller auctions the right to use an asset. When an upstream firm sells a scarce resource, the downstream firms bidding in the auction may not yet know their demand. The competitors thus bid for the right to be supplied at some future date, and subsequently choose whether or not to exercise this option after uncertainty has been resolved. Similarly, when a parent company sells a failing subsidiary, the purchaser has the right to liquidate their acquisition if they cannot turn it around. This option is frequently used: close to one-third of British private-equity-backed businesses go bankrupt.

[2] The seller is female, while buyers are male.

the contingent payment. In comparison, the seller's revenue equals the winning bidder's valuation minus his information rent. Setting the contingent payment equal to the winner's information rent thus aligns incentives in the style of a Pigouvian tax. The agent's choice of exercise time then maximises revenue—an act of perfect delegation.

The auction mechanism has one notable restriction: the allocation decision is determined at time 0, independent of the post-auction information. This auction mechanism is optimal if post-auction information has a similar effect on all agents, but may be inefficient if costs evolve differently. The final part of the paper extends the set of mechanisms to allow the identity of the winner, in addition to the exercise time, to depend on cost realisations. First, we characterise the optimal mechanism when costs are observable. Second, we extend the handicap auction of Eso and Szentes [5] to show that the seller can achieve the same revenue when she cannot observe costs.

This paper is most closely related to the literature on multi-period mechanism design. Baron and Besanko [1] and Courty and Li [4] analyse a two-period model where agents initially have an imperfect signal of their valuations, and only observe their types after signing the contract. Riordan and Sappington [12] extend the model to allow an auction between the imperfectly informed agents in the first period. They show that the problem separates: the seller first awards the good to the agent with the highest valuation and subsequently implements the optimal single-agent contract. Eso and Szentes [5] show that the seller would be better off if she waits to allocate the good in the second period, after agents have learned their types. However, as our paper highlights, the seller cannot simply wait to allocate the good if she does not know when the buyer wishes to use it.

2. Model

Time is discrete, $t \in \{0, \ldots, T\}$, where we allow $T = \infty$. At time $t = 0$, N risk-neutral agents bid for an option. The winning bidder then has the right to exercise this option at any time $t \in \{1, \ldots, T\}$. Each agent has a net valuation consisting of two elements: an ex-ante valuation, observed before the auction, and a time-varying ex-post cost.

Agent i's ex-ante valuation is given by a privately known type $\theta_i \in [\underline{\theta}, \overline{\theta}]$. Agents' types are independent and distributed according to $F(\theta_i)$ with density $f(\theta_i)$. Let $\theta := (\theta_1, \ldots, \theta_N)$ and $\theta_{-i} := (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_N)$.

The ex-post cost evolves over time, $t \in \{1, \ldots, T\}$. Agent i's cost is given by a sequence of random variables $\{c_{i,t}\}$. Let the history of i's costs be denoted $c_i^t := (c_{i,1}, \ldots, c_{i,t})$, and the entire history of costs be $c^t := (c_1^t, \ldots, c_N^t)$. These costs are independent of bidders' ex-ante types and are publicly verifiable. We place two restrictions on the stochastic structure of costs. First, $c_{i,t}$ are bounded below. Second, when predicting i's future costs, agent i's past costs, c_i^t, are a sufficient statistic for all past costs, c^t. Fluctuations in these costs can be interpreted as changes in the state of the world (e.g. input prices) or as the result of new information concerning the true cost c_i^*, where $c_{i,t} = E_t[c_i^*]$. Sometimes it will also be useful to assume costs are symmetrically distributed across agents.

Assumption (SYM). The joint distribution $\text{Prob}(c_1^T, \ldots, c_N^T)$ is invariant to permutations of $\{1, \ldots, N\}$.

An agent's net valuation equals his ex-ante valuation minus his ex-post cost, discounted at rate $\delta \in (0, 1)$. If agent i exercises the option in period $t \leqslant T$, he thus obtains

$$(\theta_i - c_{i,t})\delta^t.$$

If agent i never exercises the option, he receives 0.

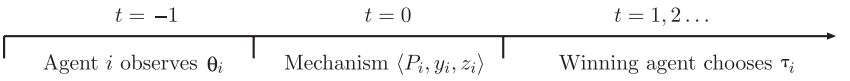

Fig. 1. Timeline.

The seller uses a direct revelation mechanism $\langle P_i, y_i, z_i \rangle$ that consists of an allocation function, an up-front payment and a contingent payment. At time $t = 0$, each agent i announces his type $\tilde{\theta}_i$. Immediately after the announcements, agent i makes up-front payment $y_i(\tilde{\theta})$ and is awarded the option with probability $P_i(\tilde{\theta})$, where $\sum_i P_i(\tilde{\theta}) \leqslant 1$. If agent i wins the auction and chooses to execute the option in period t, he makes contingent payment $z_{i,t}(\tilde{\theta}, c^t)$, where $z_i(\tilde{\theta}, c^T) := \{z_{i,t}(\tilde{\theta}, c^t)\}_t$. This contingent payment is very general and can depend upon agents' reports, the time of execution and the history of costs.

If agent i wins the auction, he subsequently chooses an exercise time, $\tau_i \in \{1, \ldots, T, \infty\}$. The exercise time is a random variable, where the decision to stop at t can only depend on the information available at time t.[3] If an agent never exercises the option, he is said to choose $\tau_i = \infty$. Agent i's optimisation problem is thus to choose τ_i to maximise his *ex-post utility*

$$u_i(\theta_i, z_i, \tau_i) := E_c[(\theta_i - c_{i,\tau_i} - z_{i,\tau_i}(\tilde{\theta}, c^{\tau_i}))\delta^{\tau_i}], \tag{1}$$

where E_c is the expectation at the start of period $t = 1$, taken over all costs. Let $\tau_i^*(\theta_i, z_i)$ denote an optimal stopping rule: Lemma 1 in Section 3 shows that such a stopping rule exists.

If the seller awards the good to agent i, her revenue equals the sum of up-front payments from the agents plus the discounted contingent payment from the winner. If the seller does not award the object, her revenue equals the sum of up-front payments from the agents plus the value she places on the option, v_0. Putting this together, the seller's revenue, when agents report truthfully, is given by

$$E_{\theta,c}\left[\sum_i y_i(\theta) + \sum_i P_i(\theta)\delta^{\tau_i^*(\theta_i,z_i)} z_{i,\tau_i^*(\theta_i,z_i)}(\theta, c^{\tau_i^*(\theta_i,z_i)}) + \left(1 - \sum_i P_i(\theta)\right) v_0\right],$$

where $E_{\theta,c}$ is the expectation at time $t = 0$, taken over both agents' costs and agents' types. The timing of the game is summarised in Fig. 1.

3. Agent's optimal stopping problem

After the auction, the winning agent chooses an exercise time τ_i to maximise his ex-post utility (1). Denote the set of maximisers by $\hat{\tau}_i(\theta_i, z_i)$. The set of stopping rules forms a lattice, where $\tau_H \geqslant \tau_L$ if τ_H stops later than τ_L for almost all sequences of costs. Comparing two sets of stopping rules, $\hat{\tau}_H \geqslant \hat{\tau}_L$ in strict set order if $\tau' \in \hat{\tau}_H$ and $\tau'' \in \hat{\tau}_L$ imply that $\tau' \vee \tau'' \in \hat{\tau}_H$ and $\tau' \wedge \tau'' \in \hat{\tau}_L$.[4]

[3] We make the natural assumption that the information available at time t equals the sigma-algebra generated by c^t. This restriction is unnecessary for our results. See Chow et al. [3] for a formal treatment of stopping times.

[4] The join $\vee$ is the least upper bound, while the meet $\wedge$ is the greatest lower bound.

Lemma 1. *Agent i's optimal exercise decision* (1) *has the following properties*:

(a) $\hat{\tau}_i(\theta_i, z_i)$ *is a nonempty sublattice and contains a greatest and least element.*
(b) *Every selection from* $\hat{\tau}_i(\theta_i, z_i)$ *is decreasing in* θ_i.
(c) *Fix* $z_{i,t}$ *and consider charging a contingent payment* $z_{i,t} + K$ $(\forall t)$, *for a constant* $K > 0$. *Then every selection from* $\hat{\tau}_i(\theta_i, z_i + K)$ *is increasing in* K.

Proof. (a) Since $\delta < 1$ and costs are bounded below, nonemptiness of $\hat{\tau}_i(\theta_i, z_i)$ follows from Klass [8, Theorem 1]. The set of optimal rules is characterised by Klass [8, Theorem 6] and contains a least and greatest element. Since the set of stopping times is a lattice and $u_i(\theta_i, z_i, \tau_i)$ is (weakly) supermodular in τ_i, the set of maximisers is a sublattice by Topkis [14, Theorem 2.7.1].

(b) $u_i(\theta_i, z_i, \tau_i)$ has strictly decreasing differences in (θ_i, τ_i), since $\delta < 1$, and is supermodular in τ_i. Hence every optimal selection is decreasing by Topkis [14, Theorem 2.8.4].

(c) $u_i(\theta_i, z_i + K, \tau_i)$ satisfies strictly increasing differences in (K, τ_i), since $\delta < 1$, and is supermodular in τ_i. Hence every optimal selection is increasing by Topkis [14, Theorem 2.8.4]. □

Lemma 1(a) examines the basic properties of the agent's choice set. For example, the least element can be found by using the rule: stop when current utility is weakly greater than the continuation utility. Lemma 1(b) says that agents with high valuations are more impatient and choose to stop earlier. Intuitively, an agent with a high valuation has an option that is more in the money, is therefore less risk-loving and is less likely to delay execution.[5] Similarly, Lemma 1(c) says that a uniform reduction in agent i's contingent payment is equivalent to an increase in his valuation, causing him to stop earlier.

Denote agent i's choice of stopping rule by $\tau_i^*(\theta_i, z_i) \in \hat{\tau}_i(\theta_i, z_i)$. From Lemma 1, $\tau_i^*(\theta_i, z_i + K)$ is decreasing in θ_i and increasing in K.

4. Optimal auctions

4.1. Information rents

If all other agents report truthfully, agent i chooses his report $\tilde{\theta}_i$ and exercise time τ_i to maximise *interim utility*,

$$U_i(\theta_i, \tilde{\theta}_i, \tau_i) = E_{\theta_{-i}}\left[P_i(\tilde{\theta}_i, \theta_{-i}) E_c\left[\left(\theta_i - c_{i,\tau_i} - z_{i,\tau_i}(\tilde{\theta}_i, \theta_{-i}, c^{\tau_i})\right)\delta^{\tau_i}\right] - y_i(\tilde{\theta}_i, \theta_{-i})\right]. \tag{2}$$

Truthful revelation is a Bayesian Nash equilibrium if interim utility (2) satisfies incentive compatibility

$$U_i(\theta_i, \theta_i, \tau_i^*(\theta_i, z_i(\theta_i, \theta_{-i}, c^T))) \geqslant U_i(\theta_i, \tilde{\theta}_i, \tau_i^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, c^T))) \tag{3}$$

[5] There is empirical support for Lemma 1(b). In OCS wildcat auctions, Porter [11] observes that the first tracts to be drilled were those with high bids; these tracts also led to higher oil revenues.

and individual rationality

$$U_i(\theta_i, \theta_i, \tau_i^*(\theta_i, z_i(\theta_i, \theta_{-i}, c^T))) \geqslant 0. \tag{4}$$

Denote the equilibrium utility of i by $V_i(\theta_i) := U_i(\theta_i, \theta_i, \tau_i^*(\theta_i, z_i(\theta_i, \theta_{-i}, c^T)))$.

To examine how agent i's utility changes with θ_i, we wish to apply an envelope theorem where agents maximise over both their report, $\tilde{\theta}_i$, and their stopping time, τ_i. The space of stopping times is too complicated for the usual envelope theorem on $\Re^N$ to be applied, so we will use the generalised envelope theorem of Milgrom and Segal [10].

Milgrom and Segal suppose an agent chooses $x \in X$ to maximise $\phi(x, \lambda)$, where $\lambda \in [0, 1]$ and X is arbitrary. Denote the set of maximisers by $\hat{x}(\lambda) := \operatorname{argmax}_x \phi(x, \lambda)$ and let $\Phi(\lambda) = \sup_{x \in X} \phi(x, \lambda)$.

Lemma 2 (*Milgrom and Segal [10, Theorem 2]*). *Suppose* (a) $\phi(x, \lambda)$ *is differentiable and absolutely continuous in* λ $(\forall x)$; (b) $|\partial\phi(x, \partial\lambda)|$ *is uniformly bounded* $(\forall x)(\forall \lambda)$, *and* (c) $\hat{x}(\lambda)$ *is nonempty. Then, for any selection* $x^*(\lambda) \in \hat{x}(\lambda)$,

$$\Phi(\lambda) = \int_0^\lambda \partial\phi(x^*(\alpha), \alpha)\, \partial\lambda\, d\alpha + \Phi(0). \tag{5}$$

Letting $\lambda = \theta_i$ and $x = (\tilde{\theta}_i, \tau_i)$, we can apply Milgrom and Segal's result. To verify the conditions of Lemma 2 observe that: (a) $U_i(\theta_i, \tilde{\theta}_i, \tau_i)$ is linear in θ_i, and hence differentiable and absolutely continuous; (b) the partial derivative is bounded by 1, and (c) the set of maximisers is nonempty by Lemma 1(a). Applying the revelation principle, the optimal choice of $\tilde{\theta}_i$ is θ_i. Hence equilibrium utility is

$$V_i(\theta_i) = E_{\theta_{-i}}\left[\int_{\underline{\theta}}^{\theta_i} P_i(\alpha, \theta_{-i}) E_c\left[\delta^{\tau_i^*(\alpha, z_i(\alpha, \theta_{-i}, c^T))}\right] d\alpha\right] + V_i(\underline{\theta}). \tag{6}$$

Eq. (6) says that agent i only obtains rents when he exercises the option. Intuitively, it is only in these states where i's information is useful. Eq. (6) also has the important implication that the seller can reduce i's rents by delaying the time when he exercises the option.

We will solve the seller's problem in two stages. First, we form the relaxed problem, replacing the incentive compatibility constraint with the integral equation (6) and replacing the individual rationality constraint with $V_i(\underline{\theta}) \geqslant 0$. We then solve for the optimal solution of the relaxed problem. Second, we check that this optimal solution satisfies incentive compatibility and individual rationality. Lemma 3 provides sufficient conditions to complete this second step.

Lemma 3. *The mechanism* $\langle P_i, y_i, z_i \rangle$ *satisfies incentive compatibility* (3) *and individual rationality* (4) *if the following three conditions hold*:

(a) *Equilibrium utility* $V_i(\theta_i)$ *satisfies* (6);
(b) *The lowest type has positive utility*, $V_i(\underline{\theta}) \geqslant 0$; *and*
(c) *The monotonicity condition holds. That is*,

$$\frac{\partial}{\partial \theta_i} U_i(\theta_i, \tilde{\theta}_i, \tau_i^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, c^T))) = E_{\theta_{-i}}\left[P_i(\tilde{\theta}_i, \theta_{-i}) E_c\left[\delta^{\tau_i^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, c^T))}\right]\right] \tag{7}$$

is increasing in $\tilde{\theta}_i$.

Proof. To verify incentive compatibility (3), pick $(\theta_i, \tilde{\theta}_i)$, where $\theta_i > \tilde{\theta}_i$ wlog.

$$\begin{aligned} U_i(\theta_i, \tilde{\theta}_i, \tau_i^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, c^T))) &= V(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\theta_i} \frac{\partial}{\partial \theta_i} U_i(\alpha, \tilde{\theta}_i, \tau_i^*(\alpha, z_i(\tilde{\theta}_i, \theta_{-i}, c^T)))\, d\alpha \\ &\leqslant V(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\theta_i} \frac{\partial}{\partial \theta_i} U_i(\alpha, \alpha, \tau_i^*(\alpha, z_i(\alpha, \theta_{-i}, c^T)))\, d\alpha \\ &= V(\theta_i), \end{aligned}$$

where the inequality comes from the monotonicity condition (7) and the last line from (6). To verify individual rationality (4), note that (6) implies $V_i(\theta_i)$ is increasing in θ_i. Hence $V_i(\underline{\theta}) \geqslant 0$ implies $V_i(\theta_i) \geqslant 0$ $(\forall \theta_i)$. □

The monotonicity condition is satisfied if reporting a higher type increases the probability of winning the object and means the option is exercised sooner. The latter aspect of the monotonicity condition might be very complicated in general, but will be simple to verify in the optimal mechanism.

Taking expectations over i's type, θ_i, and integrating (6) by parts yields ex-ante utility

$$E_{\theta_i}[V_i(\theta_i)] = E_\theta \left[P_i(\theta) E_c \left[\delta^{\tau_i^*(\theta_i, z_i(\theta_i, \theta_{-i}, c^T))} \right] \frac{1 - F(\theta_i)}{f(\theta_i)} \right] + V_i(\underline{\theta}). \tag{8}$$

4.2. Welfare maximisation

Define welfare as the sum of the agents' utilities and the seller's revenue. This equals the expected value of the option

$$\text{Welfare} = E_\theta \left[\sum_i P_i(\theta)\, E_c \left[(\theta_i - c_{i,\tau_i^*}) \delta^{\tau_i^*} \right] + \left(1 - \sum_i P_i(\theta) \right) v_0 \right]. \tag{9}$$

The welfare maximisation problem is to choose $\langle P_i, y_i, z_i \rangle$ to maximise (9) subject to incentive compatibility (3) and individual rationality (4), where the agent chooses $\tau_i^*(\theta_i, z_i(\theta, c^T))$ to maximise ex-post utility (1).

Theorem 1. *Welfare is maximised by the following mechanism*:

(a) *The contingent payment is* $z_{i,t}^W(\theta, c^t) = 0$.
(b) *The good is allocated to the agent with the largest ex-post utility* $u_i(\theta_i, z_i^W, \tau_i^*)$, *if greater than* v_0. *Otherwise the good is not awarded.*
(c) *The up-front payment* $y_i^W(\theta)$ *is such that, when* $\tilde{\theta}_i = \theta_i$ *and* $\tau_i = \tau_i^*$, *interim utility* (2) *equals equilibrium utility* (6) *and* $V_i(\underline{\theta}) \geqslant 0$.

Proof. Same as proof of Theorem 2. □

Contingent payments distort the winning agent's optimal stopping problem, delaying the exercise time. Hence the welfare-maximising auction sets contingent payments to zero and awards the object to the agent with the highest ex-post utility.

To implement the welfare-maximising auction, the seller needs to allocate the good to the agent with the highest ex-post utility. The seller can thus use an English or second-price auction. If costs

obey (SYM), the seller can use any auction that allocates the good to the agent with the highest type, such as a first-price auction. [6]

4.3. Revenue maximisation

Expected revenue equals welfare (9) minus agents' utility (8),

$$\begin{aligned}\text{Revenue} &= E_\theta\left[\sum_i P_i(\theta)\, E_c\left[(\theta_i - c_{i,\tau_i^*})\delta^{\tau_i^*}\right] + \left(1 - \sum_i P_i(\theta)\right) v_0 - \sum_i V_i(\theta_i)\right] \\ &= E_\theta\left[\sum_i P_i(\theta)\, E_c\left[(MR(\theta_i) - c_{i,\tau_i^*})\delta^{\tau_i^*}\right] + \left(1 - \sum_i P_i(\theta)\right) v_0\right] - \sum_i V_i(\underline{\theta}),\end{aligned} \tag{10}$$

where we follow Bulow and Roberts [2] in denoting agent i's marginal revenue by

$$MR(\theta_i) := \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}.$$

The revenue maximisation problem is to choose $\langle P_i, y_i, z_i\rangle$ to maximise revenue (10) subject to incentive compatibility (3) and individual rationality (4), where the agent chooses $\tau_i^*(\theta_i, z_i(\theta, c^T))$ to maximise ex-post utility (1).

Assumption (MH). The inverse hazard rate $[1 - F(\theta_i)]/f(\theta_i)$ is decreasing in θ_i.

Theorem 2. *Suppose* (*MH*) *holds. Revenue is maximised by the following mechanism*:

(a) *The contingent payment is* $z_{i,t}^R(\theta, c^t) = [1 - F(\theta_i)]/f(\theta_i)$.
(b) *The good is allocated to the agent with the largest ex-post utility* $u_i(\theta_i, z_i^R, \tau_i^*)$, *if greater than* v_0. *Otherwise the good is not awarded.*
(c) *The up-front payment* $y_i^R(\theta)$ *is such that, when* $\tilde{\theta}_i = \theta_i$ *and* $\tau_i = \tau_i^*$, *interim utility* (2) *equals equilibrium utility* (6) *and* $V_i(\underline{\theta}) = 0$.

Proof. Consider the seller's relaxed problem of choosing $\langle P_i, y_i, z_i\rangle$ to maximise revenue (10) subject to $\tau_i^*(\theta_i, z_i(\theta, c^T))$ maximising ex-post utility (1), the integral equation (6) and $V_i(\underline{\theta}) \geqslant 0$. The optimal relaxed mechanism is characterised as follows.

(a) *Contingent payment*: The contingent payment only enters revenue (10) via the agent's stopping rule $\tau_i^*(\theta_i, z_i(\theta, c^T))$. Hence it is optimal to set $z_{i,t}(\theta, c^t) = [1 - F(\theta_i)]/f(\theta_i)$, since the agent's choice of stopping rule, chosen to maximise ex-post utility (1), also maximises revenue (10).

(b) *Allocation*: Setting $z_{i,t}(\theta, c^t) = [1 - F(\theta_i)]/f(\theta_i)$, revenue (10) can be written as

$$E_\theta\left[\sum_i P_i(\theta) u_i\left(\theta_i, z_i^R(\theta, c^T), \tau_i^*(\theta_i, z_i^R(\theta, c^T))\right) + \left(1 - \sum_i P_i(\theta)\right) v_0\right] - \sum_i V_i(\underline{\theta}).$$

[6] In a second-price auction, it is a dominant strategy for each bidder i to bid his ex-post utility $u_i(\theta_i, 0, \tau_i^*)$, by Krishna [9, Proposition 2.1]. If costs obey (SYM), then each agents' ex-post utility $u_i(\theta_i, 0, \tau_i^*)$ is increasing in θ_i and symmetrically distributed. An increasing symmetric first-price equilibrium then exists by Krishna [9, Proposition 2.2].

The revenue-maximising mechanism will thus award the good to the agent with the highest ex-post utility, subject to it exceeding v_0.

(c) The up-front payment. Revenue (10) is decreasing in $V_i(\underline{\theta})$, so the seller should set $V_i(\underline{\theta}) = 0$. The up-front payment does not enter revenue (10) directly, so the seller can then choose $y_i(\theta)$ so that equilibrium utility is given by (6) and $V_i(\underline{\theta}) = 0$.

We must now verify that the relaxed mechanism does indeed satisfy incentive compatibility and individual rationality. Parts (a) and (b) of Lemma 3 are satisfied by construction. To verify the monotonicity condition (7) we break the analysis into two parts.

First, we claim that $P_i(\tilde{\theta}_i, \theta_{-i})$ is increasing in $\tilde{\theta}_i$. The auction is awarded to the agent with the highest ex-post utility. Given reports $(\tilde{\theta}_i, \theta_{-i})$, the seller calculates i's ex-post utility to be

$$u_i\left(\tilde{\theta}_i, z_i^R(\tilde{\theta}_i, \theta_{-i}, c^T), \tau_i^*(\tilde{\theta}_i, z_i^R(\tilde{\theta}_i, \theta_{-i}, c^T))\right) = \sup_{\tau_i} E_c[(MR(\tilde{\theta}_i) - c_{i,\tau_i})\delta^{\tau_i}]. \tag{11}$$

Assumption (MH) implies $MR(\tilde{\theta}_i)$ is increasing in $\tilde{\theta}_i$, so $E_c[(MR(\tilde{\theta}_i) - c_{i,\tau_i})\delta^{\tau_i}]$ is increasing in $\tilde{\theta}_i$. By the envelope theorem, ex-post utility (11) is also increasing in $\tilde{\theta}_i$, and thus $P_i(\tilde{\theta}_i, \theta_{-i})$ is increasing in $\tilde{\theta}_i$, as required.

Second, we claim that $\tau_i^*(\theta_i, z_i^R(\tilde{\theta}_i, \theta_{-i}, c^T))$ is decreasing in $\tilde{\theta}_i$. Applying (MH), an increase in $\tilde{\theta}_i$ leads to a decrease in $z_{i,t}^R(\tilde{\theta}_i, \theta_{-i}, c^t)$ that is uniform across time. Lemma 1(c) implies that $\tau_i^*(\theta_i, z_i^R(\tilde{\theta}_i, \theta_{-i}, c^T))$ then decreases. Putting the two claims together shows that (7) is increasing in $\tilde{\theta}_i$, as required. □

Theorem 2 states that the optimal contract can be separated into two parts. First, the seller awards the option to the agent with the highest ex-post utility (1). Second, the seller implements the optimal single-bidder contract, where the contingent payment is set equal to the winning agent's information rent term. Notably, the optimal contingent payment is declining in the winner's type, independent of other bidder's types, independent of the time of execution, and independent of post-auction costs.

The revenue-maximising mechanism works in a simple way. Information rents drive a wedge between welfare and revenue. The winning bidder would like to choose the exercise time to maximise his valuation minus the contingent payment. In comparison, the seller would like to choose the exercise time to maximise the winner's valuation minus his information rent. Setting the contingent payment equal to the winner's information rent thus aligns incentives in the style of a Pigouvian tax. Under the optimal contingent payment, the winning agent then chooses his exercise time to maximise the seller's revenue—an act of perfect delegation.

As stated above, the optimal contingent payment is independent of ex-post costs. This has two important implications. First, the seller can implement the optimal auction even when she does not observe cost data. Second, the seller will always release any information about future costs, whether or not she can observe the information she is releasing. Intuitively, under the revenue-maximising auction, the winning agent chooses his stopping time to maximise the seller's revenue, so giving the agent more information makes the seller better off. [7]

The revenue-maximising auction introduces two welfare distortions. First, the strike price is too high. Second, the good may be allocated to the wrong agent (if agents are asymmetric) or not awarded at all. This implies that market power induces the following dynamic distortion.

[7] See Eso and Szentes [5] for a related result.

Corollary 1. *Suppose* (*MH*) *and* (*SYM*) *hold. Then the option will be exercised earlier under the welfare-maximising auction* (*Theorem* 1) *than under the revenue-maximising auction* (*Theorem* 2). [8]

Proof. Suppose the good is awarded under the revenue-maximising auction (else there is nothing to prove). Since $z_{i,t}^R(\theta, c^t) \geqslant 0$, the good is also awarded under the welfare-maximising auction and, using (SYM), the agent with the highest type will win both auctions. Lemma 1(c) then implies that, for any optimal exercise rule, the winning agent will exercise later under the revenue-maximising auction. □

To implement the revenue-maximising mechanism, the seller needs to: (i) award the good to the agent with the highest ex-post utility; (ii) ensure the lowest type gets no rents and (iii) be able to back-out the winner's type in order to calculate the correct contingent payment. If costs obey (SYM), these objectives can be achieved by a first-price or second-price auction. First, consider the first-price auction. Assuming agents use an increasing symmetric bidding strategy, $b(\theta_i)$, Theorem 2(c) implies [9]

$$b(\theta_i) = E_c\left[\left(\theta_i - z^R(\theta_i) - c_{i,\tau_i^*(\theta_i, z^R(\theta_i))}\right)\delta^{\tau_i^*(\theta_i, z^R(\theta_i))}\right] - \frac{\int_{\underline{\theta}}^{\theta_i} E_c[\delta^{\tau_i^*(\alpha, z^R(\alpha))}]F^{N-1}(\alpha)\,d\alpha}{F^{N-1}(\theta_i)},$$

where $z^R(\theta_i) = [1 - F(\theta_i)]/f(\theta_i)$. Under (MH) one can verify that $b(\theta_i)$ is increasing, as initially assumed. Since $b(\theta_i)$ is increasing, a higher up-front bid leads to a lower contingent fee, as shown in Fig. 2. [10]

Next, consider the second-price auction. Assuming agents use an increasing symmetric bidding strategy, $B(\theta_i)$, Theorem 2(c) implies

$$B(\theta_i) = E_c\left[\left(\theta_i - z^R(\theta_i) - c_{i,\tau_i^*(\theta_i, z_i^R(\theta_i))}\right)\delta^{\tau_i^*(\theta_i, z_i^R(\theta_i))}\right] - E_c\left[\delta^{\tau_i^*(\theta_i, z^R(\theta_i))}\right]\frac{dz^R(\theta_i)}{d\theta_i}\frac{F(\theta_i)}{(N-1)f(\theta_i)}.$$

If $z^R(\theta_i) = [1 - F(\theta_i)]/f(\theta_i)$ is decreasing and concave, and $F(\theta_i)/f(\theta_i)$ is increasing, then the bidding strategy $B(\theta_i)$ is increasing, as initially assumed. These assumptions are satisfied if, for example, θ_i is uniformly distributed. Observe that, since $z^R(\theta_i)$ is decreasing, agents bid more than their ex-post utility: intuitively, a higher bid does not affect the up-front price paid, conditional on winning, but does reduce the contingent fee.

Finally, as an application of Theorem 2, consider the oil sands example discussed in the Introduction. Suppose the extraction cost $\theta_i \sim F(\cdot)$ differs among bidders, while the oil price p_t is common to all bidders and varies over time. If the seller uses a mechanism $\langle P_i, y_i, z_i\rangle$ then agent i's ex-post utility is $u_i(\theta_i, z_i, \tau_i) = E_p[(p_{\tau_i} - \theta_i - z_{i,\tau_i})\delta^{\tau_i}]$ if he uses exercise strategy τ_i.

[8] By convention, we say the option is exercised at time $t = \infty$ if the good is not awarded.

[9] The strategy satisfies the assumptions of Lemma 3, by construction, and thus constitutes an equilibrium.

[10] Fig. 2 shows the revenue-maximising first-price bidding locus for a European option ($T = 1$) where $c_i, \theta_i \sim U[0, 1]$, $v_0 = 0$, and $\delta = 1$. In this example, the contingent payment is substantial and exceeds the up-front payment for many types of agents.

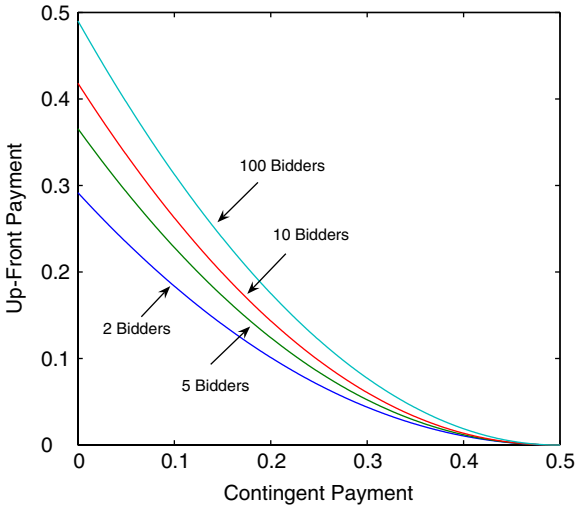

Fig. 2. First-price bidding locus for a European option ($T = 1$).

This is relabelling of (1), so the revenue-maximising contingent payment is [11]

$$z_{i,t}^{R}(\theta, p^t) = \frac{F(\theta_i)}{f(\theta_i)}. \tag{12}$$

In comparison, the OCS wildcat auctions use a contingent payment which equals 1/6th the agent's revenue, i.e. $z_{i,t} = p_t/6$ [11]. Lemma 1(c) thus implies that, under the OCS payment scheme, bidders will be mining inefficiently late when their costs are low or the oil price is high. [12,13]

5. State-dependent allocation

In Sections 2–4, the seller awards the option to an agent who subsequently chooses an exercise time. This mechanism has one notable restriction: the allocation is chosen at time $t = 0$ and does not depend on post-auction information.

This section extends the model to allow the allocation decision to depend on agents' costs. This extension is important if post-auction information affects different bidders in different ways. Section 5.1 describes the revenue-maximising state-dependent mechanism when costs are observed by the seller. Section 5.2 then describes how the seller can implement this revenue-maximising mechanism if she cannot observe agents' costs. This latter step relies on a dynamic variant of the Vickrey–Clarke–Groves (VCG) mechanism which, as far as I am aware, is original to this paper.

[11] Eq. (12) differs from that in Theorem 2 because, in the oil auction, utility is decreasing in θ_i.

[12] For the Alberta oil sands, the Alberta government currently uses a royalty-only scheme: they charge 1% of gross revenues before a project has recovered its fixed costs, and 25% of net revenues thereafter. This scheme has two problems: first, it assumes firms truthfully report their costs; second, it may not attract the most efficient firms (see [6,13]).

[13] This is clearly a highly stylised model of an oil auction, so caution is warranted. First, the bidders may have different views about how much oil is in the ground. Second, the extraction decision is more complex than a simple option, involving many investment decisions over a period of time.

5.1. State-dependent allocation with observable costs

Consider the state-dependent mechanism $\langle \sigma_i, x_i \rangle$ which works as follows:

- At time $t = 0$, agent i reports $\tilde{\theta}_i$ and pays $x_i(\tilde{\theta})$.
- At time $t \geqslant 1$, the seller observes the evolution of costs c^t. She then allocates the object to agent i at time $\sigma_i(\tilde{\theta}, c^T)$, where we say $\sigma_i(\tilde{\theta}, c^T) = \infty$ if i is never allocated the good. The exercise function $\sigma_i(\tilde{\theta}, c^T)$ is a random variable, where the decision for i to exercise at time t can only depend on the information available at time t.[14] There is only one object, so $\sum_i \mathbf{1}_{\sigma_i(\tilde{\theta}, c^T) < \infty} \leqslant 1$.

For simplicity, we suppose that the seller has no use for the good. This means we do not have to worry about the seller awarding the object to herself at some period t. This assumption implies that $v_0 = 0$.

In the Bayesian Nash equilibrium, agent i chooses his report $\tilde{\theta}_i$ to maximise interim utility

$$U_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}, c}\left[\left(\theta_i - c_{i, \sigma_i(\tilde{\theta}_i, \theta_{-i}, c^T)}\right) \delta^{\sigma_i(\tilde{\theta}_i, \theta_{-i}, c^T)} - x_i(\tilde{\theta}_i, \theta_{-i})\right]. \tag{13}$$

Denoting the equilibrium utility by $V(\theta_i) = U_i(\theta_i, \theta_i)$, the envelope theorem implies

$$V_i(\theta_i) = E_{\theta_{-i}, c}\left[\int_{\underline{\theta}}^{\theta_i} \delta^{\sigma_i(\alpha, \theta_{-i}, c^T)}\, d\alpha\right] + V_i(\underline{\theta}). \tag{14}$$

Lemma 4. *The mechanism $\langle \sigma_i, x_i \rangle$ satisfies incentive compatibility and individual rationality if the following three conditions hold*:

(a) *Equilibrium utility $V_i(\theta_i)$ satisfies* (14);
(b) *The lowest type has positive utility*, $V_i(\underline{\theta}) \geqslant 0$; *and*
(c) *The monotonicity condition holds. That is*

$$\frac{\partial}{\partial \theta_i} U_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}, c}\left[\delta^{\sigma_i(\tilde{\theta}_i, \theta_{-i}, c^T)}\right] \tag{15}$$

is increasing in $\tilde{\theta}_i$.

Proof. Same as Lemma 3. □

Welfare is the sum of the agents' utilities and the seller's revenue,

$$\text{Welfare} = \sum_i E_{\theta, c}\left[\left(\theta_i - c_{i, \sigma_i(\theta, c^T)}\right) \delta^{\sigma_i(\theta, c^T)}\right]. \tag{16}$$

As in Section 4.3, revenue equals welfare (16) minus expected rents,

$$\text{Revenue} = \sum_i E_{\theta, c}\left[\left(MR(\theta_i) - c_{i, \sigma_i(\theta, c^T)}\right) \delta^{\sigma_i(\theta, c^T)}\right] - \sum_i V_i(\underline{\theta}). \tag{17}$$

The revenue maximisation problem is to choose $\langle \sigma_i, x_i \rangle$ to maximise revenue (17) subject to incentive compatibility, individual rationality and $\sum_i \mathbf{1}_{\sigma_i(\theta, c^T) < \infty} \leqslant 1$. As in Section 4, we solve

[14] That is, $\sigma_i(\tilde{\theta}, c^T)$ is a stopping time.

the problem in two steps. First, we form the relaxed problem, replacing the incentive compatibility constraint with (14) and replacing the individual rationality constraint with $V_i(\underline{\theta}) \geqslant 0$. Second, we verify the solution to the relaxed problem satisfies conditions (a)–(c) in Lemma 4, and thus solves the full problem.

Theorem 3. *Suppose* (*MH*) *holds. Among the class of state-dependent mechanisms* $\langle \sigma_i, x_i \rangle$, *revenue is maximised by the following*:

(a) *The exercise rule* $\sigma^R(\theta, c^T)$ *is the earliest rule to maximise revenue* (17) *subject to* $\sum_i \mathbf{1}_{\sigma_i(\theta,c^T)<\infty} \leqslant 1$.

(b) *The payment* $x_i^R(\theta)$ *is such that, when* $\tilde{\theta}_i = \theta_i$, *interim utility* (13) *equals equilibrium utility* (14) *and* $V_i(\underline{\theta}) = 0$.

Proof. Let us first consider the relaxed problem, ignoring the monotonicity condition (15).

(a) *Exercise rule*: Let $\sigma^R(\theta, c^T) = \{\sigma_i^R(\theta, c^T)\}_i$ be the earliest exercise-rule that maximises revenue (17) subject to $\sum_i \mathbf{1}_{\sigma_i(\theta,c^T)<\infty} \leqslant 1$. Such a stopping rule exists, is unique and is characterised by the rule: stop when current utility is weakly greater than the continuation utility [8, Theorems 1,3 and 6].

(b) *Payment*: The payment $x_i(\theta)$ does not enter revenue (17) directly. Hence, the seller can choose $x_i(\theta)$ such that $V_i(\underline{\theta}) = 0$ and equilibrium utility is given by (14).

Next, we must verify the monotonicity condition (15) in order to show the mechanism is incentive compatible. Denote the time-t continuation value of the revenue maximisation problem when each agent reports $\tilde{\theta}_i$ by

$$\Psi_t(\tilde{\theta}) := \max_{\sigma \geqslant t+1} E_t \left[\sum_i \left(MR(\tilde{\theta}_i) - c_{i,\sigma_i} \right) \delta^{\sigma_i - t} \right] \quad \text{s.t.} \sum_i \mathbf{1}_{\sigma_i<\infty} \leqslant 1.$$

Suppose that i reports θ_i^L and $j \neq i$ report truthfully. In addition suppose that, with positive probability, i is awarded the good at time t. Then the following three facts must hold:

(i) The good is not awarded to $j \neq i$ in period $s < t$, $MR(\theta_j) - c_{j,s} < \Psi_s(\theta_i^L, \theta_{-i})$. The strict inequality comes from the fact that σ^R is the earliest optimal stopping time.
(ii) In period t, the good is awarded to i rather than j, $MR(\theta_i^L) - c_{i,t} \geqslant MR(\theta_j) - c_{j,t}$.
(iii) The good is awarded in period t rather than period $s > t$, $MR(\theta_i^L) - c_{i,t} \geqslant \Psi_t(\theta_i^L, \theta_{-i})$.

Next, suppose i reports a higher valuation, $\theta_i^H > \theta_i^L$. It must be that:

(I) The good is not awarded to $j \neq i$ in period $s < t$, $MR(\theta_j) - c_{j,s} < \Psi_s(\theta_i^H, \theta_{-i})$. This follows from (i) and the fact that $\Psi_s(\theta_i^H, \theta_{-i}) \geqslant \Psi_s(\theta_i^L, \theta_{-i})$, for $s < t$.
(II) If the good is awarded in period t, then it is awarded to bidder i rather than bidder j, $MR(\theta_i^H) - c_{i,t} > MR(\theta_j) - c_{j,t}$. This follows from (ii).
(III) If the good has not been awarded prior to period t, then it will be awarded in period t, $MR(\theta_i^H) - c_{i,t} > \Psi_t(\theta_i^H, \theta_{-i})$. To see this observe

$$\Psi_t(\theta_i^H, \theta_{-i}) = E_t \left[\left(MR(\theta_i^H) - c_{i,\sigma_i^H} \right) \delta^{\sigma_i^H - t} + \sum_{j \neq i} \left(MR(\theta_j) - c_{j,\sigma_j^H} \right) \delta^{\sigma_j^H - t} \right],$$

$$\Psi_t(\theta_i^L, \theta_{-i}) \geqslant E_t \left[\left(MR(\theta_i^L) - c_{i,\sigma_i^H} \right) \delta^{\sigma_i^H - t} + \sum_{j \neq i} \left(MR(\theta_j) - c_{j,\sigma_j^H} \right) \delta^{\sigma_j^H - t} \right], \tag{18}$$

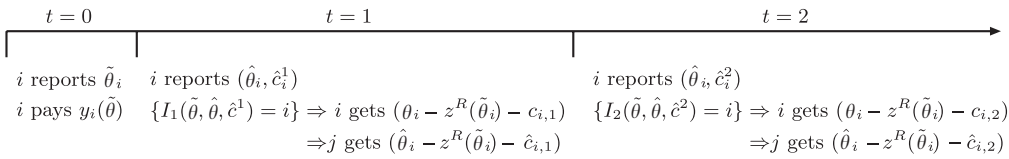

Fig. 3. Timeline.

where σ^H maximises $\Psi_t(\theta_i^H, \theta_{-i})$ subject to $\sum_i \mathbf{1}_{\sigma_i < \infty} \leqslant 1$. Hence

$$\Psi_t(\theta_i^H, \theta_{-i}) - \Psi_t(\theta_i^L, \theta_{-i}) \leqslant \left[MR(\theta_i^H) - MR(\theta_i^L)\right] \delta^{\sigma_i^H - t} < MR(\theta_i^H) - MR(\theta_i^L), \tag{19}$$

where the first inequality comes from (18), and the second comes from $\delta < 1$. The result then follows from (19) and (iii).

Putting (I)–(III) together, the good is thus awarded to agent i at or before period t. Thus, the exercise rule has the property that $\sigma_i^R(\theta_i^H, \theta_{-i}, c^T) \leqslant \sigma_i^R(\theta_i^L, \theta_{-i}, c^T)$, and satisfies the monotonicity condition (15). □

Theorem 3 characterises the seller's revenue-maximising exercise rule as the solution to a dynamic programming problem. If the same agent is allocated the good for all sequences of costs, such as in the oil example, this can be implemented by the auction in Theorem 2.

The revenue-maximising exercise rule $\sigma^R(\theta, c^T)$ introduces two kinds of distortions relative to the exercise rule that maximises welfare (16). First, the object will tend to be exercised inefficiently late, as in Corollary 1.[15] Second, under (MH), allocation is biased towards agents with higher ex-ante valuations.[16] Intuitively, biasing the allocation towards high types reduces the incentive for an agent to under-report his type. While these intertemporal and intratemporal distortions seem to be quite general, one can construct examples where these effects interact and cancel each other out.

5.2. *State dependent allocation with unobservable costs*

This section shows that the seller can obtain the revenue from Theorem 3 even if she cannot observe post-auction information. This mechanism is an extension of the handicap auction of Eso and Szentes [5].

Consider the four-part mechanism $\langle I, w_i, y_i, z_i \rangle$ consisting of an allocation function and three payments, as summarised in Fig. 3. This mechanism works as follows.

- At time $t = 0$, each agent i publicly reports his type $\tilde{\theta}_i$. The seller chooses the up-front payment $y_i(\tilde{\theta})$ and the contingent payment $z(\tilde{\theta}_i) = [1 - F(\tilde{\theta}_i)]/f(\tilde{\theta}_i)$.
- At time $t \geqslant 1$, each agent i again publicly reports his type $\hat{\theta}_i$ along with his history of costs, $\hat{c}_i^t$. These reports may differ from previous reports. The seller uses $\hat{c}_i^t$ to predict i's future costs and

[15] For example, if $c_{i,t} = c_t$ $(\forall i)$, then Corollary 1 applies directly.

[16] For example, suppose $T = 1$ and $\theta_i > \theta_j$. Then the revenue-maximising rule allocates the object to the low agent, $MR(\theta_j) - c_{j,1} \geqslant MR(\theta_i) - c_{i,1}$, only if the welfare-maximising rule also allocates it to the low agent, $\theta_j - c_{j,1} \geqslant \theta_i - c_{i,1}$.

decides whether to award the good to one of the agents. Denote this time-t allocation decision by $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) \in \{0, 1, \ldots, N\}$, where $I_t = 0$ means the good is not awarded.

- If $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = i$, for $i \in \{1, \ldots, N\}$, then i obtains utility $\theta_i - c_{i,t}$ and pays contingent fee $z(\tilde{\theta}_i)$. Each agent $j \neq i$ also receives a VCG award $w_{j,t}(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = \hat{\theta}_i - z(\tilde{\theta}_i) - \hat{c}_{i,t}$, equal to i's reported ex-post utility. The game then ends.
- If $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = 0$, then the seller forgets the reports made in period t and the game proceeds to period $t + 1$.

The allocation decision $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t)$ is chosen as follows. Define the time-t continuation value by

$$\Psi_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) := \max_{\sigma \geqslant t+1} E\left[\sum_i (\hat{\theta}_i - z(\tilde{\theta}_i) - \hat{c}_{i,\sigma_i})\delta^{\sigma_i - t} \middle| \hat{c}^t\right] \quad \text{s.t.} \sum_i \mathbf{1}_{\sigma_i < \infty} \leqslant 1. \tag{20}$$

The seller's allocation decision is then given by

$$\begin{aligned} I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = i \quad & \text{if } \hat{\theta}_i - z(\tilde{\theta}_i) - \hat{c}_{i,t} \geqslant \hat{\theta}_j - z(\tilde{\theta}_j) - \hat{c}_{j,t} \\ & \text{and } \hat{\theta}_i - z(\tilde{\theta}_i) - \hat{c}_{i,t} \geqslant \Psi_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t), \\ I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = 0 \quad & \text{otherwise.} \end{aligned} \tag{21}$$

The allocation decision thus maximises the sum of agents' utilities. With the correct contingent payment, this will also maximise revenue. First, Lemma 5 shows that, given this allocation rule, the VCG payments ensure that agents tell the truth in each period, $t \geqslant 1$.

Lemma 5. *Take the initial reports $\tilde{\theta}$ as given. It is a sequential equilibrium for each agent to truthfully report his information $(\hat{\theta}_i, \hat{c}_i^t) = (\theta_i, c_i^t)$ in each period, $t \geqslant 1$.*

Proof. Suppose we are in period $t \geqslant 1$ and the good has not yet been awarded. Suppose also that agents are playing according to the posited (truthful) strategies and agent i considers a one-period deviation. That is, i reports $(\hat{\theta}_i, \hat{c}_i^t)$ in period t, before returning to the truthtelling strategy in period $t + 1$.

$$\text{Agent } i\text{'s deviation payoff} = \begin{cases} \theta_i - z(\tilde{\theta}_i) - c_{i,t} & \text{if } I_t(\tilde{\theta}, \hat{\theta}_i, \theta_{-i}, \hat{c}_i^t, c_{-i}^t) = i, \\ \theta_j - z(\tilde{\theta}_j) - c_{j,t} & \text{if } I_t(\tilde{\theta}, \hat{\theta}_i, \theta_{-i}, \hat{c}_i^t, c_{-i}^t) = j, \\ \Psi_t(\tilde{\theta}, \theta, c^t) & \text{if } I_t(\tilde{\theta}, \hat{\theta}_i, \theta_{-i}, \hat{c}_i^t, c_{-i}^t) = 0. \end{cases}$$

Using the definition of $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t)$, it is therefore an ex-post best-response for i to report $(\hat{\theta}_i, \hat{c}_i^t)$ truthfully. Hence, truthtelling is a best-response for any beliefs i holds about his opponents' types and costs, and the posited strategies form a sequential equilibrium by the one-deviation principle [7]. □

Given Lemma 5, we can represent the equilibrium allocation strategies $I_t(\tilde{\theta}, \theta, c^t)$ by the corresponding exercise times, $\sigma_i(\tilde{\theta}, \theta, c^T) = \min\{t : I_t(\tilde{\theta}, \theta, c^t) = i\}$. By construction, this exercise time $\sigma(\tilde{\theta}, \theta, c^T)$ maximises the expected sum of ex-post utilities,

$$E_c\left[\sum_i \left(\theta_i - z(\tilde{\theta}_i) - c_{i,\sigma_i \tilde{\theta}, \theta, c^T}\right) \delta^{\sigma_i \tilde{\theta}, \theta, c^T}\right] \tag{22}$$

subject to $\sum_i \mathbf{1}_{\sigma_i(\tilde{\theta},\theta,c^T)<\infty} \leqslant 1$. Turning to period $t=0$, suppose i reports $\tilde{\theta}_i$ while agents $-i$ are truthful. Agent i's interim utility is

$$U_i(\theta_i,\tilde{\theta}_i) = E_{\theta_{-i},c}\left[\left(\theta_i - z(\tilde{\theta}_i) - c_{i,\sigma_i(\tilde{\theta}_i,\theta_{-i},\theta,c^T)}\right)\delta^{\sigma_i(\tilde{\theta}_i,\theta_{-i},\theta,c^T)} - y(\tilde{\theta}_i,\theta_{-i})\right]$$
$$+E_{\theta_{-i},c}\left[\sum_{j\neq i}\left(\theta_j - z(\theta_j) - c_{j,\sigma_j(\tilde{\theta}_i,\theta_{-i},\theta,c^T)}\right)\delta^{\sigma_j(\tilde{\theta}_i,\theta_{-i},\theta,c^T)}\right], \tag{23}$$

where the last term comes from i's VCG award. Using the fact $\sigma(\tilde{\theta},\theta,c^T)$ maximises (22), Eq. (23) can be rewritten as

$$U_i(\theta_i,\tilde{\theta}_i) = \max_{\sigma} E_{\theta_{-i},c}\left[\left(\theta_i - z(\tilde{\theta}_i) - c_{i,\sigma_i}\right)\delta^{\sigma_i}\right.$$
$$\left.+\sum_{j\neq i}\left(\theta_j - z(\theta_j) - c_{j,\sigma_j}\right)\delta^{\sigma_j} - y(\tilde{\theta}_i,\theta_{-i})\right] \tag{24}$$

subject to $\sum_i \mathbf{1}_{\sigma_i<\infty} \leqslant 1$. To examine how a change in θ_i affects equilibrium utility, $V(\theta_i) := U(\theta_i,\theta_i)$, we will apply an envelope theorem to (24). Since both $\tilde{\theta}_i$ and σ are chosen to maximise $U_i(\theta_i,\tilde{\theta}_i)$, we again use Milgrom and Segal [10, Theorem 2]. This yields

$$V_i(\theta_i) = E_{\theta_{-i},c}\left[\int_{\underline{\theta}}^{\theta_i} \delta^{\sigma_i(\alpha,\theta_{-i},\alpha,\theta_{-i},c^T)}\,d\alpha\right] + V_i(\underline{\theta}). \tag{25}$$

Lemma 6. *The mechanism $\langle I, w_i, y_i, z_i\rangle$ satisfies incentive compatibility and individual rationality if the following three conditions hold*:

(a) *Equilibrium utility $V_i(\theta_i)$ satisfies* (25);
(b) *The lowest type has positive utility, $V_i(\underline{\theta}) \geqslant 0$; and*
(c) *The monotonicity condition holds. That is,*

$$\frac{\partial}{\partial\theta_i}U_i(\theta_i,\tilde{\theta}_i) = E_{\theta_{-i},c}\left[\delta^{\sigma_i(\tilde{\theta}_i,\theta_{-i},\theta_i,\theta_{-i},c^T)}\right] \tag{26}$$

is increasing in $\tilde{\theta}_i$.

Proof. Same as proof of Lemma 3. □

As in Eq. (17), revenue equals welfare minus expected rents,

$$\text{Revenue} = \sum_i E_{\theta,c}\left[\left(MR(\theta_i) - c_{i,\sigma_i(\theta,\theta,c^T)}\right)\delta^{\sigma_i(\theta,\theta,c^T)}\right] - \sum_i V_i(\underline{\theta}). \tag{27}$$

Theorem 4. *Suppose* (*MH*) *holds. The seller can obtain the same revenue as when costs are observable* (*Theorem* 3) *by using the mechanism $\langle I, w_i, y_i, z_i\rangle$, where*

(a) *The contingent payment is $z(\tilde{\theta}_i) = [1 - F(\tilde{\theta}_i)]/f(\tilde{\theta}_i)$.*
(b) *The allocation rule $I_t(\tilde{\theta},\hat{\theta},c^T)$ is set according to* (21).

(c) *The VCG payment is* $w_{i,t}(\tilde{\theta}, \hat{\theta}, c^T) = \hat{\theta}_j - z(\tilde{\theta}_j) - \hat{c}_{j,t}$ *when* $I_t(\tilde{\theta}, \hat{\theta}, \hat{c}^t) = j$ *and* $j \neq i$.

(d) *The up-front payment* $y_i(\tilde{\theta})$ *is such that, when* $\tilde{\theta}_i = \theta_i$, *interim utility* (23) *equals equilibrium utility* (25) *and* $V_i(\underline{\theta}) = 0$.

Proof. Suppose agents tell the truth each period, $t \geqslant 0$. The mechanism $\langle I, w_i, y_i, z_i \rangle$ has two properties. First, the lowest type obtains no rents, $V_i(\underline{\theta}) = 0$. Second, given the contingent fee $z(\theta_i) = [1 - F(\theta_i)]/f(\theta_i)$, the allocation $I_t(\theta, \theta, c^T)$ induces a stopping time $\sigma(\theta, \theta, c^T)$ that maximises revenue (27) s.t. $\sum_i \mathbf{1}_{\sigma_i(\theta,\theta,c^T)<\infty} \leqslant 1$. Hence, revenue equals that in Theorem 3.

We must now check agents tell the truth in each period and that the mechanism is individually rational. At time $t \geqslant 1$, truthtelling is a sequential equilibrium by Lemma 5, for any time 0 reports. At time $t = 0$, we must verify conditions (a)–(c) in Lemma 6. Conditions (a)–(b) are satisfied by choice of the up-front payment $y_i(\tilde{\theta})$. To verify the monotonicity condition (26) one can use the same approach as in Theorem 3, using the fact that $z(\tilde{\theta}_i)$ is decreasing in $\tilde{\theta}_i$. □

Intuitively, the seller first chooses the up-front payment so that agent i truthfully reveals his type, and assigns i a contingent payment equal to his information rent. Once this contingent payment has been fixed, the agent acts as if his valuation is $MR(\theta_i)$, so his incentives coincide with those of the seller. That is, the exercise time that maximises the seller's revenue also maximises the sum of agents' ex-post utilities (22). This property means that, at time $t \geqslant 1$, the seller can use a VCG mechanism to elicit cost information.

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References

[1] D.P. Baron, D. Besanko, Regulation and information in a continuing relationship, Info. Econ. Pol. 1 (3) (1984) 267–302.
[2] J. Bulow, J. Roberts, The simple economics of optimal auctions, J. Polit. Econ. 97 (5) (1989) 1060–1090.
[3] Y.S. Chow, H. Robbins, D. Siegmund, Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin, Boston, 1971.
[4] P. Courty, H. Li, Sequential screening, Rev. Econ. Stud. 67 (4) (2000) 697–717.
[5] P. Eso, B. Szentes, Optimal information disclosure in auctions and the handicap auction, Rev. Econ. Stud., forthcoming.
[6] R.G. Hansen, Auctions with contingent payments, Amer. Econ. Rev. 75 (4) (1985) 862–865.
[7] E. Hendon, H.J. Jacobsen, B. Sloth, The one-shot-deviation principle for sequential rationality, Games Econ. Behav. 12 (2) (1996) 274–282.
[8] M. Klass, Properties of optimal extended-valued stopping rules for s_n/n, Ann. of Prob. 1 (5) (1973) 719–757.
[9] V. Krishna, Auction Theory, Academic Press, San Diego, 2002.
[10] P. Milgrom, I. Segal, Envelope theorems for arbitrary choice sets, Econometrica 70 (2) (2002) 583–601.
[11] R.H. Porter, The role of information in U.S. offshore oil and gas lease auctions, Econometrica 93 (1) (1995) 1–27.
[12] M.H. Riordan, D.E.M. Sappington, Awarding monopoly franchises, Amer. Econ. Rev. 77 (3) (1987) 375–387.
[13] W. Samuelson, Auctions with contingent payments: comment, Amer. Econ. Rev. 77 (4) (1987) 740–745.
[14] D.M. Topkis, Supermodularity and Complimentarity, Princeton University Press, New Jersey, 1998.