Selling Options

Simon Board*

This Version: November 17, 2004
First Version: June 2001

Abstract

Contracts often take the form of options: deals can be reneged on, bridges may not be built, and acquired firms might go bankrupt. This paper considers the auctioning of a dynamic option, where post–auction information interacts additively with private information but is allowed to take any stochastic form. The revenue maximising auction consists of an up–front bid and a contingent payment (strike price). The contingent payment equals the opportunity cost of exercising the contract plus a rent–tax that is time– and state–invariant and inversely related to the up–front payment. The rent–tax is designed in a Pigouvian manner so that the agents' choice of exercise decision maximises the seller's revenue; it can also be interpreted as a generalised reservation price. The revenue maximising mechanism induces a dynamic distortion: the option is exercised later than under the comparable welfare maximising mechanism.

1 Introduction

This paper derives the optimal method to sell an option, a contract that when exercised yields a state–dependent payoff. The designer of such a sales procedure has a lot of flexibility. Payment can depend on whom the good is awarded to, when the option is exercised (if ever), and the information revealed after the sale. Allowing for all these possibilities, the optimal auction is derived with only an additivity restriction on the post–auction information. Moreover, this optimal contract takes a very simple form.

The model can be applied to any kind of option, whether it is physical or financial, real or surreal. After a timber auction, the winner has the right to harvest the trees at a date of their

*Department of Economics, University of Toronto. http://www.economics.utoronto.ca/board. I am grateful for comments from David Ahn, Alan Beggs, Jeremy Bulow, Peter DeMarzo, Paul Klemperer, John McMillan, Rob McMillan, Martin Osborne, José Quintero, Ilya Segal, Daniel Sgroi, Jean Tirole and Bob Wilson. JEL classification: C7, D4, D8, D9, G3.

choosing, depending upon the market price of lumber.[1] A firm that has recently been taken over may be liquidated in a sufficiently bad state of nature.[2] Hollywood studios buy options on books and make further contingent payments if they decide to make a film.[3] Oil fields are often sold in auctions yet can be abandoned if the oil price sinks low enough.[4] Options are also the canonical example of a state–dependent decision, providing a base for more complicated decision problems.

In order to fix ideas, suppose N agents compete over the right to build houses on a disused industrial site. Before the auction, the agents have private information about their revenue from house sales, which may have common value elements. After the auction, the winning agent's costs evolve as input prices change, and as they discover more about the cost of clearing up the plot. At any time after the auction the winning agent can commence construction, or they may choose to abandon the site.

This paper considers mechanisms in which the payment to the seller consists of two components: an up–front charge and a contingent fee paid when the houses are built. The contingent payment is allowed to depend upon (1) who wins the good; (2) all the agents' types; (3) when the option is exercised; and (4) any information revealed after the auction.

The crucial difference between up–front and contingent payments is that the latter introduces a distortion. Up–front payments are sunk and do not affect when the houses are built. In comparison, a positive contingent payment will delay construction (Proposition 1). Welfare is therefore maximised by setting contingent payments equal to zero and using an up–front scheme, such as an English auction (Theorem 1).

In contrast, the revenue maximising auction involves a positive contingent payment (Theorem 2). For a loose intuition, notice that a high value agent is more likely to exercise the option than a low value agent, and therefore cares more about the size of the contingent payment. Making low types pay a contingent fee thus helps the seller separate agents, reducing the incentive for a high value agent to copy a low value agent.

More precisely, contingent fees have the effect of reducing information rents. An agent's payoff is independent of their private information when they do not exercise the option, so agents extract information rents only when they choose to exercise. Since rents are increasing in the probability of exercising the option, raising contingent payments makes the option less

[1] For a description of this market see Haile (2001).

[2] In 1990 a banking consortium, Old Bond Street Holdings, bought Yardley, a struggling British cosmetics company. In 1999, after failing to rebrand its image, the owners called in the receivers. In a similar story, ITV Digital paid £315 million for the right to broadcast Football League games. In 2002 the owners, two other TV companies, put the firm into administration after subscriptions fell below expectations.

[3] These options normally last for one or two years while the producer arranges for a cast, crew and acceptable screenplay. If filming goes ahead, the option is usually exercised on the first day of principal photography; otherwise the property rights revert back to the author (Litwak (1999)).

[4] Over 1954–79, 22% of 5–year OCS wildcat leases expired without any wells being drilled (Porter (1995)).

desirable, lowering the probability of exercise and reducing rents.

Contingent payments thus lower welfare by distorting the exercise decision, but reduce information rents, giving rise to a tradeoff. An agent chooses an exercise time to maximise their valuation minus the contingent payment, while revenue from this agent equals the valuation minus the expected information rent. Setting the contingent payment equal to this information rent thus aligns incentives in the style of a Pigouvian tax. The agent's choice of exercise time then maximises revenue—an act of perfect delegation. Moreover, this revenue maximising contingent fee is (1) positive, (2) declining in valuations, (3) independent of when the option is exercised, and (4) independent of post–auction information. This last property is particularly attractive: it means the seller does not have to observe post–auction data in order to implement the optimal mechanism. Moreover, if the seller can commit to release extra information they will always choose to do so, even if they cannot observe its effect. Hence the seller of the industrial site should commit to release any reports on the levels of contamination.

The revenue maximising auction introduces inefficiencies. In addition to an excessive reservation price and a bias against larger agents, the positive contingent payment means the option will be exercised later than is socially optimal (Proposition 2). This extends a result of Stokey (1979) in the context of durable goods monopolies.

The paper considers two extensions of the basic model.

The first extension is to allow the good to be allocated to different agents depending upon the state of the world. The welfare maximising mechanism can be implemented by the welfare maximising auction in Theorem 1 if the auction is held at the right time. It can also be implemented via a Vickrey mechanism when agents have private values. Similar results apply to the revenue maximising auction.

The second extension allows for non–additive valuations. If the seller can observe the state of the world, the revenue maximising mechanism sets the contingent payment equal to the agent's information rent. Unlike the additive case, revenue will generally be reduced if the seller cannot observe the state of the world. However, in special cases, the mechanism without observability can attain the maximal revenue with observability.

The paper is structured as follows: Section 2 introduces the model. Section 3 derives the welfare and revenue maximising auctions, and discusses the interpretation. Section 4 explores the two extensions, and Section 5 concludes. Omitted proofs are contained in Appendix A.

1.1 Literature

Milgrom and Weber (1982) show revenue is increased when the sale price is linked to information correlated with agents' private information. Riley (1988) used this linkage principle to prove royalties can increase revenue in mineral rights auctions. An extreme example of this was given by Hansen (1985) where the winning agent's valuation was ex–post observable, leading to fullextraction. In contrast to these models, post–auction information will tell the seller nothing about the agent's private information; however, the *act* of exercising the option will.

The paper is most closely related to the optimal regulation literature. Baron and Myerson (1982) consider procuring from a supplier with unknown cost. This is extended by Baron and Besanko (1984) who suppose the supplier's cost may change after the contract is signed. The optimal procurement auction is considered by McAfee and McMillan (1987), Laffont and Tirole (1987) and Riordan and Sappington (1987) who show that the problem separates—the contract should be given to the best type, and then the optimal single–agent regulation contract should be implemented. Analogous reasoning is applied to price discrimination by Mussa and Rosen (1978) and Courty and Li (2000). Section 3.8 considers these related ideas in more detail.

The problem of selling an option is identical to the problem of selling a durable good where agents' valuations vary over time. When there is one agent, the model of this paper can thus be interpreted as the optimal sales contract for a durable good monopolist with elastic demand. To illustrate, consider a car manufacturer whose sales depend on the (uncertain) level of interest rates. Rather than charging a sequence of prices, Theorem 2 suggests the manufacturer may do better by asking for a down–payment when a new model first comes out in exchange for a reduced sale price. This problem was first considered by Conlisk (1984). More recently, Laffont and Tirole (1996) and Biehl (2001) have derived the optimal rental contracts for such a durable goods monopolist in a two–period model.

Other authors have considered constrained mechanisms. Stokey (1979) analyses a multi–period model, but assumes payments are only contingent, i.e. the firm charges a sequence of prices. Hansen (1988), Board (2003), and DeMarzo, Kremer, and Skrzypacz (2004) also analyse contingent payment schemes, while Waehrer (1995) supposes that the contingent payment is independent of types.

Haile (2001, 2003) and McAfee, Takacs, and Vincent (1999) consider a two–period game where a good is sold in period 1. New information then becomes available and, in period 2, agents may engage in resale. This is a hard problem since the opportunity to purchase in the resale market affects bidding in the original auction. The optimal auctions are, however, more straightforward. Welfare is maximised by selling the good in the second period, after the uncertainty has resolved. Similarly, revenue is maximised by running an auction in the second period and also demanding the winner make a contingent payment equal to their information rent (Eso and Szentes (2003) and Section 4.1). However this solution is not without problems. Schwarz and Sonin (2001) observe that the seller cannot simply wait until the second period if an agent must make investments before they can harvest the tract. In addition, as this paper highlights, the seller cannot simply wait if they do not know *when* the agents will exercise their option.

2 Model

Suppose each of N agents has a net valuation consisting of two elements: an ex–ante valuation and an ex–post cost.

The ex–ante valuation of agent i is determined by their privately observed type $\theta_i \in [\underline{\theta}, \overline{\theta}]$. Agents' types are mutually independent, where θ_i is distributed according to $F_i(\theta_i)$ with density $f_i(\theta_i)$. Agent i's ex–ante valuation is denoted $v_i(\theta_i, \theta_{-i})$, where $\theta_{-i} := (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_N)$. The function $v_i(\cdot)$ has several standard properties: it is continuously differentiable in all arguments, its derivative $\partial v_i / \partial \theta_i$ is strictly positive and bounded above, and $\partial v_j / \partial \theta_i \leq \partial v_i / \partial \theta_i$ for $j \neq i$. This formulation includes private values ($v_i(\theta) = \theta_i$) and common values ($v_i(\theta) = \sum_j \theta_j$) as special cases, where $\theta := (\theta_1, \ldots, \theta_N)$

The ex–post cost evolves over time, which is discrete and finite, $t \in \{1, \ldots, T\}$. The cost of each agent i constitutes a sequence $\{c_{i,t}\}$ of random variables, determined by the state of the world, $\omega \in \Omega$. These costs are unrestricted in their dependence across time and across agents, although they are independent of the agents' types. The information about costs possessed by agent i after the sale is described by a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_{i,t}\}, Q)$, where Ω are the states of the world, $\mathcal{F}$ the measurable sets, $\{\mathcal{F}_{i,t}\}$ the information partitions, which grow finer over time, and Q the probability measure. At each time t, the agent knows their current costs, $\{\omega : c_{i,t} \leq c\} \in \mathcal{F}_{i,t}$ for each $c \in \Re$. That is $c_{i,t}$ is $\mathcal{F}_{i,t}$–measurable ($\forall t$), or more simply, $c_{i,t}$ is $\mathcal{F}_{i,t}$–adapted. Let $\mathcal{F}_t = \cap_i \mathcal{F}_{i,t}$ be the total information available at time t, and assume $\mathcal{F}_t$ contains no more information about i's costs than $\mathcal{F}_{i,t}$, i.e. $\mathcal{E}[c_{i,t'} \mid \mathcal{F}_t] = \mathcal{E}[c_{i,t'} \mid \mathcal{F}_{i,t}]$ for $t' \geq t$ ($\forall i$), where "$\mathcal{E}$" denotes the expectation over ex–post costs.

In terms of the example, mentioned in the Introduction, of a firm who has the right to build a housing estate on a piece of land, one can think of $v_i(\theta)$ as the revenue from the sale of the houses. The cost $c_{i,t}$ can either be interpreted as (a) the material cost of starting construction at time t, or (b) the time–t estimate of the ultimate clean–up cost c_i^*, where $c_{i,t} = \mathcal{E}[c_i^* \mid \mathcal{F}_{i,t}]$.

Denote the common discount factor by $\delta \leq 1$. If agent i exercises the option in period $t \leq T$ their net valuation is

$$(v_i(\theta) - c_{i,t})\delta^t$$

and if they never exercise the option, they receive 0. This additive formulation is important for the simplicity of the revenue maximising auction. Section 4.2 examines the robustness of the results to relaxing this assumption.

Absent any payments, the problem of an agent i, who knows their valuation $v_i(\theta)$, is to choose a stopping time (exercise time) τ_i to solve the problem:

$$\max_{\tau_i} \mathcal{E}[(v_i(\theta) - c_{i,\tau_i})\delta^{\tau_i}]$$

The stopping time depends upon the random sequence of costs, so is itself a random variable taking values in $\{1,\ldots,T\} \cup \{\infty\}$, where $\tau_i = \infty$ means that the option is never exercised. In addition, the decision to stop at time t can depend only on information available at time t, $\{\omega : \tau_i \leq t\} \in \mathcal{F}_{i,t}$ $(\forall t)$. Abusing terminology slightly, we will say the such a stopping time τ_i is $\mathcal{F}_{i,t}$–adapted. Since $\mathcal{F}_{i,t}$ is as informative about agent i's costs as $\mathcal{F}_t$, the agent's choice of stopping time also maximises utility amongst the class of $\mathcal{F}_t$–adapted stopping rules.

Now consider the seller's problem of designing a sales procedure. The seller observes the evolution of costs, $\{\mathcal{F}_t\}$, and chooses a three–part direct revelation mechanism $\langle P_i, y_i, z_i\rangle$ consisting of an allocation function, an up–front payment and a contingent payment. After the mechanism is determined, each agent i of type θ_i reports $\tilde{\theta}_i$, where the vector of reports is denoted $\tilde{\theta}$. After the the option is awarded to an agent and the up–front fees are paid, the seller publicly releases the reports $\tilde{\theta}$.[5] The winning agent then chooses whether and when to exercise the option, making a contingent payment when they do so. More formally, the mechanism is defined as follows:

- The allocation function $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0,1]$ is the probability that agent i wins the object, where $\sum_i P_i(\tilde{\theta}) \leq 1$.
- The up–front payment function $y_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$ is a transfer to the seller made by agent i immediately after the auction and is independent of post–auction costs.
- The contingent payment function $z_{i,t} : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \Re$ is a $\mathcal{F}_t$–adapted transfer made by agent i if they win and exercise the option in period t. One should note that the contingent payment is allowed to depend upon: (i) the identity of the agent, (ii) all the agents' reports, (iii) the time the option is exercised, and (iv) the entire history of the stochastic process. This contingent payment can also be interpreted as the strike price. Let $z_i = \{z_{i,t}\}_t$.

An equivalent, and more standard, way to formulate the problem is to have the seller choose a direct revelation mechanism $\langle P_i, x_i, \tau_i\rangle$ consisting of an allocation function $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0,1]$ determining which agent obtains the good, a payment function $x_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$, and a $\mathcal{F}_t$–adapted stopping time $\tau_i : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \{1,\ldots,T\}$ for the winning agent. The equivalence of these mechanisms follows from the revelation and taxation principles and proved in Appendix A.1. This latter mechanism, however, is harder to interpret and implement than the contingent payment mechanism, $\langle P_i, y_i, z_i\rangle$.

[5] Releasing this information will always make the seller better off since they can contract on $\tilde{\theta}$. Any allocation implemented when $\tilde{\theta}$ is not released can be implemented when $\tilde{\theta}$ is released by choosing appropriate payments. When agents have private values ($v_i(\theta) = \theta_i$) there is no need for the seller to release these reports.

The mechanism $\langle P_i, y_i, z_i\rangle$ contains one notable restriction: the allocation of the good does not depend upon post–auction costs. This restriction may be serious if the winner turns out to have substantially higher costs than a losing agent. However, if the agents' post–auction costs are similar, reallocation will not be desirable. The issue is further explored in Section 4.1.

If all agents report truthfully and the option is allocated to agent i, who exercises the option using stopping rule τ_i, they obtain *ex–post utility*

$$u_i(\theta, z_i, \tau_i) := \mathcal{E}[(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i}(\theta))\delta^{\tau_i}] \tag{2.1}$$

The auction mechanism can be summarised by the following steps:

- Time $t = -1$. Each agent i observes type θ_i. The seller chooses the mechanism $\langle P_i, y_i, z_i\rangle$ to maximise welfare or revenue (as defined in Section 3).
- Time $t = 0$. Each agent i reports their type $\tilde{\theta}_i$. The good is allocated to agent i with probability $P_i(\tilde{\theta})$ and they make up–front payment $y_i(\tilde{\theta})$. The seller then reveals $\tilde{\theta}$ to the winning agent.
- Time $t \in \{1, \ldots, T\}$. After the auction, costs $\{c_{i,t}\}$ are revealed, while the agent observes $\mathcal{F}_{i,t}$. Agent i may then choose exercise at time t and make contingent payment $z_{i,t}(\tilde{\theta})$, or they may choose never to exercise the option.

2.1 Optimal Stopping Problem

Before analysing the mechanism design problem, it will be useful to establish some properties of the winning agent's optimal exercise decision. In stage 3 of the game, the mechanism $\langle P_i, y_i, z_i\rangle$ has been determined, and the option has been awarded to some agent i. This agent must then choose a stopping time τ_i to maximise their ex–post utility (2.1).

Denote the set of maximisers by $\hat{\tau}_i$. The set of stopping rules forms a lattice, where $\tau^H \geq \tau^L$ if $\tau^H(\omega) \geq \tau^L(\omega)$ (a.e. $\omega \in \Omega$). Comparing two sets of stopping rules, $\hat{\tau}^H \geq \hat{\tau}^L$ in strict set order if $\tau' \in \hat{\tau}^H$ and $\tau'' \in \hat{\tau}^L$ imply that $\tau' \vee \tau'' \in \hat{\tau}^H$ and $\tau' \wedge \tau'' \in \hat{\tau}^L$.

Proposition 1. *The solution $\hat{\tau}$ of agent i's optimal stopping problem (2.1) has the following properties:*
(a) $\hat{\tau}_i$ is a nonempty sublattice containing a greatest and least element.
(b) $\hat{\tau}_i$ is decreasing in θ_i in strict set order.
(c) Fix $z_{i,t}$ and consider charging a contingent payment $z_{i,t} + K$ ($\forall t$), for a constant K. Then $\hat{\tau}_i$ is increasing in K in strict set order.

Proof. (a) Nonemptiness follows from backwards induction (e.g. Chow, Robbins, and Siegmund (1971, Theorem 3.2)). This construction also implies the set of maximisers has a greatest and least element. To find the least (greatest) element use the rule: stop when current payoffs are weakly (strictly) larger than the continuation utility. Since the set of stopping times is a lattice and $u_i(\theta, z_i, \tau_i)$ is modular in τ_i, the set of maximisers is a sublattice by Topkis (1998, Theorem 2.7.1).

(b) $u_i(\theta, z_i, \tau_i)$ satisfies decreasing differences in (θ_i, τ_i), since $\delta \leq 1$, and is modular in τ_i. Hence the optimal solution is decreasing by Topkis (1998, Theorem 2.8.1).

(c) $u_i(\theta, z_i + K, \tau_i)$ satisfies increasing differences in (K, τ_i), since $\delta \leq 1$, and is modular in τ_i. Hence the optimal solution is increasing by Topkis (1998, Theorem 2.8.1). $\square$

Proposition 1(b) says that agents with high valuations are more impatient and choose to stop earlier.[6] Similarly, Proposition 1(c) says that when the contingent payment decreases the agent's effective valuation increases and they stop earlier. Let τ_i^* be the least element from $\hat{\tau}_i$. From Proposition 1, this exists, is decreasing in θ_i and increasing in K.

3 Optimal Auctions

This section characterises the mechanisms $\langle P_i, y_i, z_i \rangle$ that maximise revenue and welfare.

3.1 Information Rents

Agent i chooses their stopping time τ_i and their reported type $\tilde{\theta}_i$ to maximise interim utility

$$E_{\theta_{-i}} \left[P_i(\tilde{\theta}_i, \tilde{\theta}_{-i}) \, \mathcal{E} \left[\Big(v_i(\theta_i, \tilde{\theta}_{-i}) - c_{i,\tau_i} - z_{i,\tau_i}(\tilde{\theta}_i, \tilde{\theta}_{-i}) \Big) \, \delta^{\tau_i} \right] - y_i(\tilde{\theta}_i, \tilde{\theta}_{-i}) \right]$$

where $E_{\theta_{-i}}$ is the expectation over other agent's types. Truthful revelation is a Bayesian Nash equilibrium if the direct revelation mechanism satisfies incentive compatibility and individual rationality. From agent i's perspective, if the other agents report truthfully, their interim utility given type θ_i, report $\tilde{\theta}_i$ and stopping time τ_i is,

$$U_i(\theta_i, \tilde{\theta}_i, \tau_i) = E_{\theta_{-i}} \left[P_i(\tilde{\theta}_i, \theta_{-i}) \, \mathcal{E} \left[\Big(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i}(\tilde{\theta}_i, \theta_{-i}) \Big) \, \delta^{\tau_i} \right] - y_i(\tilde{\theta}_i, \theta_{-i}) \right] \tag{3.1}$$

Incentive compatibility then says $U_i(\theta_i, \theta_i, \tau_i^*) \geq U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$, while individual rationality states $U_i(\theta_i, \theta_i, \tau_i^*) \geq 0$.

In order to examine how utility is affected by a change in θ_i, we would like to use an envelope theorem where the agent optimises over $(\tilde{\theta}_i, \tau_i)$. The space of stopping times is too complicated for the usual envelope theorem on $\Re^N$ to be applied, so we will use the generalised

[6]There is empirical support for Proposition 1(b). In OCS wildcat auctions, Porter (1995) observes that the first tracts to be drilled were those with high bids; these tracts also led to higher oil revenues.

envelope theorem of Milgrom and Segal (2002).[7] Interim utility, when the agent chooses the optimal stopping rule, τ_i^*, and reports their type truthfully, $\tilde{\theta}_i = \theta_i$, can then be expressed as the integral equation,

$$U_i(\theta_i, \theta_i, \tau_i^*) = E_{\theta_{-i}} \left[\int_{\underline{\theta}}^{\theta_i} P_i(s, \theta_{-i}) \mathcal{E}[\delta^{\tau_i^*}] \frac{\partial}{\partial \theta_i} v_i(s, \theta_{-i})\, ds \right] + U_i(\underline{\theta}_i, \underline{\theta}_i, \tau_i^*) \tag{3.2}$$

Equation (3.2) implies that an agent's information is useful only when they execute the option, so this is the only time when they collect rents. Thus by delaying when the agent exercises, the seller can reduce the agent's utility and potentially increase revenue.

Incentive compatibility implies that interim utility can be expressed by the integral representation (3.2). It also implies that interim utility $U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$ is supermodular in $(\theta_i, \tilde{\theta}_i)$. This is the *monotonicity condition* and implies that

$$E_{\theta_{-i}} \left[P_i(\tilde{\theta}_i, \theta_{-i}) \mathcal{E}[\delta^{\tau_i^*}] \frac{\partial}{\partial \theta_i} v_i(\theta) \right] \tag{3.3}$$

is increasing in $\tilde{\theta}_i$. Together, (3.2) and (3.3) are necessary and sufficient for incentive compatibility, as shown in Appendix A.2. Generally, the monotonicity condition might be very complicated (e.g. if contingent payments differ over time), but is simple for the optimal mechanism.

Individual rationality states that interim utility is positive, $U_i(\theta_i, \theta_i, \tau_i^*) \geq 0$ $(\forall \theta_i)$. Since transferring money from the seller to the agents will not improve welfare or revenue, we henceforth assume that incentive compatibility binds for the lowest type, $U_i(\underline{\theta}_i, \underline{\theta}_i, \tau_i^*) = 0$.

Taking expectations with respect to type θ_i and integrating by parts yields ex–ante utility:

$$E_{\theta_i}[U_i(\theta_i, \theta_i, \tau_i^*)] = E_\theta \left[P_i(\theta) \mathcal{E}[\delta^{\tau_i^*}] \frac{\partial}{\partial \theta_i} v_i(\theta) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right]$$

3.2 Seller's Payoffs

Consider selling a tract of land to an oil company, a logging firm, or a construction company, that goes to waste if not used. The seller's payoff equals the sum of the up–front and contingent payments when the option is sold. When the option is not sold the seller obtains v_0, which may

[7] The theorem requires three conditions be met: (1) U_i is differentiable with respect to θ_i, which holds because v_i is differentiable, (2) The derivative U_i is bounded, which holds because $\partial v_i(\theta)/\partial \theta_i$ is bounded, (3) The maximum is attained. Proposition 1(a) says the stopping rule attains its supremum for any given $\tilde{\theta}_i$ while, in a Bayesian Nash equilibrium, the agent's payoff is maximised at $\tilde{\theta}_i = \theta_i$ by the revelation principle.

be stochastic, but is independent of θ. To summarise, the payoffs are:

	Agent i	Seller
Not Awarded	0	v_0
Exercised at t	$(v_i - c_{i,t} - z_{i,t})\,\delta^t - y_i$	$z_{i,t}\delta^t + y_i$
Not Exercised	$-y_i$	y_i

Some examples may have different payoff structures. This is discussed in Section 3.7.

3.3 Welfare Maximisation

Define welfare as the sum of the agents' utilities and the seller's revenue. This equals the value of the option:

$$\text{Welfare} = E_\theta\left[\sum_i P_i(\theta)\,\mathcal{E}[(v_i(\theta) - c_{i,\tau_i^*})\delta^{\tau_i^*}]\right] + E_\theta\left[\left(1 - \sum_i P_i(\theta)\right)\mathcal{E}[v_0]\right] \tag{3.4}$$

The welfare maximisation problem is to maximise (3.4) subject the to monotonicity condition (3.3) and τ_i^* maximising ex–post utility $u_i(\theta, z_i, \tau_i)$, as defined by (2.1). Notice that if there is no contingent payment then utility coincides with welfare and τ_i^* is welfare–optimal. Hence:

Theorem 1. *Suppose either: (a) agents have private values, or (b) the distribution of costs is the same for all agents. Then welfare is maximised by a mechanism with contingent payments* $z_{i,t}^W(\theta) = 0$ $(\forall\theta)$ *and allocation*

$$P_i^W(\theta) = \begin{cases} 1 & \textit{if} \quad u_i(\theta, z_i^W, \tau_i^*) > u_j(\theta, z_j^W, \tau_j^*) \quad \forall j \neq i \\ & \textit{and} \quad u_i(\theta, z_i^W, \tau_i^*) > \mathcal{E}(v_0) \\ 0 & \textit{otherwise} \end{cases} \tag{3.5}$$

Proof. See Appendix A.3. □

Corollary 1 (Symmetry)**.** *Suppose the valuation function $v_i(\cdot)$ and the distribution of costs are the same for all agents. Then welfare is maximised by awarding the option to the agent with the highest type, θ_i, if the ex–post utility (2.1) from the option exceeds the value to the seller.*

Proof. Under symmetry, $\theta_i \geq \theta_j$ implies $u_i(\theta, z_i^W, \tau_i^*) \geq u_j(\theta, z_j^W, \tau_j^*)$. □

Contingent payments distort the winning agent's optimal stopping problem, delaying the exercise time. Hence the welfare maximising auction sets contingent payments to zero and awards the object to the agent with the highest ex–post utility.

With private values the seller can use any mechanism that awards the good to the agent with the highest utility, such as an English or second price auction. If agents are symmetric (as is Corollary 1) the seller can use any standard auction that allocates the good to the agent with the highest type, such as a first price, second price, or all–pay, even is there are common value components. With asymmetries and common values one can use the scheme of Dasgupta and Maskin (2000).

For the welfare maximising mechanism to satisfy the monotonicity condition (3.3) we require that the probability i is awarded the good to increase in i's reported type. This is equivalent to assuming $u_i(\theta, z_i^W, \tau_i^*) - u_j(\theta, z_i^W, \tau_j^*)$ is quasi–increasing in θ_i $(\forall \theta_{-i})$.[8] While the conditions in Theorem 1 are sufficient for monotonicity to be satisfied, the efficient auction may be impossible to implement with asymmetric costs and common values.

Example 1. Suppose a European option, where $T = 1$, is auctioned to two agents, A and B, who have common values. Assume only agent A has any post–auction uncertainty. Payoffs are

$$u_A = \mathcal{E} \max\{\theta_A + \theta_B - c, 0\} \quad \text{and} \quad u_B = \theta_A + \theta_B$$

where c may be negative. Define θ^* by

$$E_{\theta_A}[\mathcal{E} \max\{\theta_A + \theta^* - c, 0\}] = E_{\theta_A}[\theta_A + \theta^*] \tag{3.6}$$

The welfare maximising auction then sets contingent payments to zero and allocates the contract to A if $\theta_B \leq \theta^*$, and to B if $\theta_B > \theta^*$. This allocation function is independent of A's type.[9] $\triangle$

3.4 Revenue Maximisation

Expected revenue equals welfare minus agents' total utility.

$$\begin{aligned} \text{Revenue} &= E_\theta \left[\sum_i P_i(\theta)\, \mathcal{E}[(v_i(\theta) - c_{i,\tau_i^*})\delta^{\tau_i^*} - v_0] - \sum_i U_i(\theta_i, \theta_i, \tau_i^*) \right] + \mathcal{E}[v_0] \\ &= E_\theta \left[\sum_i P_i(\theta)\, \mathcal{E}[(MR_i(\theta) - c_{i,\tau_i^*})\delta^{\tau_i^*} - v_0] \right] + \mathcal{E}[v_0] \end{aligned} \tag{3.7}$$

where we follow Bulow and Roberts (1989) in denoting i's marginal revenue (or virtual utility) by

$$MR_i(\theta) := v_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\theta)}{\partial \theta_i} \tag{3.8}$$

[8] A function $f : \Re \to \Re$ is quasi increasing if $f(x) \geq 0$ implies $f(y) \geq 0$ for $y \geq x$. This condition is relatively common, e.g. Dasgupta and Maskin (2000).

[9] See Appendix A.4 for a proof.

Equation (3.7) yields a revenue equivalence result: any mechanism that has the same allocation function $P_i(\cdot)$, gives the lowest type no surplus and induces the same optimal stopping rule yields the same revenue. Consequently the revenue maximising auction will not be able to pin down the up–front payment scheme, just the expected up–front payment.

With this in mind, the seller's aim is to pick $\langle P_i, z_i\rangle$ to maximise revenue (3.7) subject to the monotonicity condition (3.3) and τ^* maximising ex–post utility (2.1).

Assumption (MON). The information rent (or marginal consumer surplus)

$$\frac{1-F_i(\theta_i)}{f_i(\theta_i)}\frac{\partial v_i(\theta)}{\partial \theta_i}$$

is differentiable and decreasing in θ_i. If $v_i(\theta)$ is concave in θ_i ($\forall\, \theta_{-i}$), (MON) is implied by the usual monotone hazard condition.

The winning agent's ex–post utility (2.1) equals their net valuation minus their contingent payment. Revenue (3.7) equals the winning agent's net valuation minus their information rent. Hence setting the contingent payment equal to the information rent term induces the agent to choose the optimal stopping rule.

Theorem 2. *Suppose (MON) holds and either: (a) agents have private values, or (b) the distribution of costs is the same for all agents and $MR_i(\theta) - MR_j(\theta)$ is quasi–increasing in θ_i ($\forall i, j$). Then revenue is maximised by a mechanism with contingent payments*

$$z^R_{i,t}(\theta) = \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\frac{\partial v_i(\theta)}{\partial \theta_i} \tag{3.9}$$

and allocation

$$P^R_i(\theta) = \begin{cases} 1 & \text{if} \quad u_i(\theta, z^R_i, \tau^*_i) > u_j(\theta, z^R_j, \tau^*_j) \quad \forall j \neq i \\ & \text{and} \quad u_i(\theta, z^R_i, \tau^*_i) > \mathcal{E}(v_0) \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

Proof. (a) Contingent payments. Setting contingent payments according to (3.9) implies

$$\begin{aligned} \tau^*_i &\in \arg\max \mathcal{E}[(v_i(\theta) - c_{i,\tau_i} - z^R_{i,\tau_i}(\theta))\delta^{\tau_i}] \\ &\in \arg\max E_\theta\left[\sum_i P_i(\theta)\mathcal{E}[(MR_i(\theta) - c_{i,\tau_i})\delta^{\tau_i} - v_0]\right] + \mathcal{E}[v_0] \end{aligned}$$

so when agents choose τ_i to maximise ex–post utility (2.1) they also maximise revenue (3.7).

(b) Allocation rule. The option should be awarded to the agent who induces the highest revenue, subject to this value exceeding the seller's valuation, yielding equation (3.10).

(c) Finally, we need to check the monotonicity condition (3.3). This is done through three claims.

Claim 1: $\mathcal{E}[\delta^{\tau_i^*}]$ *is increasing in* $\tilde{\theta}_i$.

Proof: An increase in $\tilde{\theta}_i$ reduces $z_{i,t}^R$ uniformly across time, by (MON). By Proposition 1(c), this reduces τ_i^* and therefore increases $\mathcal{E}[\delta^{\tau_i^*}]$.

Claim 2: $P_i^R(\tilde{\theta}_i, \theta_{-i})$ *is increasing in* $\tilde{\theta}_i$.

Proof: In the revenue–maximising auction ex–post utility is

$$u_i(\theta, z_i^R, \tau_i^*) = \mathcal{E}\left[(MR_i(\theta) - c_{i,\tau_i^*})\delta^{\tau_i^*}\right]$$

Let us establish two facts.

(1) $u_i(\theta, z_i^R, \tau_i^*)$ is increasing in θ_i. The envelope theorem implies

$$u_i(\theta_i, \theta_{-i}, z_i^R, \tau_i^*) = u_i(\underline{\theta}, \theta_{-i}, z_i^R, \tau_i^*) + \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial \theta_i} MR_i(s, \theta_{-i}) \mathcal{E}[\delta^{\tau_i^*(s,\theta_{-i})}]\, ds$$

which is increasing in θ_i since marginal revenue $MR_i(\theta)$ is increasing in θ_i, by (MON).

(2) $u_i(\theta, z_i^R, \tau_i^*) - u_j(\theta, z_j^R, \tau_j^*)$ is quasi–increasing in θ_i. Applying the envelope theorem to $u_j(\theta, z_j^R, \tau_j^*)$,

$$u_j(\theta_i, \theta_{-i}, z_j^R, \tau_j^*) = u_j(\underline{\theta}, \theta_{-i}, z_j^R, \tau_j^*) + \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial \theta_i} MR_j(s, \theta_{-i}) \mathcal{E}[\delta^{\tau_j^*(s,\theta_{-i})}]\, ds$$

If agents have private values then $u_j(\theta, z_j^R, \tau_j^*)$ is constant in θ_i, so (1) implies (2). Next, suppose all types have the same distribution of costs and $u_i(\theta, z_i^R, \tau_i^*) = u_j(\theta, z_j^R, \tau_j^*) > 0$. Then $MR_i(\theta) = MR_j(\theta)$ and $\mathcal{E}[\delta^{\tau_i^*}] = \mathcal{E}[\delta^{\tau_j^*}]$. The assumption that $MR_i(\theta) - MR_j(\theta)$ is quasi–increasing in θ_i means $\frac{\partial}{\partial \theta_i}[MR_i(\theta) - MR_j(\theta)] \geq 0$ so that $\frac{\partial}{\partial \theta_i}[u_i(\theta, z_i^R, \tau_i^*) - u_j(\theta, z_j^R, \tau_j^*)] \geq 0$ which implies (2).

Claim 3: $U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$ *is supermodular in* $(\theta_i, \tilde{\theta}_i)$.

Proof: The two claims then imply that $\frac{\partial}{\partial \theta_i} U_i(\theta_i, \tilde{\theta}_i, \tau_i^*) = E_{\theta_{-i}}\left[P_i^R(\tilde{\theta}_i, \theta_{-i})\mathcal{E}[\delta^{\tau_i^*}]\frac{\partial}{\partial \theta_i} v_i(\theta)\right]$ is increasing in $\tilde{\theta}_i$, as required. □

Corollary 2 (Symmetry)**.** *Suppose (MON) holds, the valuation function* $v_i(\cdot)$*, the distribution of types* $F_i(\theta_i)$ *and the distribution of costs are the same for all agents, and* $MR_i(\theta) - MR_j(\theta)$ *is quasi–increasing in* θ_i*. Then revenue is maximised by awarding the option to the agent with the highest type, if the ex–post utility (2.1) from the option exceeds the value to the seller.*

Proof. Under symmetry, $\theta_i \geq \theta_j$ implies $u_i(\theta, z_i^R, \tau_i^*) \geq u_j(\theta, z_j^R, \tau_j^*)$. □

Theorem 2 states that the optimal contingent payment is set according to equation (3.9), and the option then allocated to the agent with the highest ex–post utility. There is more flexibility over the up–front payment which can be determined in any number of ways so long as interim utility (3.1) equals the integral equation, (3.2).

If agents have private values the mechanism can be implemented by a second price auction which reveals the highest agent's type and allocates the good to the agent with the highest ex–post utility. If agents are also symmetric (as in Corollary 2), the seller can use any standard auction that reveals the winning agent's type, such as a first price, second price or all–pay auction (but not an English auction). The optimal contingent payment can then be deduced from the winner's up–front bid. With common values, the seller needs to use a mechanism that reveals all agents' types. Under symmetry, the first price, second price and all–pay auctions also satisfy this criterion.

The optimal mechanism in Theorem 2 is not unique. For example, the usual mechanism design approach is to derive the optimal stopping time τ_i^* and to use a forcing contract, demanding the agent pay an infinite amount if they choose anything other than τ_i^*. This mechanism, however, is hard to interpret, harder to implement, and depends on post–auction costs.

In contrast, the contingent payment in Theorem 2 is independent of post–auction costs. This means that the optimal mechanism can be implemented when the seller cannot observe, or cannot contract on, cost data.

Suppose the seller has some extra information about costs $\{c_{i,t}\}$ not possessed by the agents. That is, the seller knows $\mathcal{F}'_t \supset \mathcal{F}_t$. If the seller can contract on $\mathcal{F}'_t$ then they should automatically release this information since they can always implement the outcome without revelation by choosing appropriate payments. After the information is released, the revenue maximising mechanism is given by Theorem 2, which is independent of costs. Consequently the seller always wishes to release cost information, even when they *cannot* observe the information they are releasing.

The revenue maximising auction introduces two distortions. First, the strike price is too high. Second, the good may be allocated to the wrong agent (if agents are asymmetric) or not awarded at all. This tells us about the type of dynamic distortion that market power can induce.

Proposition 2. *If the distribution of costs is the same for all agents then the option will be exercised earlier under the welfare maximising auction (Theorem 1) than under the revenue maximising auction (Theorem 2).*

Proof. Suppose agent i wins the welfare maximising auction and let $\hat{\tau}_i^W$ be the set of stopping rules that maximise ex–post utility (2.1). Similarly, suppose agent j (who may be the same as i) wins the revenue maximising auction and let $\hat{\tau}_j^R$ be the set of stopping rules that maximise ex–post utility (2.1). Since the distribution of costs is the same for all agents, $v_i(\theta) \geq v_j(\theta) \geq MR_j(\theta)$. Proposition 1(b) then implies that $\hat{\tau}_j^R \geq \hat{\tau}_i^W$ in strict set order, and the least stopping time is greater under the revenue maximising mechanism. $\square$

Setting $N = 1$, Proposition 2 implies that a durable goods monopolist with varying demand (or varying costs) always sells later than a perfectly competitive firm. This extends the result of Stokey (1979), who assumes the sequence of costs $\{c_{i,t}\}$ is deterministic and gradually decreases.

3.5 Interpretation

Theorem 2 says that the problem separates. The option is first allocated to the agent with the highest ex–post utility (2.1). In the second stage, the contract implements the optimal single–agent contract, where the contingent payment (strike price) is positive, declining in the agent's type, independent of post–auction costs, and independent of the time of execution.

The revenue–optimal mechanism works in a simple way. Information rents drive a wedge between welfare and revenue. When the agent chooses the stopping time they are interested in maximising their discounted valuation, whereas the seller is interested in maximising their discounted marginal revenue. Inserting a Pigouvian tax equal to the expected information rents means the agent's and seller's problems coincide—an act of perfect delegation.

As shown in Section 3.1, agents collect rents only when they exercise the option. Thus the Pigouvian tax, the *rent–tax*, will be charged only when an agent executes the option, and can be fully captured by the contingent payment. Rents are smaller for higher types, since there are fewer agents who wish to copy them, so agent i's contingent payment is decreasing in θ_i. Consequently the highest type makes no contingent payment (i.e. no distortion at the top). Rents are linear in the discount factor, so the contingent payment is independent of time. And post–auction costs are additive, so rents do not depend on the cost, nor does the contingent payment.

3.6 Some Numbers

The contingent payment can be substantial. *Figure 1* shows the revenue–maximising first–price bidding locus for a European option ($T = 1$) with private values ($v_i(\theta) = \theta_i$), where $c_i, \theta_i \sim U[0, 1]$, $v_0 = 0$, and $\delta = 1$. In this example, the contingent payment is larger than the up–front payment for many types of agents.

While this provides some indication about the size of the optimal contingent payment, it says nothing about the welfare and revenue effects of running such a mechanism. For example, consider a European option ($T = 1$) with private values ($v_i(\theta) = \theta_i$), where $\ln c_i \sim N(0, 1)$, $\theta_i \sim \exp(1)$, $v_0 = 0$ and $\delta = 1$. The following table compares the welfare–maximising auction (Theorem 1) and the revenue–maximising auction (Theorem 2):[10]

[10]These numbers are based on 10,000 simulations.

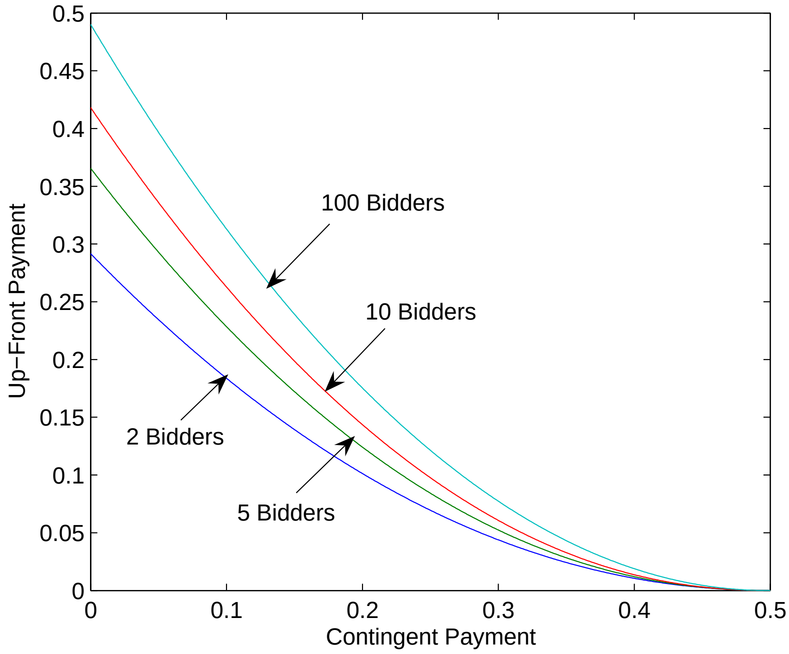

Figure 1: First Price Bidding Locus for a European Option

	Revenue			Welfare		
#of agents	2	10	50	2	10	50
Revenue–Max	0.28	0.95	2.18	0.55	1.60	3.05
Welfare–Max	0.25	0.89	2.16	0.69	1.70	3.08
Change	+12.8%	+6.1%	+1.1%	−20.1%	−5.7%	−1.0%

3.7 Generalising the Seller's Payoff

In Section 3.2 we assumed the seller's payoffs are fully captured by the transfer payments. However, this is not always the case. If a oil company fails to drill within five years of acquiring an OCS lease, the tract returns to the government. Similarly, before a financial option is exercised the dividends accrue to the seller. Instead, suppose payoffs are given by:

	agent	Seller
Not Awarded	0	v_0
Exercised in t	$(v_i - c_{i,t} - z_{i,t})\,\delta^t - y_i$	$a_t + z_{i,t}\delta^t + y_i$
Not Exercised	$-y_i$	$b + y_i$

where b and a_t may be random and are independent of agent's types.

When the agent chooses to stop at time t the seller loses $b - a_t$, so the welfare maximising auction inserts a Pigouvian tax equal to the opportunity cost of the option.

$$z_{i,t}^W(\theta) = \mathcal{E}[b - a_t|\mathcal{F}_t]$$

The contingent payment may now depend on time and the post–auction costs. Similarly, the optimal contingent payment under revenue maximisation is

$$z_{i,t}^R(\theta) = \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\theta)}{\partial \theta_i} + \mathcal{E}[b - a_t|\mathcal{F}_t] \tag{3.11}$$

Hence the revenue maximising strike price equals the opportunity cost plus the rent–tax.

3.8 Discussion

The idea of a rent–tax can also be used to reinterpret the solutions to a number of other classic economic problems.

First, consider the textbook monopolist, where consumer i has valuation, θ_i, distributed according to $F(\theta_i)$. The profit maximising solution is to sell to the agent if marginal revenue exceeds costs, $MR(\theta_i) \geq c$. The usual way to implement this outcome is to charge a price $MR^{-1}(c)$. Alternatively the firm could set a contingent payment $z_i = c + (1 - F(\theta_i))/f(\theta_i)$ in addition to an up–front fee. Agents would then purchase if $\theta_i - z_i \geq 0$, i.e. if $MR(\theta_i) \geq c$.

Similarly, one can use the rent–tax to interpret the reserve price in a standard auction as an option contract. The seller should charge a contingent payment $z_i = v_0 + (1 - F(\theta_i))/f(\theta_i)$ and then award the good to the highest agent without a reserve price. Agents will then only execute the option if $\theta_i - z_i \geq 0$, i.e. if $MR(\theta_i) \geq v_0$. Hence a reserve price is a special case of an option. Moreover, this mechanism is robust to additive uncertainty. In addition, if a price is more credible than an allocation, this method of implementation might help overcome the Coase conjecture (McAfee and Vincent (1997)).[11]

Theorem 2 can be used to derive the optimal contingent payment in a oil tract auction. Suppose agent i has estimate, $\theta_i \sim F_i$, of the extraction cost, $k_i(\theta)$, which may have common value components. Oil prices, p_t, are uncertain, while the amount of oil in the ground is normalized to 1. Given contingent payments $z_{i,t}(\theta)$, suppose agent i's ex–post utility from extracting the oil using exercise rule τ_i is $u_i(\theta, z_i, \tau_i) := \mathcal{E}[(p_{\tau_i} - k_i(\theta) - z_{i,\tau_i}(\theta))\delta^{\tau_i}]$. This equation is just a relabeling of (2.1) and, correspondingly, the revenue maximising contingent

[11] For example, selling an option is likely to give the winning agent a lot of bargaining power. In comparison, not awarding the good keeps any bargaining power in the hands of the seller.

payment is

$$z_{i,t}^R(\theta) = \frac{F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial k_i(\theta)}{\partial \theta_i}$$

The winning agent chooses τ_i to maximise $u_i(\theta, z_i^R, \tau_i) = \mathcal{E}[(p_{\tau_i} - k_i(\theta) - z_{i,t}^R(\theta))\delta^{\tau_i}]$. In comparison, the OCS wildcat auctions use a contingent payment of $z_{i,t} = p_t/6$ (Porter (1995)). The winning agent then chooses τ_i to maximise $u_i(\theta, z_i^R, \tau_i) \propto \mathcal{E}[(p_{\tau_i} - (6/5)k_i(\theta))\delta^{\tau_i}]$. While the optimal exercise decisions may coincide, there is no reason to believe they will do so.

Next, consider a second–degree price discriminating monopolist who can award different quality, q, to different agents. Suppose that for agent i the marginal benefit from quality is $\theta_i \sim F(\theta_i)$, and the cost of quality is given by the convex function $c(q)$. Mussa and Rosen (1978) show the principal equates marginal costs to marginal benefit minus the information rent:

$$c'(q) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

This outcome can be implemented by charging a price, $z_i(q) = c(q) + q(1 - F(\theta_i))/f(\theta_i)$, equal to cost plus the rent–tax, in addition to an up–front payment. Once the rent–tax has been imposed, the principal's problem has been perfectly delegated and the agent's choice of q maximises the principal's revenue.

Baron and Myerson (1982) consider a principal who is trying to procure quantity q of an input from an agent with unknown cost. If $V(q)$ is the principal's benefit and θ_i the marginal cost, this is the reverse of the price discrimination story. The optimal solution can then be implemented by paying $z_i(q) = V(q) - qF(\theta_i)/f(\theta_i)$ for quantity q, and setting an appropriate up–front payment. This is an adjusted version of the "selling the firm" strategy suggested by Loeb and Magat (1979) and induces the agent to supply the optimal quantity.

Since a durable good is an option, Theorem 2 derives the optimal *sales contract* for a single durable good with stochastic valuations. The optimal sales contract for a durable goods monopolist with elastic supply can then be obtained by setting $N = 1$, where v_0 is interpreted as the marginal cost. Agent i is awarded an option if $u_i(\theta_i, z_i^R, \tau_i^*) \geq v_0$. The contract then consists of a contingent payment $z_{i,t}^R(\theta)$ along with an up–front payment such that interim utility (3.1) equals the integral equation, (3.2).

This sales contract can be compared with Laffont and Tirole (1996) and Biehl (2001) who analyse the optimal *rental contract* with stochastic valuations. Their models allow for non–additive valuations and consider only two periods, however, for the sake of comparison suppose the rental value each period t is $(1 - \delta)(\theta_i - c_t)$. Using the rent–tax approach, the optimal rental contract then charges a contingent payment $z_i = (1 - \delta)(1 - F(\theta_i))/f(\theta_i)$ every period the good is rented, along with an up–front fee.[12]

[12]The choice between a rental and sale contract depends upon the circumstances. While renting will generally be preferred to selling, it may not be possible. Many goods are irreversible: once a bridge has been built there

These examples show how the rent–tax approach can be used in a variety of different situations. In the simple examples, such as textbook monopoly pricing, there are simpler schemes. However, in more complex problems, rent–taxation seems appealing and has the advantage that it is independent of additive shocks to valuations.

4 Extensions

This section looks at two extensions of the basic model. Section 4.1 considers the question of auction timing and the optimal state–dependent allocation function. Section 4.2 analyses the payoffs with non–additive valuations.

The section paints a general picture of these extensions, but will only consider the relaxed problems, omitting the monotonicity condition. This caveat should be born in mind when interpreting the results, but should not affect the basic intuition.

4.1 Auction Timing and State–Dependent Allocation

One advantage of a multi–period model is that is allows us to analyse when the auction should be held. Suppose the seller contracts at time 0 (as in Section 3), but chooses to hold the auction at time s, which itself is an $\mathcal{F}_t$–adapted stopping time, $\{\omega : s \leq t\} \in \mathcal{F}_t$.[13] Define the information available at the start of the auction to be $\mathcal{F}_s := \{A : A \cap \{s = t\} \in \mathcal{F}_t \, \forall t\}$. The mechanism can be described by the triple $\langle P_i, y_i, z_i \rangle$, where $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0, 1]$ is a $\mathcal{F}_s$–measurable allocation function, $y_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$ is the up–front payment made at time 0, and $z_{i,t} : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \Re$ is a $\mathcal{F}_t$–adapted contingent payment. As in Section 3, the revenue maximisation problem is

$$\begin{aligned} \text{Revenue} \quad &= \quad E_\theta \mathcal{E}\left[\sum_i P_i(\theta) \mathcal{E}\left[(MR_i(\theta) - c_{i,\tau_i^*})\delta^{\tau_i^*} - v_0 \mid \mathcal{F}_s\right]\right] + \mathcal{E}[v_0] \\ \text{s.t.} \quad & \tau_i^* \in \arg\max_{\tau_i \geq s} \mathcal{E}[(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i})\delta^{\tau_i}] \end{aligned}$$

where $MR_i(\theta)$ is defined by (3.8). Given a starting time s, revenue is maximised by the policy is Theorem 2. The contingent payment is set equal to the information rent (3.9), and the good given to the agent with the highest time–s ex–post utility, $\mathcal{E}[(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i}(\theta))\delta^{\tau_i} \mid \mathcal{F}_s]$.[14]

With private values this can be implemented using the handicap auction of Eso and Szentes

is little point in taking it down in the off–season. Other goods are one–time experiences: watching Citizen Kane is a certainly durable event yet can only be rented if the agent suffers from severe amnesia.

[13]The fact the contract is signed at time 0 is important and means that the agents do not gain rents from knowledge of post–auction costs.

[14]Similarly, welfare is maximised by the policy in Theorem 1 when implemented at time s.

(2003). At time 0 each agent is assigned contingent payment $z^R_{i,t}(\theta)$ and makes up–front payment $\hat{y}_i(\theta)$ chosen so that incentive compatibility holds. At time s the auctioneer than holds a second price auction that allocates the good to the agent with the highest time–s ex–post utility.

When choosing the starting time s, there are two main effects at work.

First, the decision effect. By delaying the auction the agent may have missed good opportunities to execute the option, reducing revenue.

Second, the allocation effect. By delaying the auction, information is revealed and the seller increases revenue.

In the case of a timber auctions, contract terms are generally longer than the time needed to harvest, so most agents wait until the until the end of their contract (Haile (2001)). By delaying the auction, a more efficient allocation would be possible. However, delaying the auction may also mean some tracts were harvested too late.

The begs the question: when is the best time to hold the auction? In order to answer this, let us consider the optimal state dependent mechanism.

4.1.1 State Dependent Allocation

Define the state–dependent mechanism by a pair $\langle \sigma_i, x_i \rangle$. The exercise function $\sigma_i : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \{1, \dots, T\} \cup \{\infty\}$ describes when agent i exercises the object, where $\{\omega : \sigma_i(\tilde{\theta}) \leq t\} \in \mathcal{F}_t$. If $\sigma_i(\tilde{\theta}) = \infty$ then agent i never exercises, and $\sum_i \mathbf{1}_{\sigma_i(\tilde{\theta})<\infty} \leq 1$ since only one agent can win.[15] Let the seller be agent 0. Payment is given by $x_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$.

In the Bayesian Nash equilibrium, agent i chooses their reported type $\tilde{\theta}_i$ to maximise interim utility

$$U_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}} \mathcal{E}[(v_i(\theta) - c_{i,\sigma_i(\tilde{\theta}_i,\theta_{-i})})\delta^{\sigma_i(\tilde{\theta}_i,\theta_{-i})} - x_i(\tilde{\theta}_i, \theta_{-i})]$$

Following Section 3, welfare is the sum of the seller's and agents' utilities,

$$\text{Welfare} = \sum_i E_\theta \mathcal{E}[(v_i(\theta) - c_{i,\sigma_i(\theta)})\delta^{\sigma_i(\theta)}]$$

Let $\sigma^W(\theta) = \{\sigma^W_i(\theta)\}_i$ be the welfare–optimal stopping rule.[16] Similarly, revenue equals welfare minus rents,

$$\text{Revenue} = \sum_i E_\theta \mathcal{E}[(MR_i(\theta) - c_{i,\sigma_i(\theta)})\delta^{\sigma_i(\theta)}]$$

[15]In Section 3 the allocation was determined by two functions: $P_i(\cdot)$ says which agent wins the option, and τ_i says when the winning agent executes. The $\sigma_i(\cdot)$ formulation merges these two functions into one.

[16]Under private values the welfare maximising policy satisfies the monotonicity constraint, that $U_i(\theta_i, \tilde{\theta}_i)$ is supermodular in $(\theta_i, \tilde{\theta}_i)$. Intuitively, when agent i's type rises they are allocated the good more often and are allocated it earlier (by Proposition 1)(b). With common value components, monotonicity will require other assumptions, as in Theorem 1.

where the seller's marginal revenue is $MR_0(\theta) = v_0$. Let $\sigma^R(\theta)$ be the revenue–optimal stopping rule.

The welfare maximising rule can be implemented by the mechanism in Theorem 1 if the auction is started at time $s = \min_i\{\sigma_i^W(\theta)\}$.[17] Similarly, the revenue maximising rule can be implemented by the mechanism in Theorem 2 if the auction is started at time $s = \min_i\{\sigma_i^R(\theta)\}$. Intuitively, once the option is in the hands of the right agent they will execute at the time that maximises the seller's objective. The auctioneer thus delays the auction until they have enough information to allocate the option to the right agents in the each state.

While it is easier to run an auction than directly implement the stopping function $\sigma(\theta)$, the seller still requires a lot of knowledge. In particular, they need to know cost information, $\mathcal{F}_t$, in order start the auction at the correct time. The next section considers how to implement these mechanisms when the seller cannot observe post–auction costs.

4.1.2 State Dependent Allocation with Unobservable State

The first task is to derive the welfare maximising mechanism when costs are not observable. Recall that each agent observes information $\mathcal{F}_{i,t}$, and assume agents have private values, $v_i(\theta) = \theta_i$.[18]

The welfare maximising mechanism, $\sigma^W(\theta)$, can then be implemented through a Vickrey auction. Let $\{\sigma_j^-(\theta_{-i})\}_{j\neq i}$ maximise welfare in a world without agent i. Suppose at time 0 each agent i reports their type $\tilde{\theta}_i$. In every subsequent period, $t \in \{1, \ldots, T\}$, agent i then reports an information partition $\tilde{\mathcal{F}}_{i,t}$ which implies reported costs $\tilde{c}_{i,t}$. Eventually, if i is awarded the good, they pay the externality they exert on agents $j \neq i$,

$$\sum_{j\neq i} \left[(\tilde{\theta}_j - \tilde{c}_{j,\sigma_j^-})\delta^{\sigma_j^-} - (\tilde{\theta}_j - \tilde{c}_{j,\sigma_j^W})\delta^{\sigma_j^W} \right]$$

Truthful revelation is a weakly dominant strategy in each period. See Appendix A.6 for a proof.

To implement the the revenue maximising mechanism, we can use a version of the handicap mechanism of Eso and Szentes (2003) which is a natural compliment to the Theorem 2. At time 0 agents make a report of their types θ_i, are assigned a contingent payment $z_i^R(\theta_i)$, equal to the rent–tax (3.9), and make an up–front payment $\hat{y}_i(\theta)$ chosen so that incentive compatibility holds. Once the contingent payment has been introduced each agent i acts *as if* their valuation is $MR_i(\theta_i)$, so we can run the above Vickrey auction where valuations θ_i are replaced by marginal revenues $MR_i(\theta_i)$.

[17] See Appendix A.5 for a proof.

[18] One may be able to extend the treatment to common values using mechanisms such as Dasgupta and Maskin (2000).

4.2 Options Without Additivity

So far we have assumed private information is additively separable from post–auction information. Suppose instead that agent i's net valuation in period t is given by the $\mathcal{F}_{i,t}$-adapted function $v_{i,t} : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \Re_+$.

The welfare maximising mechanism (Theorem 1) awards the option to the agent with the highest ex–post utility and so is unaffected by dropping the additivity assumption. Similarly, as shown in Section 4.2.1, if the seller can contract on the state then the revenue maximising mechanism is essentially the same as Theorem 2. However, the optimal contingent payment may now be state dependent. Therefore, if the seller cannot observe the state, the optimum with observability can only be achieved in special cases.

4.2.1 Observable State

First suppose the total post–auction information, $\mathcal{F}_t = \cap_i \mathcal{F}_{i,t}$, is observed by the seller. Agents choose their stopping time to maximise ex–post utility,

$$u_i(\theta, z_i, \tau_i) := \mathcal{E}[(v_{i,\tau_i}(\theta) - z_{i,\tau_i}(\theta))\delta^{\tau_i}] \tag{4.1}$$

The derivation of revenue is the same as in Section 3 and is given by

$$\text{Revenue} = E_\theta \left[\sum_i P_i(\theta)\, \mathcal{E} \left[\left(v_{i,\tau_i^*}(\theta) - \frac{\partial v_{i,\tau_i^*}(\theta)}{\partial \theta_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \delta^{\tau_i^*} - v_0 \right] \right] + \mathcal{E}[v_0] \tag{4.2}$$

where τ_i^* is chosen to maximise (4.1). Comparing equations (4.1) and (4.2) one can see that the agent's choice of stopping rule coincides with the seller's choice if the contingent payment equals information rents,

$$z^O_{i,t}(\theta) = \frac{\partial v_{i,t}(\theta)}{\partial \theta_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$$

where the "O" superscript stands for observable. So the optimal contingent payment is time (state) independent if i's rents are independent of the time (state).

4.2.2 Unobservable State

Next, suppose the state of the world is not observed by the seller. If the seller can learn agent i's state from agent j, then it is easy to implement the optimum under observability. To avoid this trivial case, suppose the set of states is a product space, where $\omega = \{\omega_1, \ldots, \omega_N\}$ and $\mathcal{F}_{j,t}$ is independent of the sigma–algebra generated by ω_i for $j \neq i$. In this situation revenue still is given by equation (4.2); however, the contingent payment $z_{i,t}$ can no longer depend upon the state. Consequently the optimum with observability may not be attainable.

Example 2. Suppose the option is European ($T = 1$), agent i wins the auction, and their state is represented by $\omega_i = (\omega_{i,1}, \omega_{i,2}) \in [0,1]^2$, while $\theta_i \sim U[0,1]$. Let $v(\theta_i, \omega_{i,1}, \omega_{i,2}) = \theta_i \omega_{i,1} + \omega_{i,2}$ so that $MR_i(\theta_i, \omega_{i,1}, \omega_{i,2}) = (2\theta_i - 1)\omega_{i,1} + \omega_{i,2}$. The seller would then like to execute if $(2\theta_i - 1)\omega_{i,1} + \omega_{i,2} \geq 0$, while the agent executes if $\theta_i \omega_{i,1} + \omega_{i,2} \geq z_i$, given some contingent payment z_i. With observability the optimum can be implemented by choosing $z_i = (1 - \theta_i)\omega_{i,1}$; however, there is no state independent contingent payment z_i which will align these incentives. Hence principal chooses z_i to maximise $\mathcal{E}\left[[(2\theta_i - 1)\omega_{i,1} + \omega_{i,2}]\mathbf{1}_{\{\theta_i \omega_{i,1} + \omega_{i,2} \geq z_i\}}\right]$. △

Under some circumstances the maximum revenue with observability can be attained without observability. For example, consider a European option ($T = 1$) where $\omega_i \in [0,1]$ and $v_i(\theta, \omega_i)$ is increasing in the state, ω_i. Define the marginal revenue from agent i in state ω_i to be

$$MR_i(\theta, \omega_i) := v_i(\theta, \omega_i) - \frac{\partial v_i(\theta, \omega_i)}{\partial \theta_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$$

If $MR_i(\theta, \omega_i)$ is quasi–increasing in ω_i then the states where the seller wishes to exercise, $\{\omega_i : MR_i(\theta, \omega_i) \geq 0\}$, is an increasing set of the form $\{\omega_i \geq \overline{\omega}_i\}$. The revenue maximising exercise rule can then be implemented under non–observability by setting the contingent payment

$$z_{i,t}^{NO}(\theta) := \frac{\partial v_{i,t}(\theta, \overline{\omega}_i)}{\partial \theta_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$$

That is, the seller introduces a Pigouvian tax equal to the information rent in the marginal state. Since $v_i(\theta, \omega_i)$ is increasing in ω_i, the winning agent will choose the revenue maximising exercise decision in the infra–marginal states.[19]

The crucial assumption is that there is only one marginal state in the seller's and agent's exercise decisions. Then the contingent payment can be chosen to align incentives in the marginal states.

This approach can also be extended to multiple periods.[20] Suppose agent i's state is given by $\omega_i = (\omega_{i,1}, \ldots, \omega_{i,T}) \in [0,1]^T$, and that payoffs are markov, so the decision to stop at time t depends only on the state $\omega_{i,t}$. Also suppose that the seller's and agents' stopping problems are monotone, so that $\{\omega_{i,t} : \tau = t\}$ are increasing sets. Intuitively, this means that payoffs display "mean reversion," so an increase in today's state increases today's payoff more than future payoffs. Under these assumptions, the maximum revenue with observability can be implemented without observability by choosing the contingent payments to align incentives in these (unique) marginal states.

[19]This one–period result is similar to Baron and Besanko (1984), Laffont and Tirole (1996) and Courty and Li (2000). These authors work with the induced distribution of valuations, rather than specifying the underlying state–space.

[20]Monotone markov stopping problems are further examined by Friedman and Johnson (1997).

While this result is encouraging, it is also limited. States are often multi–dimensional, as in Example 2 and in non–markov multi–period models. In addition, the assumption that the stopping problems are monotone puts strong restrictions on the evolution of payoffs. The general point is that once the additivity assumption is dropped the effect of observability depends upon the nature of the optimal stopping problem.

5 Concluding Remarks

This paper has derived the optimal way to sell an option, where post–auction information is allowed to follow any stochastic process. This introduces timing considerations into the allocation decision, allowing the agents to choose both whether to initiate a project and when to do so. In analysing this we have seen that the theory of optimal stopping is a natural complement to the theory of mechanism design.

The revenue maximising auction consists of an up–front and contingent payment, in the style of a two–part tariff. The contingent payment acts like a Pigouvian rent–tax so the winning agent's choice of stopping time is optimal for the seller. This perfect delegation property means that the seller will always reveal information that is valuable to the agents. Under an additivity assumption, we also showed that this contingent payment is time– and state–invariant, so the seller does not need to contract on the post–auction information.

The introduction of multiple periods allows us to assess dynamic aspects of auction design. In Proposition 2, we demonstrated that market power induces an allocative distortion whereby the winning agent will exercise the option too late. We also analysed the optimal timing of an auction, showing the seller could attain the maximum revenue available through a state–dependent mechanism if they hold the auction at the right time.

There are other aspects of dynamic auction design that we have said little about. If the seller cannot run the optimal mechanism they would wish to influence the winning agent's choice of execution time through bureaucratic means or by hiding information. We have also said little about resale, which may enhance revenue by increasing efficiency while undermining discrimination between agents.

A Omitted Proofs

A.1 Equivalence of Mechanisms

Lemma 1. *Any direct mechanism $\langle P_i, x_i, \tau_i\rangle$ can be replaced by an indirect mechanism $\langle P_i, y_i, z_i\rangle$ that has an equilibrium giving rise to the same outcome (expected payoffs, stopping time and allocation). Conversely, any indirect mechanism $\langle P_i, y_i, z_i\rangle$ can be replaced by a direct mechanism $\langle P_i, x_i, \tau_i\rangle$ that has an equilibrium giving rise to the same outcome.*

Proof. Consider three mechanisms:

Mechanism 1 consists of an allocation function $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0,1]$, a $\mathcal{F}_t$–adapted payment $x_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$, and a $\mathcal{F}_t$–adapted stopping time $\tau_i : [\underline{\theta}, \overline{\theta}]^N \times \Omega \to \Re$ for the winner.

Mechanism 2 allocates the object according to $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0,1]$. The seller then reveals reported types $\tilde{\theta}$ and charges a $\mathcal{F}_t$–adapted payment $\hat{x}_i : [\underline{\theta}, \overline{\theta}]^N \times \{1,\dots,T\} \cup \{\infty\} \times \Omega \to \Re$ for agent i, if they stop at time t. If the agent loses then let $\tau_i = \infty$.

Mechanism 3 allocates the object according to $P_i : [\underline{\theta}, \overline{\theta}]^N \to [0,1]$, and charges an up–front payment $y_i : [\underline{\theta}, \overline{\theta}]^N \to \Re$. The seller then reveals reported types $\tilde{\theta}$ and charges a $\mathcal{F}_t$–adapted contingent payment $z_i : [\underline{\theta}, \overline{\theta}]^N \times \{1,\dots,T\} \times \Omega \to \Re$ for the winner.

Mechanisms 1 and 2 are equivalent by the taxation principle and the revelation principle (Salanie (1997)). An outcome implemented by mechanism 3 can be implemented via mechanism 2 by setting $\hat{x}_{i,t} = y_i + z_{i,t}$. Next, take an outcome implemented under mechanism 2 and construct the following payments. For a losing agent let $y_i(\theta) = \mathcal{E}[\hat{x}_{i,\infty}(\theta)]$. For a winning agent define y_i and $z_{i,t}$ by

$$\begin{aligned} y_i(\theta) &= \mathcal{E}[\hat{x}_{i,\infty}(\theta)] \\ z_{i,t}(\theta) &= \delta^{-t}\mathcal{E}[\hat{x}_{i,t}(\theta) - \hat{x}_{i,\infty}(\theta) \mid \mathcal{F}_t] \end{aligned}$$

This implements the same stopping time as mechanism 2 since, agent i's ex–post utility is

$$\begin{aligned} u_i(\theta, z_i, \tau_i) &= \mathcal{E}[(v_i(\theta) - c_{i,\tau_i})\delta^{\tau_i} - \hat{x}_{i,\tau_i}] \\ &= \mathcal{E}[(v_i(\theta) - c_{i,\tau_i})\delta^{\tau_i} - \hat{x}_{i,\tau_i} - \hat{x}_{i,\infty}] + \mathcal{E}[\hat{x}_{i,\infty}] \\ &= \mathcal{E}[(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i})\delta^{\tau_i}] + y_i(\theta) \end{aligned}$$

The first inequality adds and subtracts $\hat{x}_{i,\infty}$. The second equality comes from the definition of $z_{i,t}$ and conditioning $\hat{x}_{i,\infty}$ on $\mathcal{F}_{\tau_i}$, which we can do since this is the only part of $\hat{x}_{i,\infty}$ relevant for the stopping decision. $\square$

A.2 Necessary and Sufficient Conditions for Incentive Compatibility

Lemma 2. *The mechanism is incentive compatible if and only if equation (3.2) holds and* $U_i(\theta_i, \theta_i, \tau_i^*)$ *is supermodular in* $(\theta_i, \tilde{\theta}_i)$*.*

Proof. Necessity. By the envelope theorem, (IC) implies equation (3.2). (IC) also implies

$$U_i(\theta_i, \theta_i, \tau_i^*) - U_i(\theta_i, \tilde{\theta}_i, \tau_i^*) \geq 0 \geq U_i(\tilde{\theta}_i, \theta_i, \tau_i^*) - U_i(\tilde{\theta}_i, \tilde{\theta}_i, \tau_i^*)$$

which states $U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$ is supermodular in $(\theta_i, \tilde{\theta}_i)$.

Sufficiency. Let $\theta_i \geq \tilde{\theta}_i$ without loss. From equation (3.2)

$$\begin{aligned} U_i(\theta_i, \theta_i, \tau_i^*) &= U_i(\tilde{\theta}_i, \tilde{\theta}_i, \tau_i^*) + \int_{\tilde{\theta}_i}^{\theta_i} \frac{\partial}{\partial \theta_i} U_i(s, s, \tau_i^*)\, ds \\ &\geq U_i(\tilde{\theta}_i, \tilde{\theta}_i, \tau_i^*) + \int_{\tilde{\theta}_i}^{\theta_i} \frac{\partial}{\partial \theta_i} U_i(s, \tilde{\theta}_i, \tau_i^*)\, ds \\ &= U_i(\tilde{\theta}_i, \tilde{\theta}_i, \tau_i^*) + \left[U_i(\theta_i, \tilde{\theta}_i, \tau_i^*) - U_i(\tilde{\theta}_i, \tilde{\theta}_i, \tau_i^*) \right] \\ &= U_i(\theta_i, \tilde{\theta}_i, \tau_i^*) \end{aligned}$$

where the inequality comes from the supermodularity of $U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$ in $(\theta_i, \tilde{\theta}_i)$. □

A.3 Proof of Theorem 1

(a) Contingent payments. Setting $z = 0$ implies

$$\begin{aligned} \tau_i^* &\in \arg\max \mathcal{E}\left[(v_i(\theta) - c_{i,\tau_i} - z_{i,\tau_i}^W(\theta)) \delta^{\tau_i} \right] \\ &\in \arg\max E_\theta \left[\sum_i P_i(\theta) \mathcal{E}\left[(v_i - c_{i,\tau_i}) \delta^{\tau_i} - v_0 \right] \right] + \mathcal{E}[v_0] \end{aligned}$$

so when agents choose τ_i to maximise ex–post utility (2.1) they will also maximise welfare (3.4).

(b) Allocation rule. The option should be awarded to the agent who has the largest addition to welfare, subject to this value exceeding the seller's valuation, yielding equation (3.5).

(c) Finally, we need to check the monotonicity condition (3.3). This is done through two claims.

Claim 1: $P_i^W(\tilde{\theta}_i, \theta_{-i})$ is increasing in $\tilde{\theta}_i$.

Proof: In the welfare–maximising auction ex–post utility is

$$u_i(\theta, z_i^W, \tau_i^*) = \mathcal{E}\left[(v_i(\theta) - c_{i,\tau_i^*}) \delta^{\tau_i^*} \right]$$

Let us establish two facts.

(1) $u_i(\theta, z_i^W, \tau_i^*)$ is increasing in θ_i. The envelope theorem implies

$$u_i(\theta_i\theta_{-i}, z_i^W, \tau_i^*) = u_i(\underline{\theta}\theta_{-i}, z_i^W, \tau_i^*) + \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial \theta_i} v_i(s, \theta_{-i}) \mathcal{E}[\delta^{\tau_i^*(s,\theta_{-i})}]\, ds$$

and we are done since $v_i(\theta)$ is increasing in θ_i

(2) $u_i(\theta, z_i^W, \tau_i^*) - u_j(\theta, z_j^W, \tau_j^*)$ is quasi–increasing in θ_i. Applying the envelope theorem to $u_i(\theta, z_j^W, \tau_j^*)$,

$$u_j(\theta_i\theta_{-i}, z_j^W, \tau_j^*) u_j(\underline{\theta}, \theta_{-i}, z_j^W, \tau_j^*) + \int_{\underline{\theta}}^{\theta_i} \frac{\partial}{\partial \theta_i} v_j(s, \theta_{-i}) \mathcal{E}\left[\delta^{\tau_j^*(s,\theta_{-i})}\right]\, ds$$

If agents have private values then $u_j(\theta, z_j^W, \tau_j^*)$ is constant in θ_i, and (1) implies (2). Next suppose all types have the same distribution of costs and $u_i(\theta, z_i^W, \tau_i^*) = u_j(\theta, z_j^W, \tau_j^*) > 0$. Then $v_i(\theta) = v_j(\theta)$ and $\mathcal{E}[\delta^{\tau_i^*}] = \mathcal{E}[\delta^{\tau_j^*}]$. The assumption that $\frac{\partial}{\partial\theta_i}[v_i(\theta) - v_j(\theta)] \geq 0$ means $\frac{\partial}{\partial\theta_i}[u_i(\theta, z_i^W, \tau_i^*) - u_j(\theta, z_j^W, \tau_j^*)] \geq 0$ which implies (2).

Claim 2: $U_i(\theta_i, \tilde{\theta}_i, \tau_i^*)$ is supermodular in $(\theta_i, \tilde{\theta}_i)$.

Proof: The above claim implies $\frac{\partial}{\partial\theta_i} U_i(\theta_i, \tilde{\theta}_i, \tau_i^*) = E_{\theta_{-i}}\left[P_i^W(\tilde{\theta}_i, \theta_{-i}) \mathcal{E}[\delta^{\tau_i^*}] \frac{\partial}{\partial\theta_i} v_i(\theta)\right]$ is increasing in $\tilde{\theta}_i$, as required. □

A.4 Derivation of Example 1

Let $P_A(\theta_A, \theta_B)$ be the probability that A is allocated the contract, and denote the difference is expected utility by $\Delta = \max\{\theta_A + \theta_B - c, 0\} - [\theta_A + \theta_B]$. Welfare maximisation is equivalent to maximising, $E_\theta P_A(\theta_A, \theta_B)\mathcal{E}[\Delta]$, subject to the monotonicity condition (3.3). Notice $\mathcal{E}[\Delta]$ is decreasing in θ_A, while monotonicity implies $P_A(\theta_A, \theta_B)$ is increasing in θ_A. Consequently, $P_A(\theta_A, \theta_B)$ will be independent of θ_A in the welfare maximising auction.

The principal should set $z_A = 0$ so that the choice of stopping time maximises $\mathcal{E}[\Delta]$. Welfare is thus maximised by allocating the contract to B when $E_{\theta_A}\mathcal{E}[\Delta] < 0$ which can be rewritten as (3.6). Since $\mathcal{E}[\Delta]$ is decreasing in θ_B, B's monotonicity condition is satisfied. □

A.5 State Dependent Allocation and Auction Timing

Suppose in a given state the welfare maximising mechanism, $\sigma^W(\theta)$, awards the good to agent i in period s. Since this is welfare maximising we know

$$(v_i(\theta) - c_{i,s})\delta^s = \max_{\sigma_j(\theta) \geq s} \sum_j \mathcal{E}[(v_j(\theta) - c_{j,\sigma_j(\theta)})\delta^{\sigma_j(\theta)} \mid \mathcal{F}_s] \tag{A.1}$$

Now consider starting the mechanism in Theorem 1 at time s. Equation (A.1) has two implications. First, agent i will win the auction since, $(\forall j \neq i)$,

$$(v_i(\theta) - c_{i,s})\delta^s \geq \max_{\sigma_j(\theta)\geq s} \mathcal{E}[(v_j(\theta) - c_{j,\sigma_j(\theta)})\delta^{\sigma_j(\theta)} \mid \mathcal{F}_s]$$

Second, agent i will execute immediately since

$$(v_i(\theta) - c_{i,s})\delta^s \geq \max_{\sigma_i(\theta)\geq s} \mathcal{E}[(v_i(\theta) - c_{i,\sigma_i(\theta)})\delta^{\sigma_i(\theta)} \mid \mathcal{F}_s]$$

Thus we implement the optimal allocation under the state dependent allocation function. $\square$

A.6 Vickrey Auction

To show that truthful revelation is a weakly dominant strategy under the Vickrey auction suppose agents $j \neq i$ pretend to have type $\tilde{\theta}_j$ and costs $\tilde{c}_{j,t}$. Suppose agent i lies about their information and induces allocation $\sigma'(\theta)$. Their benefit from doing this at time t is the induced change in expected welfare,

$$\sum_{j=1}^{N} \mathcal{E}\left[(\tilde{\theta}_j - \tilde{c}_{j,\sigma_j^W})\delta^{\sigma_j^W} - (\tilde{\theta}_j - \tilde{c}_{j,\sigma_j'})\delta^{\sigma_j'} \mid \mathcal{F}_t\right]$$

This is negative by the definition of the welfare maximising mechanism, $\sigma^W(\theta)$. $\square$

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