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REPUTATION FOR QUALITY

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We propose a model of firm reputation in which a firm can invest or disinvest in product quality and the firm's reputation is defined as the market's belief about this quality. We analyze the relationship between a firm's reputation and its investment incentives, and derive implications for reputational dynamics.

Reputational incentives depend on the specification of market learning. When consumers learn about quality through perfect good news signals, incentives decrease in reputation and there is a unique work–shirk equilibrium with ergodic dynamics. When learning is through perfect bad news signals, incentives increase in reputation and there is a continuum of shirk–work equilibria with path-dependent dynamics. For a class of imperfect Poisson learning processes and low investment costs, we show that there exists a work–shirk equilibrium with ergodic dynamics. For a subclass of these learning processes, any equilibrium must feature working at all low and intermediate levels of reputation and shirking at the top.

KEYWORDS: Reputation, monitoring processes, firm dynamics, investment.

1. INTRODUCTION

IN MOST INDUSTRIES, firms can invest in the quality of their products through human capital investment, research and development, and organizational change. Often these investments and the resulting quality are imperfectly observable, giving rise to a moral hazard problem. The firm then invests so as to build a reputation for quality, justifying premium prices in the future. This paper analyzes the investment incentives in such a market, examining how they depend on the current reputation of the firm and the information structure of market observations.

The key innovation of our paper is to adopt a capital-theoretic approach to modeling both quality and reputation. Specifically, our model considers a firm that can invest or disinvest in product quality and whose revenue depends on its reputation, which is defined as the market's belief about this quality. This setting seems realistic in many contexts: car makers engage in research

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and development to increase safety, with customers updating from news reports; electronics firms organize production to increase reliability, with buyers learning through product reviews; academics invest in their human capital, with employers inferring their ability from publications. In traditional models of reputation, a firm's reputation is the market's belief about some exogenous type, and the firm exerts effort to signal this type, for example, Kreps, Milgrom, Roberts, and Wilson (1982). In our model, the firm's quality serves as an endogenous type and the firm invests to actually build this type. As the firm's quality is persistent, the effects of investment are long lasting and firm revenue is less sensitive to the market's belief about its actions than in traditional reputation models. Hence investment today yields rewards tomorrow, even if the firm is believed to be shirking tomorrow.

The model gives rise to simple Markovian equilibria in which investment incentives are determined by the present value of future reputational dividends. We investigate these incentives for a class of Poisson learning processes and determine when a firm builds a reputation, when it invests to maintain its reputation, and when it chooses to deplete its reputation. Our results suggest that incentives for academics, where the market learns through good news events like publications, are different from the incentives for clinical doctors, where the market learns through bad news events like malpractice suits.[2]

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of the firm's past investments. More specifically, *technology shocks* arrive with Poisson intensity λ. If there is no technology shock at time t, the firm's quality at time t is the same as its quality at time $t - dt$; if there is a technology shock at time t, then the firm's quality becomes high if it is investing/working and low if it is disinvesting/shirking. Thus, quality at time t is an additively separable function of quality at $t - dt$ and investment at time t. Consumers' expected utility is determined by the firm's quality, so their willingness to pay equals the market belief that quality is high; we call this belief the *reputation* of the firm and denote it by x_t.

Consumers observe neither quality nor investment directly, but learn about the firm's quality through *signals* with Poisson arrival times. A signal arrives at rate μ_H if the firm has high quality and rate μ_L if the firm has low quality. If the net arrival rate $\mu := \mu_H - \mu_L$ is positive, the signal is *good news* and indicates high quality; if μ is negative, the signal is *bad news* and indicates low quality. Market learning is *imperfect* if the signal does not reveal the firm's quality perfectly. Together with the market's belief about firm investment, the presence and absence of signals determines the evolution of the firm's reputation.

[2]Other examples for good news learning include the bio-tech industry when a trial succeeds and actors when they win an Oscar. Other examples for bad news learning include the computer industry when batteries explode and the financial sector when a borrower defaults. MacLeod (2007) coined the terms "normal goods" for experience goods that are subject to bad news learning and "innovative goods" for experience goods that are subject to good news learning.

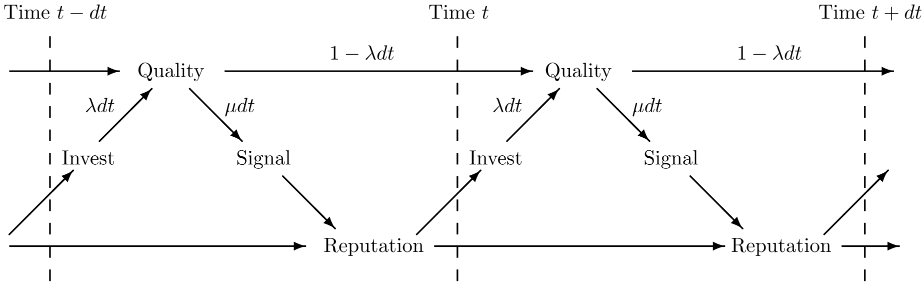


FIGURE 1.—Timeline. Quality is persistent and depends stochastically on past investments. The market learns about the firm's quality through Poisson signals. Reputation then evolves as a function of market learning and equilibrium beliefs about the firm's investments. As quality is persistent, today's investment affects all future signals, generating a stream of reputational dividends.

In Section 3 we characterize the firm's investment incentives. By investing, the firm raises its quality in the event of a technology shock; investment incentives are thus governed by the difference in value between a high and a low quality firm, which we term the *value of quality*. As quality is persistent and controls the arrival of future signals, we can represent the value of quality as a present value of future *reputational dividends*. This representation differs from traditional reputation models in which an increase in effort leads to a one-time boost in reputation.

In Section 4, we characterize equilibria under perfect Poisson learning. For perfect good news, where high quality gives rise to product *breakthroughs* that boost its reputation to $x = 1$, reputational dividends and investment incentives decrease in the firm's reputation. Equilibrium therefore must be *work–shirk* in that the firm works when its reputation lies below some cutoff x^* and shirks above this cutoff. Intuitively, a firm with low reputation has stronger incentives to invest in quality because it benefits more from the reputational boost due to a breakthrough. Reputational dynamics are ergodic in a work–shirk equilibrium because believed investment induces an upward drift for low reputations and a downward drift for high reputations. When technology shocks are sufficiently frequent, this work–shirk equilibrium is unique.

For perfect bad news signals, where high quality insures the firm against product *breakdowns* that destroy its reputation, reputational dividends and investment incentives increase in the firm's reputation. Equilibrium therefore must be *shirk–work* in that the firm works when its reputation lies above some cutoff x^* and shirks below this cutoff. Intuitively, a firm with high reputation has a stronger incentive to invest in quality because it has more to lose from the collapse in reputation following a breakdown. Reputational dynamics are path-dependent in a shirk–work equilibrium because the firm's reputation never recovers after a breakdown, while a high quality firm that keeps investing can avoid a breakdown with certainty. There may be a continuum of shirk–work equilibria: the multiplicity is caused by the divergent reputational drift around the cutoff that creates a discontinuity in the value functions and investment incentives. Intuitively, the favorable beliefs above the shirk–work cutoff give the firm strong incentives to invest and protect its appreciating reputation, creating a self-fulfilling prophecy.

In Section 5, we analyze imperfect Poisson learning processes when the cost of investment is low. Bayesian learning based on imperfect signals ceases for extreme reputations and reputational dividends tend to be hump-shaped. While this suggests a shirk–work–shirk equilibrium, we surprisingly show the existence of a work–shirk equilibrium. The work–shirk result relies on a fundamental asymmetry between investment incentives at high and low reputations. For $x \approx 1$, work is not sustainable: If the firm is believed to work, its reputation stays high and reputational dividends stay small, undermining incentives to actually invest. For $x \approx 0$, work is sustainable: If the firm is believed to work, its reputation drifts up and reputational dividends increase, generating incentives to invest. Crucially, a firm with a low reputation works not because of the small immediate reputational dividends, but because of the larger future dividends it expects after its reputation has drifted up. Thus, the work–shirk incentives are sustained by a combination of persistent quality and endogenous reputational drift.

If learning is such that the firm's reputation may rise even when it is believed to be shirking, then any equilibrium must feature work at low and intermediate levels of reputation and shirking at the top. This condition, that we call HOPE, is satisfied if learning is either via good news or if learning is via bad news and signals are sufficiently frequent. Under HOPE, putative shirk–work–shirk equilibria unravel as the favorable beliefs in the work region guarantee high investment incentives for a firm in a neighborhood of the shirk–work cutoff. In contrast, if HOPE is not satisfied, adverse beliefs below a shirk–work cutoff are self-fulfilling and can support a shirk region at the bottom.

1.1. *Literature*

The key feature that distinguishes our paper from traditional models of reputation and models of repeated games is that product quality is a function of past investments rather than current effort. This difference is important. In classical models, the firm exerts effort to convince the market that it will also exert effort in the future. In our model, a firm's investment increases its quality; since quality is persistent, this increases future revenue even if the firm is believed to be shirking in the future.

The two reputation models closest to ours are Mailath and Samuelson (2001) and Holmström (1999), both of which model reputation as the market's belief about an exogenous state variable. The mechanisms that link effort, types, and signals are illustrated in Figure 2. In Mailath and Samuelson (2001), a competent firm can choose to work so as to distinguish itself from an incompetent firm

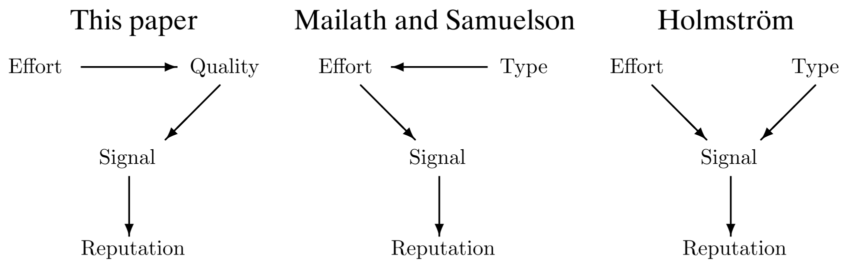


FIGURE 2.—Comparison of reputation models. The literature generally models reputation as belief over some exogenous type. This type affects consumer utility either directly, as in Holmström (1999), or indirectly through the cost of effort, as in Mailath and Samuelson (2001). In contrast, our firm controls its endogenous type through its investment.

that always shirks. A reputation for competence benefits the firm to the degree that the market expects a competent firm to work. With imperfect monitoring, a firm with a high reputation shirks because updating is slow; effort then unravels from the top, since a firm just below a putative work–shirk cutoff finds it unprofitable to further invest in its reputation. In our model, persistent quality prevents this unraveling because current investment affects the firm's future reputation and revenue, regardless of beliefs about its future investments.

Holmström's (1999) signal-jamming model is similar to ours in that the firm's type directly affects consumers' utility. In his model, the firm works to induce erroneous market beliefs that its exogenous ability type is higher than in reality. This is in contrast to our model, where a firm invests to actually improve its endogenous quality type.

In both of these papers, learning about a fixed type eventually vanishes, as do reputational incentives. These are instances of a more general theme: Cripps, Mailath, and Samuelson (2004) show that with imperfect monitoring and fixed types, reputation is a short-run phenomenon. Long-run relevance of reputation requires "some mechanism by which the uncertainty about types is continually replenished." Stochastic investment in quality, as featured in our paper, is a natural candidate for this mechanism. Unlike models of exogenous shocks (e.g., Mailath and Samuelson (2001), Holmström (1999)) in which reputation trails the shocks, the reputational dynamics of our model are endogenously determined by the forward-looking reputational incentives.[3]

There is a wider literature on life cycle effects in reputation models, as surveyed in Bar-Isaac and Tadelis (2008). Some of these results can be understood through our analysis of different learning processes: with perfect good news learning, firms with low reputation try to build or buy a reputation, as in Tadelis (1999). With perfect bad news learning, firms with high reputation have high

[3]Liu (2011) gave an alternative explanation of long-run reputational dynamics where uncertainty is replenished by imperfect, costly recall.

incentives to maintain them, as in Diamond (1989). With imperfect learning, reputational incentives are hump-shaped, as in Benabou and Laroque (1992) and Mailath and Samuelson (2001).

Compared to the repeated games literature (e.g., Fudenberg, Kreps, and Maskin (1990)), our model is distinguished by an evolving state variable. Investment directly feeds through to future reputation and revenue in our model, rather than preventing deliberate punishment by a counterparty.

Our model is related to other literatures. Rob and Fishman (2005) used a repeated game with imperfect monitoring to explain the dynamics of firm size. In the contract design literature, models with persistent effort have been studied by Fernandes and Phelan (2000) and Jarque (2010). The contrast between classical reputation models and our model is analogous to the difference between models of industry dynamics with exogenous types (Jovanovic (1982) and Hopenhayn (1992)) and those with endogenous capital accumulation (Ericson and Pakes (1995)). Finally, with its focus on Poisson signals, our paper is related to the strategic learning literature, for example, Keller, Rady, and Cripps (2005), Keller and Rady (2010), Klein and Rady (2011).

Our model has clear empirical predictions concerning reputational dynamics. While there is a growing empirical literature concerning reputation, surveyed in Bar-Isaac and Tadelis (2008), most of these papers are static and focus on quantifying the value of reputation. One notable exception is Cabral and Hortaçsu (2010), who showed that an eBay seller who receives negative feedback becomes more likely to receive additional negative feedback and is more likely to exit. This is consistent with our bad news case, where a seller who receives negative feedback stops investing.

2. MODEL

Players and actions: There is one firm and a competitive market of identical consumers. Time $t \in [0, \infty)$ is continuous. At every time t, the firm chooses an *investment* level $a_t \in [0, 1]$ and sells one unit of its product to consumers. Following Holmström (1999) and Mailath and Samuelson (2001), we do not model consumers' actions explicitly, but rather assume directly that at any time t they purchase the firm's single unit of output at a price equal to expected quality (see below).

At time t, the firm's product *quality* is $\theta_t \in \{L, H\}$, where $L = 0$ and $H = 1$. Initial quality θ_0 is exogenous; subsequent quality depends on investment and technology shocks. Specifically, shocks are generated according to a Poisson process with arrival rate $\lambda > 0$. Quality θ_t is constant between shocks and is determined by the firm's investment at the most recent technology shock $s \leq t$; that is, $\theta_t = \theta_s$ and $\Pr(\theta_s = H) = a_s$.

Information: Consumers observe neither quality nor investment, but learn about quality through public signals. Given quality θ, public signals are generated according to a Poisson process with arrival rate μ_θ.[4] A public history h^t at time t consists of a sequence of past signal arrival times $0 \leq t_1 \leq \cdots \leq t_n \leq t$; we write h for infinite histories and $\varnothing$ for the history with no signal. We say that learning is via *good news* if the net arrival rate $\mu := \mu_H - \mu_L$ is strictly positive, *perfect good news* if $\mu_H > \mu_L = 0$, *bad news* if $\mu < 0$, and *perfect bad news* if $\mu_L > \mu_H = 0$. Learning is *imperfect* if $\mu_L, \mu_H > 0$ and $\mu \neq 0$.

In addition to public signals, the firm observes product quality and chooses an investment plan $a = \{a_t\}_{t\geq 0}$ that is predictable with respect to the associated filtration. Intuitively, the firm conditions investment on realizations of quality $\theta^{t-} := \{\theta_s\}_{s\in[0,t)}$ and public signals h^{t-} up to but not including time t, $a_t = a_t(\theta^{t-}, h^{t-})$. The market's belief about the firm's investment $\tilde{a} = \{\tilde{a}_t\}_{t\geq 0}$ is predictable with respect to public histories; intuitively, $\tilde{a}_t = \tilde{a}_t(h^{t-})$.

From the markets' perspective, believed investment $\tilde{a}$ and the exogenous initial belief about quality $x_0 \in [0, 1]$ control the joint distribution of quality $\{\theta_t\}_{t\geq 0}$ and histories h; we write $\mathbb{E}^{\tilde{a},x_0}$ for expectations under this measure. The market's belief about product quality at time t is called the firm's *reputation* and is denoted by $x_t = x_t(x_0, h, \tilde{a}) := \mathbb{E}^{\tilde{a},x_0}[\theta_t|h^t]$. We will also use two other probability measures in the paper. From the firm's perspective, actual investment a and initial quality θ_0 control the distribution over $\{\theta_t\}_{t\geq 0}$ and h; we write $\mathbb{E}^{a,\theta_0}$ for the firm's expectation. Finally, we write $\mathbb{E}^{\theta}$ for the expectation over histories h when signals arrive at constant rate μ_θ.

Payoffs: The firm and consumers are risk-neutral and discount future payoffs at rate $r > 0$. At time t, the firm produces one unit with flow value θ_t. Given the public information h^{t-}, consumers' willingness to pay is then given by $\mathbb{E}^{\tilde{a},x_0}[\theta_t|h^{t-}]$; this equals the firm's reputation x_t as long as no signal arrives at t. We assume that the price equals the willingness to pay, so consumers' expected utility is 0. Investment has a marginal flow cost of $c > 0$. The firm's flow profits are thus given by $x_t - ca_t$ for almost all t.

Given the firm's investment strategy $a = \{a_t\}_{t\geq 0}$ and the market's belief about this strategy $\tilde{a} = \{\tilde{a}_t\}_{t\geq 0}$, the firm's expected present value equals

$$\mathbb{E}^{a,\theta_0}\left[\int_{t=0}^{\infty} e^{-rt}(x_t - ca_t)\,dt\right].$$

The believed investment process $\tilde{a}_t = \tilde{a}_t(h^{t-})$ determines the firm's revenue $x_t = x_t(x_0, h, \tilde{a}) = \mathbb{E}^{\tilde{a},x_0}[\theta_t|h^t]$ for a given history h, while the actual investment

[4]Formally, we consider a probability space $(\Omega, \mathcal{F}, P)$ together with the following independent random variables: (i) an $\{L, H\}$-valued random variable that determines initial quality θ_0 with $\Pr(\theta_0 = H) =: x_0$; (ii) a sequence of uniform $[0, 1]$ random variables whose realizations control quality at the technology shocks (jointly with investment at those times); (iii) three Poisson processes—the technology shock process with rate λ and two processes that generate unobservable "θ events" with rates μ_θ for $\theta \in \{L, H\}$. An observable public signal realizes at time t if a θ_t event arrives at time t.

process $a_t = a_t(\theta^{t-}, h^{t-})$ determines the distribution over quality $\{\theta_t\}_{t\geq 0}$ and histories h.[5]

2.1. *Markov Perfect Equilibrium*

To define equilibrium in a concise way, we first introduce some convenient notions that pertain to reputational dynamics and analyze optimal investment.

Reputational dynamics: We assume that market beliefs about investment are Markovian and depend on calendar time and public history only via the left-sided limit of reputation $x_{t-} = \lim_{\varepsilon\to 0} x_{t-\varepsilon}$.[6] That is, there exists a function $\tilde{A} : [0, 1] \to [0, 1]$ such that $\tilde{a}_t = \tilde{A}(x_{t-})$. To analyze the trajectory of reputation $\{x_t\}_{t\geq 0}$, suppose $\tilde{A}$ is continuous at $x_t \in [0, 1]$. If no signal arrives in $[t, t + \delta]$, then Bayes' rule implies

$$x_{t+\delta} = \lambda\delta\tilde{A}(x_t) + (1 - \lambda\delta)\frac{x_t(1 - \mu_H\delta)}{x_t(1 - \mu_H\delta) + (1 - x_t)(1 - \mu_L\delta)} + o(\delta). \tag{2.1}$$

The term $\lambda\delta\tilde{A}(x_t)$ reflects the possibility that quality switches in $[t, t + \delta]$ and is newly determined based on $\tilde{A}(x_t)$. The second term reflects market learning about quality based on the absence of a signal in $[t, t+\delta]$. In the limit as $\delta \to 0$, reputation x_t (in the absence of signals) is then governed by the autonomous ordinary differential equation (ODE) $\dot{x} = g(x)$, where *reputational drift* is given by

$$g(x) = \lambda\big(\tilde{A}(x) - x\big) - \mu x(1 - x). \tag{2.2}$$

We next impose an assumption on beliefs $\tilde{A}$ to ensure that $\dot{x} = g(x)$ admits a solution. Inspired by Klein and Rady (2011), we say that $\tilde{A}$ is *admissible at* $x^* \in [0, 1]$ if $\tilde{A}$ and the associated drift g satisfy one of the following three conditions

(a) $g(x^*) = 0$

(b) $g(x^*) > 0$ and $\tilde{A}$ (and therefore g) is right-continuous at x^*[7]

(c) $g(x^*) < 0$ and $\tilde{A}$ (and therefore g) is left-continuous at x^*.

[5] Below, we restrict attention to investment functions that are independent of realized quality. Then x_t and a_t only depend on histories h and we reinterpret $\mathbb{E}^{a,\theta_0}$ as expectation over h with respect to the marginal distribution of h.

[6] By definition, $x_{0-} = x_0$; note that $x_{t-} = x_t$ for almost all t.

[7] Note that $g(x^*) > 0$ implies $x^* < 1$, so right continuity at x^* is well defined. Similarly, $g(x^*) < 0$ implies $x^* > 0$.

We call $\tilde{A}$ *admissible* if there is a finite number of cutoffs x_i^* with $0 \leq x_1^* < \cdots < x_n^* \leq 1$, such that $\tilde{A}$ is Lipschitz-continuous on any interval $[0, x_1^*), \dots, (x_i^*, x_{i+1}^*), \dots, (x_n^*, 1]$ and $\tilde{A}$ is admissible at every cutoff.[8]

For admissible beliefs $\tilde{A}$, we show in Appendix A.1 that for any initial reputation x_0, there exists a solution to the ODE $\dot{x} = g(x)$, that is, a trajectory $\{x_t^\varnothing\}_{t\geq 0}$ such that $s \mapsto g(x_s^\varnothing)$ is Riemann-integrable and $x_t^\varnothing = x_0 + \int_0^t g(x_s^\varnothing)\,ds$ for all $t \geq 0$; the superscript $\varnothing$ indicates a history with no signals, so we have $x_t^\varnothing = x_t(x_0, \varnothing, \tilde{A})$. If there are multiple solutions, we select the unique solution $\{x_t^\varnothing\}_{t\geq 0}$ that is consistent with a discrete-time approximation. We also show in the Appendix that believed investment as a function of time in the absence of signals, $t \mapsto \tilde{A}(x_t^\varnothing)$, is right-continuous.[9]

If there is a signal at time t, then reputation jumps from the limit of the reputation before the jump x_{t-} to $x_t = j(x_{t-})$, where the jump function j is given by Bayes' rule

$$j(x) := \frac{\mu_H x}{\mu_H x + \mu_L(1-x)} = x + \frac{\mu x(1-x)}{\mu_H x + \mu_L(1-x)}. \tag{2.3}$$

With good news, the signal indicates high quality and $j(x) > x$ for $x < 1$; with bad news, we have $j(x) < x$ for $x > 0$; with perfect good (resp. bad) news, we have $j(x) = 1$ (resp. $j(x) = 0$) for any $x \in [0, 1]$.

Optimal investment: Given beliefs $\tilde{A}$, we can write the firm's continuation value at time t as a function of its current reputation and quality:

$$V_{\theta_t}(x_t) = \sup_{a=\{a_s\}_{s\geq t}} \mathbb{E}^{a,\theta_t}\left[\int_{s=t}^{\infty} e^{-r(s-t)}(x_s - ca_s)\,ds\right]. \tag{2.4}$$

Let $D(x) = V_H(x) - V_L(x)$ be the *value of quality*. Truncating the integral expression (2.4) for $V_{\theta_0}(x_0)$ at the first arrival of a technology shock, we get

$$\begin{aligned} V_{\theta_0}(x_0) &= \sup_a \mathbb{E}^{\theta_0}\left[\int_0^\infty e^{-(r+\lambda)t}\left[x_t - ca_t + \lambda\big(a_t V_H(x_t) + (1-a_t)V_L(x_t)\big)\right]dt\right] \\ &= \sup_a \mathbb{E}^{\theta_0}\left[\int_0^\infty e^{-(r+\lambda)t}\left[x_t + a_t\big(\lambda D(x_t) - c\big) + \lambda V_L(x_t)\right]dt\right], \end{aligned} \tag{2.5}$$

[8]We view admissibility as a mild restriction on beliefs $\tilde{A}$. Specifically, we show in Section 3.3 that any cutoff $x^* \in (0, 1)$ is compatible with admissibility in a sense made precise in that section.

[9]This definition of admissibility differs from Klein and Rady (2011) in two respects. First, our admissibility condition pertains to believed investment $\tilde{A}$ rather than actual investment a. Second, we impose explicit conditions on the function $\tilde{A} : [0, 1] \to [0, 1]$ that ensure not only that $\dot{x} = g(x)$ admits a solution, but also that $t \mapsto \tilde{A}(x_t^\varnothing)$ is right-continuous, which we will use in the proof of Lemma 1.

where $\mathbb{E}^\theta$ is the expectation over histories when signals arrive at constant rate μ_θ.

Now consider a strategy $\{a_t\}_{t\geq 0}$ that satisfies $a_t = 1$ if $\lambda D(x_{t-}) > c$ and $a_t = 0$ if $\lambda D(x_{t-}) < c$. For any history h, such a strategy maximizes the integrand in (2.5) pointwise for all t with $x_{t-} = x_t$, that is, all t except at the discrete signal arrival times. Hence, any measurable function $A : [0, 1] \to [0, 1]$ that satisfies

$$(2.6)\qquad A(x) = \begin{cases} 1, & \text{if } \lambda D(x) > c, \\ 0, & \text{if } \lambda D(x) < c, \end{cases} \qquad \text{for all } x \in [0, 1]$$

defines an optimal strategy $\{A(x_{t-})\}_{t\geq 0}$. Thus, for any admissible $\tilde{A}$, there exists a quality-independent Markovian strategy A that maximizes firm value; we simplify our analysis by restricting attention to equilibria in such strategies.

Intuitively, condition (2.6) compares the marginal cost and benefit of investment. The marginal cost of investment over $[t, t + dt]$ is $c\,dt$. The marginal benefit of investment equals the probability $\lambda\,dt$ of a technology shock times the gain from investing in this event $D(x)$, that is, the net value of having high rather than low quality at $t + dt$.

Markov perfect equilibrium: A Markov perfect equilibrium $\langle A, \tilde{A}\rangle$, or simply *equilibrium*, consists of a quality-independent Markovian investment function $A : [0, 1] \to [0, 1]$ for the firm and admissible Markovian market beliefs about investment $\tilde{A} : [0, 1] \to [0, 1]$ such that: (a) given beliefs $\tilde{A}$, investment A maximizes firm value (2.4) for all $x_0 \in [0, 1]$ and $\theta_0 \in \{L, H\}$; and (b) beliefs are correct, $\tilde{A} = A$.

We next show that condition (2.6) characterizes Markov perfect equilibria.

LEMMA 1: *Fix admissible beliefs $\tilde{A}$. Then $\langle A, \tilde{A}\rangle$ is a Markov perfect equilibrium if and only if $\tilde{A} = A$ and condition* (2.6) *is satisfied.*

PROOF: The "if" part follows by the paragraphs preceding condition (2.6). For the "only if" part, we will establish that the functions $t \mapsto V_\theta(x_t^\varnothing)$ and $t \mapsto A(x_t^\varnothing)$ are right-continuous for any x_0 and t. Then if A violates (2.6) at some x_0 with, for example, $\lambda D(x_0) > c$ but $A(x_0) = 0$, we have $\lambda D(x_t^\varnothing) > c$ and $A(x_t^\varnothing) < 1$ at all $t \in [0, \delta]$ for some $\delta > 0$. Such a strategy is not optimal because the firm can strictly increase profits in (2.5) by setting $A(x) = 1$ for $x \in \{x_t^\varnothing : t \in [0, \delta]\}$.

To see that $t \mapsto A(x_t^\varnothing)$ is right-continuous, it suffices to show that $t \mapsto \tilde{A}(x_t^\varnothing)$ is right-continuous because we have assumed correct beliefs, $\tilde{A} = A$. Right continuity of beliefs in turn follows directly from admissibility.

To see that $t \mapsto V_\theta(x_t^\varnothing)$ is Lipschitz-continuous (and a fortiori right-continuous), first note that value functions are bounded, $V_\theta(x) \in [-c/r, 1/r]$ for all θ, x, because the flow profit is bounded, $x_t - ca_t \in [-c, 1]$. Then we truncate the integral expression of $V_\theta(x_t^\varnothing)$ at the first technology shock, the first signal, and $t' > t$, whichever is earliest:

$$V_\theta(x_t^\varnothing) = \int_t^{t'} e^{-(r+\lambda+\mu_\theta)(s-t)}\big[x_s^\varnothing - A(x_s^\varnothing)c + \lambda\big(A(x_s^\varnothing)D(x_s^\varnothing) + V_L(x_s^\varnothing)\big) + \mu_\theta V_\theta\big(j(x_s^\varnothing)\big)\big]\,ds + e^{-(r+\mu_\theta+\lambda)(t'-t)}V_\theta(x_{t'}^\varnothing).$$

This establishes Lipschitz continuity of $t \mapsto V_\theta(x_t^\varnothing)$ because value functions are bounded, the integral is of order $t'-t$, and $e^{-(r+\mu_\theta+\lambda)(t'-t)} \in [1-(r+\mu_\theta+\lambda)(t'-t), 1]$. *Q.E.D.*

First-best solution: As a benchmark, suppose that product quality θ_t is publicly observed at time t, so that price equals quality. Then the benefit of investing equals the rate at which investment affects quality λ times the price differential of 1 divided by the effective discount rate $r + \lambda$. Thus, first-best investment is given by

$$a = \begin{cases} 1, & \text{if } c < \dfrac{\lambda}{r+\lambda}, \\ 0, & \text{if } c > \dfrac{\lambda}{r+\lambda}. \end{cases}$$

In our model, there is no equilibrium with positive investment if $c > \frac{\lambda}{r+\lambda}$: Investment decreases welfare and consumers receive zero utility in equilibrium, so firm profits would be negative. The firm, therefore, prefers to shirk and guarantee itself a nonnegative payoff. Our results are, therefore, nontrivial only if $c \leq \frac{\lambda}{r+\lambda}$.

Discussion: Several remarks are in order concerning our modeling assumptions. First, our model has an obvious analogue in discrete time (see Figure 1). However, simple properties like monotonicity of the value function (Lemma 2) may fail in discrete time: a low reputation firm that is believed to be investing may leapfrog over a high reputation firm that is believed not to be investing. It is, therefore, more convenient to work in continuous time.

Second, as optimal investment does not depend on quality, our results do not rely on the assumption that the firm knows its own quality. In a companion paper, Board and Meyer-ter-Vehn (2010b), we extend the current model to allow for entry and exit; as the firm's exit decision depends on its quality, the results in that paper depend on whether or not the firm knows its own quality.

Third, our model assumes that quality at time t is based on investment at the time of the last technology shock. One can interpret such investment as the choice of absorptive capacity, determining the ability of a firm to recognize and apply new external information (Cohen and Levinthal (1990)). However, two alternative model formulations with different interpretations give rise to the same value functions and investment incentives. First, in a Markovian spirit, we could assume that a low quality firm can purchase the arrival rate λa_t of a quality improvement while a high quality firm is abating the arrival rate $\lambda(1-a_t)$ of a quality deterioration. Second, modeling investment as lumps rather than flow, we could assume the firm observes the arrival of technology shocks, and then chooses whether to adopt the new technology at cost $k = c/\lambda$ to become high quality or to forgo the opportunity and become low quality.

3. PRELIMINARY ANALYSIS

In this section we derive expressions for the value of quality $D(x)$, establish monotonicity of the value functions $V_\theta(x)$, and analyze reputational dynamics.

3.1. *Value of Quality*

Investment incentives are determined by the value of quality $D(x)$, which in turn is determined by the persistent effects of quality on future reputation and revenue. We now show heuristically how to express $D(x)$ as the present value of future reputational dividends. This derivation assumes that value functions are smooth; the formal derivation is given in Appendix A.2.[10]

Fix beliefs $\tilde{A}$ and let A be an optimal investment strategy. Then value functions satisfy the system of ordinary differential-difference equations

$$\begin{aligned} rV_L(x) &= x - cA(x) \\ &\quad + \mu_L\big(V_L\big(j(x)\big) - V_L(x)\big) + V_L'(x)g(x) + \lambda A(x)D(x), \end{aligned} \tag{3.1}$$

$$\begin{aligned} rV_H(x) &= x - cA(x) \\ &\quad + \mu_H\big(V_H\big(j(x)\big) - V_H(x)\big) + V_H'(x)g(x) - \lambda\big(1 - A(x)\big)D(x). \end{aligned} \tag{3.2}$$

Intuitively, the interest on the firm's value $rV_\theta(x)$ equals the sum of its flow profits $x - cA(x)$ and the expected appreciation due to reputational jumps $\mu_\theta(V_\theta(j(x)) - V_\theta(x))$, reputational drift $V_\theta'(x)g(x)$, and changing quality; the latter equals $\lambda A(x)D(x)$ for a low quality firm and equals $-\lambda(1-A(x))D(x)$ for a high quality firm. Subtracting (3.1) from (3.2), we get

$$\begin{aligned} &(r+\lambda)D(x) \\ &\quad = \mu_H\big(V_H\big(j(x)\big) - V_H(x)\big) - \mu_L\big(V_L\big(j(x)\big) - V_L(x)\big) + D'(x)g(x) \\ &\quad = \mu\big(V_H\big(j(x)\big) - V_H(x)\big) + \mu_L\big(D\big(j(x)\big) - D(x)\big) + D'(x)g(x), \end{aligned} \tag{3.3}$$

[10] In general, value functions need not be smooth and may even be discontinuous (as we will see in Section 4.2). The proof of Theorem 1 shows how to interpret the term $V_\theta'(x)g(x)$ in equations (3.1) and (3.2) in that case.

where the second line adds and subtracts $\mu_L(V_H(j(x)) - V_H(x))$.

Intuitively, the interest on the value of quality $(r+\lambda)D(x)$—computed at rate $r+\lambda$ to account for quality obsolescence—equals the sum of appreciation due to jumps $\mu_L(D(j(x)) - D(x))$ and drift $D'(x)g(x)$, plus a flow payoff $\mu(V_H(j(x)) - V_H(x))$. We call this flow payoff the *reputational dividend* of quality because it represents the flow benefit of high quality; namely the incremental rate $\mu = \mu_H - \mu_L$ at which a signal arrives times the value of the reputational jump. If signals indicate good news, $\mu > 0$, the dividend is due to the increased probability of upward jumps in reputation; if signals indicate bad news, $\mu < 0$, the dividend is due to the decreased probability of downward jumps in reputation.

We then integrate (3.3) to express the value of quality as the present value of these reputational dividends in (3.4). The alternative expression, (3.5), follows from the analogue of (3.3) when we add and subtract $\mu_H(V_L(j(x)) - V_L(x))$ instead of $\mu_L(V_H(j(x)) - V_H(x))$.

THEOREM 1—Value of Quality: *For any admissible beliefs $\tilde{A}$, the value of quality is given by*

$$D(x_0) = \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\mu\left[V_H\big(j(x_t)\big) - V_H(x_t)\right]dt\right] \tag{3.4}$$

$$= \mathbb{E}^H\left[\int_0^\infty e^{-(r+\lambda)t}\mu\left[V_L\big(j(x_t)\big) - V_L(x_t)\right]dt\right]. \tag{3.5}$$

See Appendix A.2 for the proof.

While standard reputation models incentivize effort through an immediate effect on the firm's reputation, investment in our model pays off through quality with a delay. Once quality is established, it is persistent and generates a stream of reputational dividends until it becomes obsolete. Theorem 1 shows that we must accordingly evaluate reputational incentives at future levels of reputation x_t, rather than at the current level x_0.

Theorem 1 is the workhorse for our main results. Specifically, we use the drift and jump equations (2.2) and (2.3) to analyze reputational trajectories $\{x_t\}_{t\geq 0}$, then substitute these into value functions (2.4), and, finally, substitute value functions into (3.4) and (3.5) to analyze investment incentives. This method of directly analyzing path integrals stands in contrast to the standard Hamilton–Jacobi–Bellman (HJB) approach employed in recent continuous-time reputation models (Faingold and Sannikov (2011), Atkeson, Hellwig, and Ordonez (2012)), and continuous-time Poisson learning models (Keller and Rady (2010), Klein and Rady (2011)). This standard approach shows that the HJB equations (3.1), (3.2) together with appropriate boundary conditions admit a unique solution which corresponds to the firm's value function, and then uses the HJB equations to derive a closed-form solution or monotonicity/concavity properties. We have not adopted this approach because our system of coupled HJB equations has no clear boundary conditions when reputational drift is inward-pointing, and—even for constant beliefs—does not admit a closed-form solution.

3.2. *Properties of the Value Functions*

Here we show that firm value is increasing in reputation and quality, and we give sufficient conditions for continuity of the value functions. Reputation is valuable because it determines the firm's revenue; quality in turn is valuable because it determines future reputation. Specifically, Lemma 2 shows that $V_\theta(x)$ is strictly increasing in x. Thus, reputational dividends $\mu(V_\theta(j(x)) - V_\theta(x))$ are nonnegative, so by Theorem 1, the value of quality is nonnegative.

LEMMA 2: *For any admissible beliefs $\tilde{A}$, the firm's value function $V_\theta(x)$ is strictly increasing in reputation x.*

PROOF: Consider two hypothetical firms, "high" and "low," with the same initial quality θ but different initial levels of reputation $\hat{x}_0 > x_0$. We prove $V_\theta(\hat{x}_0) > V_\theta(x_0)$ by showing that the high firm can secure itself higher flow profits than the low firm by mimicking the low firm's optimal strategy.

More precisely, let $x_{t-}(x_0, h, \tilde{A})$ be the left-sided limit of the low firm's reputation in history h at time t, let $a_t = A(x_{t-}(x_0, h, \tilde{A}))$ be the low firm's optimal investment at this reputation, and assume that the high firm adopts investment strategy $a = \{a_t\}_{t\geq 0}$; note that this strategy is generally not Markovian with respect to the high firm's reputation $x_{t-}(\hat{x}_0, h, \tilde{A})$. When the high and low firms both follow strategy a, they have the same quality $\{\hat{\theta}_t\}_{t\geq 0} = \{\theta_t\}_{t\geq 0}$ for any realization of the technology shocks. Thus, they face the same distribution over signal histories h.

We now show that for any history $h = (t_1, \dots)$ the high firm's reputation never falls behind the low firm's reputation

$$x_t(\hat{x}_0, h, \tilde{A}) \geq x_t(x_0, h, \tilde{A}). \tag{3.6}$$

For $t = 0$, (3.6) holds by assumption as $\hat{x}_0 > x_0$. Before the first signal, for $t \in (0, t_1)$, (3.6) follows because both trajectories are smooth and are governed by the same law of motion $\dot{x} = g(x)$. At the first signal $t = t_1$, (3.6) follows by the monotonicity of the jump function j, defined in (2.3). Repeating the last two steps inductively on all intervals (t_i, t_{i+1}) and all signal arrival times t_i implies that (3.6) holds for all $t \geq 0$. Furthermore, there almost surely exists $\delta = \delta(h) > 0$ such that the inequality (3.6) is strict for all $t \in [0, \delta]$.[11]

[11] In particular, in the absence of signals, the inequality is strict for small t; we shall use this fact in the proofs of Theorems 2 and 3.

Thus, abbreviating $x_t = x_t(x_0, h, \tilde{A})$ and $\hat{x}_t = x_t(\hat{x}_0, h, \tilde{A})$, and writing $\hat{a} = \{\hat{a}_t\}_{t\geq 0}$ for the high firm's optimal strategy, we get

$$V_\theta(\hat{x}_0) = \mathbb{E}^{\hat{a},\theta}\left[\int_0^\infty e^{-rt}(\hat{x}_t - c\hat{a}_t)\,dt\right] \geq \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}(\hat{x}_t - ca_t)\,dt\right]$$
$$> \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}(x_t - ca_t)\,dt\right] = V_\theta(x_0)$$

as required. *Q.E.D.*

Value functions $V_\theta(x)$ are not generally continuous in x; discontinuities arise in equilibrium under perfect bad news learning. The next lemma shows that such discontinuities cannot arise at x if the reputational drift is positive (or negative) and bounded away from zero in a neighborhood of x.

LEMMA 3: *Fix admissible beliefs $\tilde{A}$, an interval $[\underline{x}, \overline{x}]$, and $\varepsilon > 0$ such that either $g(x) > \varepsilon$ for all $x \in [\underline{x}, \overline{x})$ or $g(x) < -\varepsilon$ for all $x \in (\underline{x}, \overline{x}]$. Then $V_\theta(x)$ is continuous on $[\underline{x}, \overline{x}]$.*

PROOF: If $g(x) > \varepsilon$ for all $x \in [\underline{x}, \overline{x})$, then the trajectory $t \mapsto x_t^\varnothing$ with $x_0 = \underline{x}$ induces a homeomorphism between some time interval $[0, \tau]$ and $[\underline{x}, \overline{x}]$. By the proof of Lemma 1, the function $t \mapsto V_\theta(x_t^\varnothing)$ is continuous, so the concatenation $x_t^\varnothing \mapsto t \mapsto V_\theta(x_t^\varnothing)$ is continuous on $[\underline{x}, \overline{x}]$. The proof for $g(x) < -\varepsilon$ is analogous. *Q.E.D.*

Lemma 3 has a useful implication: If drift g is continuous and nonzero at x, then $V_\theta(x)$ is continuous at x.

3.3. *Work–Shirk Terminology*

We now introduce some convenient terminology for reputational dynamics. Fix beliefs $\tilde{A} : [0, 1] \to [0, 1]$. We say that $x^* \in (0, 1)$ is a *work–shirk cutoff* if $\tilde{A}$ is equal to 1 on an interval below x^* and equal to 0 on an interval above x^*. Conversely, $x^* \in (0, 1)$ is a *shirk–work cutoff* if $\tilde{A}$ is equal to 0 on an interval below x^* and equal to 1 on an interval above x^*. At the boundary, $x^* = 0$ is a shirk–work cutoff if $\tilde{A}(0) = 0$ and $\tilde{A}$ is equal to 1 on an interval above $x^* = 0$. Similarly, $x^* = 1$ is a shirk–work cutoff if $\tilde{A}(1) = 1$ and $\tilde{A}$ is equal to 0 in an interval below $x^* = 1$.[12]

[12]There can be no extremal work–shirk cutoffs $x^* \in \{0, 1\}$ because $g(x^*) = \lambda(\tilde{A}(x^*) - x^*)$; hence admissibility implies $\tilde{A}(0) = 0$ if the firm is believed to shirk on an interval above $x^* = 0$ and implies $\tilde{A}(1) = 1$ if the firm is believed to work on an interval below $x^* = 1$.

To describe reputational dynamics at a cutoff $x^* \in (0,1)$, we write $g(x^*-) = \lim_{\varepsilon\to 0} g(x^* - \varepsilon)$ and $g(x^*+) = \lim_{\varepsilon\to 0} g(x^* + \varepsilon)$ for the left- and right-sided limits of reputational drift. We say that x^* is

$$\begin{array}{ll} \textit{convergent} & \text{if } g(x^*-) \geq 0 \geq g(x^*+), \text{ with at most one equality,} \\ \textit{permeable} & \text{if } g(x^*-), g(x^*+) > 0 \text{ or } g(x^*-), g(x^*+) < 0, \\ \textit{divergent} & \text{if } g(x^*-) \leq 0 \leq g(x^*+), \text{ with at most one equality.} \end{array}$$

A work–shirk cutoff $x^* \in (0,1)$ is either convergent or permeable because $g(x^*-) > g(x^*+)$; a shirk–work cutoff $x^* \in (0,1)$ is divergent or permeable because $g(x^*-) < g(x^*+)$.

Convergent, permeable, and divergent cutoffs give rise to qualitatively different reputational dynamics. To see this, assume that no signals arrive. If x^* is convergent, then any small neighborhood of x^* is absorbing; if x^* is permeable with positive (resp. negative) drift, then a trajectory originating just below x^* (resp. just above x^*) drifts through x^* in finite time; if x^* is divergent, then two reputational trajectories originating just below and just above x^* drift apart from each other. At the boundaries, we call a shirk–work cutoff $x^* = 0$ divergent since $g(0) = 0$, while the drift above 0 is strictly positive; similarly, we call a shirk–work cutoff $x^* = 1$ divergent since $g(1) = 0$, while the drift below 1 is strictly negative.

At a cutoff, believed investment $\tilde{A}$ is constrained by admissibility. At a convergent cutoff $x^* \in (0,1)$, the positive drift below x^* rules out $g(x^*) < 0$ and the negative drift above x^* rules out $g(x^*) > 0$. Thus, admissibility implies $g(x^*) = 0$, pinning down $\tilde{A}(x^*) = x^*(1 + \frac{\mu}{\lambda}(1 - x^*)) \in [0,1]$. At a permeable cutoff $x^* \in (0,1)$ with strictly positive drift, admissibility requires $\tilde{A}(x^*) = \tilde{A}(x^*+)$; at a permeable cutoff $x^* \in (0,1)$ with strictly negative drift, admissibility requires $\tilde{A}(x^*) = \tilde{A}(x^*-)$. At a divergent cutoff $x^* \in [0,1]$, multiple values for $\tilde{A}(x^*)$ are possible. Specifically, admissibility requires $\tilde{A}(x^*) \in \{0, x^*(1 + \frac{\mu}{\lambda}(1 - x^*)), 1\}$. These choices of $\tilde{A}(x^*)$ are sufficient as well as necessary for admissibility at x^*, so any work–shirk cutoff $x^* \in (0,1)$ and any shirk–work cutoff $x^* \in [0,1]$ is consistent with admissibility.

Believed investment $\tilde{A}$ is *work–shirk* if there exists a single work–shirk cutoff $x^* \in (0,1)$, so $\tilde{A} = 1$ on $[0, x^*)$ and $\tilde{A} = 0$ on $(x^*, 1]$. Conversely, $\tilde{A}$ is *shirk–work* if there exists a single shirk–work cutoff $x^* \in [0,1]$, so $\tilde{A} = 0$ on $[0, x^*)$ and $\tilde{A} = 1$ on $(x^*, 1]$. Finally, $\tilde{A}$ is *full work* if $\tilde{A} = 1$ and *full shirk* if $\tilde{A} = 0$. An equilibrium $\langle A, \tilde{A}\rangle$ is work–shirk (resp. shirk–work, full work, full shirk) if believed investment $\tilde{A}$ is work–shirk (resp. shirk–work, full work, full shirk).

3.4. *Ergodic Dynamics*

One goal of this paper is to analyze long-term reputational dynamics. In particular, we would like to understand whether the impact of initial reputation x_0 is permanent or transitory. We say that reputational dynamics for beliefs $\tilde{A}$ are *ergodic* if there exists a probability distribution F over $[0, 1]$ such that for any starting value x_0, the process $\{x_t\}_{t\geq 0}$ converges to F in distribution, that is, $\lim_{t\to\infty} F_t(x) = F(x)$ for all points of continuity x of F, where $F_t(x) := \mathbb{E}^{\tilde{A},x_0}[\mathbb{I}_{\{x_t\leq x\}}]$ is the distribution of reputation at time t from the market's perspective.[13] Otherwise, we say that reputational dynamics are *path-dependent.* If reputational dynamics are ergodic, the effect of initial reputation is transitory.

LEMMA 4: *For any admissible beliefs $\tilde{A}$ that are work–shirk, full work, or full shirk, reputational dynamics are ergodic.*

PROOF: Consider first work–shirk beliefs with cutoff x^*. Let

$$T = T(x_0, x^*) = \inf\{t > 0 : x_t = x^*, \exists t' \in [0, t) : x_{t'} \neq x^*\}$$

be the time at which $\{x_t\}_{t\geq 0}$ first hits/returns to x^*. In Appendix A.3, we show that $\mathbb{E}^{\tilde{A},x_0}[T(x_0, x^*)]$ is bounded for all x_0. Since the cycle length $T(x^*, x^*)$ is nonlattice,[14] Asmussen (2003, Chapter VI, Theorem 1.2) implies that for any starting value x_0, the process $\{x_t\}_{t\geq 0}$ converges in distribution to F, defined by

$$F(x) = \frac{1}{\mathbb{E}^{\tilde{A},x^*}[T(x^*, x^*)]}\mathbb{E}^{\tilde{A},x^*}\left[\int_0^{T(x^*,x^*)} \mathbb{I}_{\{x_t\leq x\}}\, dt\right]$$

as required. Intuitively, the convergent reputational dynamics imply that reputation eventually hits the cutoff; then the randomness of the signal arrivals implies that the process $\{x_t\}_{t\geq 0}$ eventually "forgets" the initial condition x_0.

In a full-shirk equilibrium with $\tilde{A} = 0$, the process $\{x_t\}_{t\geq 0}$ is a bounded supermartingale with $\mathbb{E}^{\tilde{A},x_0}[x_t] = x_0 - \mathbb{E}^{\tilde{A},x_0}[\int_{s=0}^{t} \lambda x_s\, ds]$. Hence the martingale convergence theorem implies that $\{x_t\}_{t\geq 0}$ converges to 0 almost surely. Similarly, in a full-work equilibrium with $\tilde{A} = 1$, $\{x_t\}_{t\geq 0}$ converges to 1 almost surely. *Q.E.D.*

4. PERFECT POISSON LEARNING

We first consider learning processes where a Poisson signal arrival perfectly reveals the firm's quality. Theorems 2 and 3 highlight how different learning processes lead to opposite investment incentives and reputational dynamics.

[13] Lemma 4 obtains with the same limit distribution F if we assume correct beliefs, $\tilde{A} = A$, and take expectations from the firm's perspective, conditioning expectations on actual investment A and initial quality θ_0, rather than on $\tilde{A}$ and initial reputation x_0.

[14] A real-valued random variable Y is *lattice* if there is a constant $\kappa > 0$ such that Y almost surely equals a multiple of that constant, $\sum_{n=0}^{\infty} \Pr(Y = n\kappa) = 1$.

These cases are highly tractable and help to build intuition for learning processes with imperfect signals, which are considered in Section 5.

4.1. *Perfect Good News*

Suppose that consumers learn about quality via product *breakthroughs* that reveal high quality with arrival rate μ_H, while low quality products never enjoy breakthroughs, $\mu_L = 0$. When a breakthrough occurs, the firm's reputation jumps to $j(x) = 1$. Absent a breakthrough, updating evolves deterministically according to

$$\dot{x} = g(x) = \lambda\big(\tilde{A}(x) - x\big) - \mu_H x(1 - x). \tag{4.1}$$

The reputational dividend is the value of having high quality in the next instant. This equals the value of increasing the reputation from its current value to 1, multiplied by the probability of a breakthrough, $\mu_H(V_H(1) - V_H(x))$. Using equation (3.4), the value of quality is the present value of these dividends,

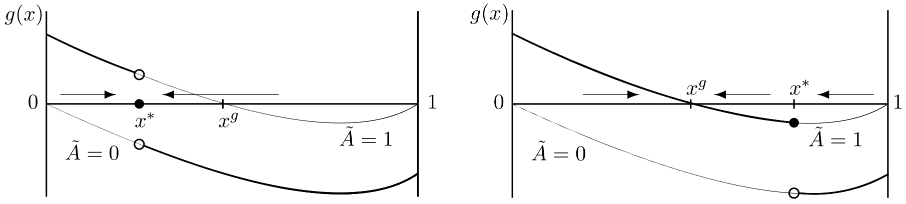
$$\begin{aligned} D(x_0) &= \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\mu_H\big[V_H(1) - V_H(x_t)\big]\,dt\right] \\ &= \int_0^\infty e^{-(r+\lambda)t}\mu_H\big[V_H(1) - V_H\big(x_t^\varnothing\big)\big]\,dt, \end{aligned} \tag{4.2}$$

where $\{x_t^\varnothing\}_{t\geq 0}$ is the solution of (4.1) with initial value x_0. We can drop the expectation because there are no breakthroughs conditional on low quality, so reputation at time t equals $x_t^\varnothing$.

The reputational dividend $V_H(1) - V_H(x_t^\varnothing)$ is decreasing in $x_t^\varnothing$ by Lemma 2. Intuitively, a breakthrough that boosts the firm's reputation to 1 is most valuable for a firm with a low reputation. Thus, investment incentives are decreasing in reputation, so any equilibrium must be work–shirk, full shirk, or full work.

In a work–shirk equilibrium with cutoff x^*, reputational dynamics eventually cycle. When $x > x^*$, reputation drifts down toward x^* due to both believed investment and learning through the absence of breakthroughs. When $x < x^*$, the effect of believed investment is reversed and reputation drifts toward the stationary point $\min\{x^g, x^*\}$, where $x^g = \min\{\lambda/\mu_H, 1\}$ is the stationary point under full-work beliefs. At a breakthrough, the firm's reputation jumps to 1, whereupon it starts drifting back toward the stationary point. In the long run, the firm's reputation therefore cycles over the range $[\min\{x^g, x^*\}, 1]$.

Reputational dynamics around the work–shirk cutoff depend on the relative positions of x^* and x^g. If $x^* < x^g$, the cutoff x^* is convergent and, absent a breakthrough, reputation x_t reaches the cutoff x^* in finite time; at the cutoff, admissibility implies $\tilde{A}(x^*) = x^*(1 + \frac{\mu_H}{\lambda}(1 - x^*))$ and reputational drift vanishes. This case is illustrated in Figure 3(a). If $x^* > x^g$, the cutoff x^* is permeable with strictly negative drift and admissibility implies $A(x^*) = 1$. Reputation



(a) Convergent (b) Permeable

FIGURE 3.—Reputational drift under perfect good news learning and work–shirk beliefs. This figure illustrates reputational drift as a function of reputation. Both panels assume $\lambda < \mu_H$, so that $x^g = \lambda/\mu_H < 1$. The dark line shows equilibrium drift and the arrows show its direction. The bullets show the drift at the cutoff. In the left panel, the cutoff $x^* < x^g$ is convergent and admissibility requires $g(x^*) = 0$. In the right panel, the cutoff $x^* > x^g$ is permeable and admissibility requires that drift is left-continuous.

thus drifts through the cutoff, into the work region $[0, x^*]$ and toward x^g. The favorable beliefs on $[0, x^*]$ slow down the reputational decline but do not reverse it because they are outweighed by the lack of breakthroughs. This case is illustrated in Figure 3(b).

To highlight the dependence of reputation at time t on the work–shirk cutoff x^*, we write $x_t = x_t(x_0, h, x^*)$ if beliefs $\tilde{A}$ are work–shirk with cutoff x^*.

THEOREM 2—Perfect Good News: *Assume perfect good news learning.*
(a) *Every equilibrium is work–shirk or full shirk.*
(b) *An equilibrium exists.*
(c) *In any equilibrium, reputational dynamics are ergodic.*
(d) *If* $\lambda \geq \mu_H$, *the equilibrium is unique.*

PROOF: Part (a). Fix beliefs $\tilde{A}$. Absent a breakthrough, reputation equals $x_t^{\varnothing}$ and is governed by (4.1). By footnote 11, $x_t^{\varnothing}$ is weakly increasing in the initial value x_0 for all t and is strictly increasing for small values of t. By Lemma 2, $V_H(x)$ is strictly increasing in x, so equation (4.2) implies that $D(x_0)$ is strictly decreasing in x_0, and (2.6) means optimal investment $A(x_0)$ is weakly decreasing in x_0. Therefore, any equilibrium must be work–shirk, full shirk, or full work.

We can rule out a full-work equilibrium: If $\tilde{A} = 1$, then $x_0 = 1$ implies $x_t^{\varnothing} = 1$ for all t by (4.1); thus, $V_\theta(1) = 1/r$ is independent of θ, so that $D(1) = 0$ and a firm with perfect reputation strictly prefers to shirk.

Part (b). We establish equilibrium existence by finding a work–shirk cutoff x^* with indifference at the cutoff, $\lambda D(x^*) = c$, and invoking the monotonicity of D to conclude that the optimality condition (2.6) is satisfied for all $x \in [0, 1]$. For any $x^* \in (0, 1)$, let $D_{x^*}(x)$ be the value of the quality of a firm with reputation x when beliefs $\tilde{A}$ are work–shirk with cutoff x^*. For $x^* = 0$ (resp. $x^* = 1$), let $D_{x^*}(x)$ be the value of quality when beliefs $\tilde{A}$ are full shirk (resp. full work).

Appendix B.1 shows that value functions at the cutoff, and thus the value of quality $D_{x^*}(x^*)$ at the cutoff, is continuous in x^*. We showed in part (a) that $D_1(1) = 0$. If $\lambda D_0(0) > c$, then the intermediate value theorem implies that there exists $x^* \in (0, 1)$ with $\lambda D_{x^*}(x^*) = c$. Otherwise, if $\lambda D_0(0) \leq c$, full shirk is an equilibrium.

Part (c) follows from part (a) and Lemma 4.

Part (d). If $\lambda \geq \mu_H$, then the direction of drift $g(x) = \lambda(\tilde{A}(x) - x) - \mu_H x(1 - x)$ is determined by believed investment $\tilde{A}(x)$, so any work–shirk cutoff x^* is convergent. In Appendix B.2, we show that in a work–shirk equilibrium with convergent cutoff x^*, the value of quality at the cutoff is given by

$$D_{x^*}(x^*) = \frac{\mu_H}{r+\lambda} \int_{t=0}^{\infty} (x_t^{\varnothing} - x^*) e^{-rt} \left[\frac{\lambda}{\lambda + \mu_H} + \frac{\mu_H}{\lambda + \mu_H} e^{-(\mu_H+\lambda)t} \right] dt, \tag{4.3}$$

where $x_t^{\varnothing} = x_t(1, \varnothing, x^*)$ is the reputational trajectory starting at $x_0 = 1$, absent signals and with drift determined by work–shirk beliefs with cutoff x^*. This trajectory drifts from $x_0 = 1$ to x^* and remains there forever. The term $x_t^{\varnothing} - x^*$ is weakly decreasing in x^* for all $t \geq 0$ and strictly decreasing for small t, so (4.3) is strictly decreasing in x^*. Thus, equilibrium is unique. *Q.E.D.*

As intuition for the uniqueness result, Theorem 2(d), note that the work–shirk cutoff x^* is a lower bound for future reputation x_t when $x_0 \geq x^*$ and $\lambda \geq \mu_H$. An increase in this lower bound is more valuable to a low quality firm than to a high quality firm as the former is less likely to enjoy breakthroughs, so the lower bound is more likely to be binding. Thus, $D_{x^*}(x)$ is decreasing both in the firm's reputation x and the work–shirk cutoff x^*, so the indifference condition $\lambda D_{x^*}(x^*) = c$ cannot be satisfied for multiple values of x^*.

4.2. *Perfect Bad News*

Suppose that consumers learn about quality via product *breakdowns* that reveal low quality with arrival rate μ_L, while high quality products never suffer breakdowns, $\mu_H = 0$. When a breakdown occurs, the firm's reputation jumps to $j(x) = 0$. Absent a breakdown, updating evolves deterministically according to

$$\dot{x} = g(x) = \lambda(\tilde{A}(x) - x) + \mu_L x(1 - x). \tag{4.4}$$

The reputational dividend is the value of having high quality in the next instant. Quality insures the firm against breakdowns, so the value of this instantaneous insurance equals $\mu_L(V_L(x) - V_L(0))$. Using equation (3.5), the value of quality is given by the present value of these dividends,

$$\begin{aligned} D(x_0) &= \mathbb{E}^H\left[\int_0^{\infty} e^{-(r+\lambda)t} \mu_L [V_L(x_t) - V_L(0)]\, dt\right] \\ &= \int_0^{\infty} e^{-(r+\lambda)t} \mu_L [V_L(x_t^{\varnothing}) - V_L(0)]\, dt, \end{aligned} \tag{4.5}$$

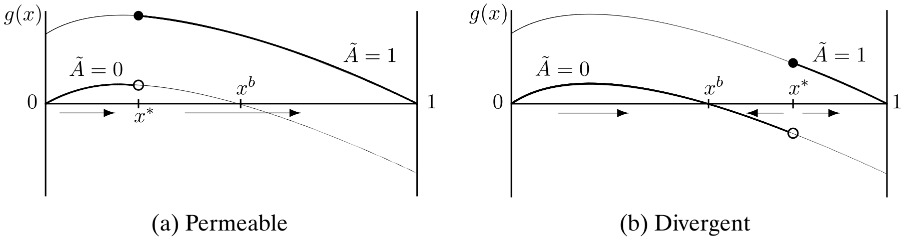


FIGURE 4.—Reputational drift under perfect bad news learning and shirk–work beliefs. This figure illustrates reputational drift as a function of reputation. Both panels assume $\lambda < \mu_L$, so that $x^b > 0$. The dark line shows equilibrium drift and the arrows show its direction. The bullets show the drift at the cutoff. In the left panel, the cutoff $x^* < x^b$ is permeable and admissibility requires $\tilde{A}(x^*) = 1$. In the right panel, the cutoff $x^* > x^b$ is divergent and drift could either be right-continuous (as illustrated), equal to zero, or left-continuous.

where $\{x_t^\varnothing\}_{t\geq 0}$ is the deterministic solution of the ODE (4.4) with initial value x_0. We can drop the expectation because there are no breakdowns conditional on high quality, so reputation at time t equals $x_t^\varnothing$.

The reputational dividend $V_L(x_t^\varnothing) - V_L(0)$ is increasing in $x_t^\varnothing$. Intuitively, a breakdown that destroys the firm's reputation is most damaging for a firm with a high reputation. Thus, investment incentives are increasing in reputation and any equilibrium must be shirk–work, full work, or full shirk.

To analyze reputational dynamics in a shirk–work equilibrium with cutoff x^*, consider first a firm that starts with reputation above x^*. Absent a breakdown, its reputation converges to $x = 1$; upon a breakdown, its reputation drops to zero and is trapped there forever. A firm with reputation below x^* initially shirks and its reputation drifts toward $x^b = \max\{1 - \lambda/\mu_L, 0\}$, the stationary point under full-shirk beliefs. Thus, reputational dynamics are path-dependent and converge to either 0 or 1.

Reputational dynamics around the shirk–work cutoff depend on the relative positions of x^* and x^b. If $x^* < x^b$, the cutoff is permeable and the absence of breakdowns outweighs the effect of adverse market beliefs; reputation can drift through the permeable cutoff x^* into the work region where favorable market beliefs accelerate the reputational ascent. We call such an equilibrium *permeable*; see Figure 4(a). Otherwise, if $x^* \geq x^b$, the cutoff is divergent and reputation cannot escape the shirk region. We call such an equilibrium *divergent*; see Figure 4(b).

THEOREM 3—Perfect Bad News: *Assume perfect bad news learning.*

(a) *Every equilibrium is shirk–work, full shirk, or full work.*

(b) *An equilibrium exists.*

(c) *In any shirk–work equilibrium, reputational dynamics are path-dependent.*

(d) *Assume* $\lambda \geq \mu_L$ *and* $c < \frac{\lambda\mu_L}{(r+\lambda)(r+\mu_L)}$. *There exist* $a < b$ *such that every* $x^* \in [a, b]$ *is the cutoff of a shirk–work equilibrium.*

PROOF: Part (a). Fix beliefs $\tilde{A}$. Absent a breakdown, reputation equals $x_t^{\varnothing}$ and is governed by (4.4). By footnote 11, $x_t^{\varnothing}$ is weakly increasing in the initial value x_0 for all t and is strictly increasing for small values of t. By Lemma 2, $V_L(x)$ is strictly increasing in x, so equation (4.5) implies that $D(x_0)$ is strictly increasing in x_0, and (2.6) means investment $A(x_0)$ is weakly increasing in x_0. Therefore, any equilibrium must be shirk–work, full shirk, or full work.

Part (b). In the perfect good news case, we established equilibrium existence by finding a work–shirk cutoff x^* with indifference at the cutoff and invoking the monotonicity of D to conclude that the optimality condition (2.6) is satisfied at all x. We now apply the same logic to shirk–work cutoffs in the perfect bad news case. However, the arguments are more complicated because value functions are shown to be discontinuous at the shirk–work cutoff when the cutoff is divergent.

For shirk–work beliefs with a permeable cutoff $x^* \in (0, 1)$, admissibility at x^* requires $\tilde{A}(x^*) = 1$. For shirk–work beliefs with a divergent cutoff $x^* \in (0, 1)$, admissibility allows for $\tilde{A}(x^*) \in \{0, x^*(1 + \frac{\mu}{\lambda}(1 - x^*)), 1\}$; for this proof, we focus on the case $\tilde{A}(x^*) = 1$ at both cutoff types. We write $V_\theta^{x^*}(x)$ and $D^{x^*}(x)$ for firm value and the value of quality, given such beliefs.[15] We call beliefs $\tilde{A}$ "full work except at 0" if $\tilde{A}(0) = 0$ and $\tilde{A}(x) = 1$ for $x \in (0, 1]$; we write $V_\theta^0(x)$ and $D^0(x)$ for firm value and the value of quality in this case. Finally we write $V_\theta^{\text{fs}}(x)$ and $D^{\text{fs}}(x)$ in the case of full-shirk beliefs. We will show that at least one of these beliefs defines an equilibrium.

To analyze possible discontinuities at the cutoff, we define for any $x^* \in [0, 1)$, the value of quality just above the cutoff, $D^{x^*}(x^*+) = \lim_{\varepsilon \to 0} D^{x^*}(x^* + \varepsilon)$, and for any $x^* \in (0, 1)$, the value of quality just below the cutoff, $D^{x^*}(x^*-) = \lim_{\varepsilon \to 0} D^{x^*}(x^* - \varepsilon)$. In Appendix B.3, we establish the following properties of $D^{x^*}(x^*\pm)$:

(P+) Properties of $D^{x^*}(x^*+)$: The function $x^* \mapsto D^{x^*}(x^*+)$ is continuous and strictly increasing on $[0, 1)$, with $\lim_{x^* \to 1} D^{x^*}(x^*+) = D^0(1)$.

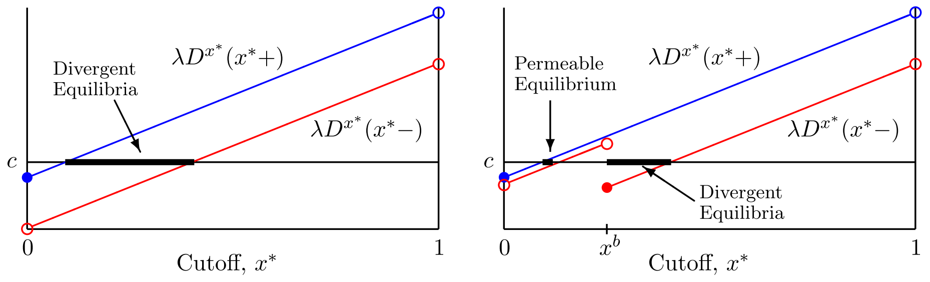
(P−) Properties of $D^{x^*}(x^*-)$: $\lim_{x^* \to 1} D^{x^*}(x^*-) = D^{\text{fs}}(1)$. If $\lambda \geq \mu_L$, then the function $x^* \mapsto D^{x^*}(x^*-)$ is continuous and strictly increasing on $(0, 1)$ with $\lim_{x^* \to 0} D^{x^*}(x^*-) = 0$.

(PD) Discontinuity of $D^{x^*}(x)$ at $x = x^*$: At any divergent cutoff $x^* \in (x^b, 1)$, we have $D^{x^*}(x^*+) > D^{x^*}(x^*-)$.

The functions $x^* \mapsto D^{x^*}(x^*\pm)$ are illustrated in Figure 5. We now use these functions to formulate necessary and sufficient conditions for equilibrium. First, we claim that shirk–work with cutoff $x^* \in (0, 1)$ is an equilibrium if and only if

$$\lambda D^{x^*}(x^*-) \leq c \leq \lambda D^{x^*}(x^*+). \tag{4.6}$$

[15] We use superscripts here to distinguish the value of quality $D^{x^*}(x)$ under shirk–work beliefs from the value of quality $D_{x^*}(x)$ under work–shirk beliefs in the perfect good news case.



(a) Divergent equilibria (b) Permeable and divergent equilibria

FIGURE 5.—Incentives and equilibrium sets under perfect bad news learning. This figure shows the value of quality around the shirk–work cutoff $D^{x^*}(x^*\pm)$, illustrating some properties established in the proof of Theorem 3 and some additional properties. The left panel assumes $\lambda \geq \mu_L$ and $x^b = 0$; the right panel assumes $\lambda < \mu_L$ and $x^b > 0$. For every divergent cutoff $x^* \in [x^b, 1)$, the value of quality $D^{x^*}(x)$ is discontinuous at $x = x^*$, so $D^{x^*}(x^*-) < D^{x^*}(x^*+)$. For every permeable cutoff $x^* \in (0, x^b)$, the value of quality $D^{x^*}(x)$ is continuous at $x = x^*$ with $D^{x^*}(x^*-) = D^{x^*}(x^*+)$. A cutoff $x^* \in (0, 1)$ defines an equilibrium if $\lambda D^{x^*}(x^*-) \leq c \leq \lambda D^{x^*}(x^*+)$. There is a continuum of divergent equilibria in the left panel and a continuum of divergent equilibria together with a unique permeable equilibrium in the right panel.

As $D^{x^*}(\cdot)$ is monotone, this condition is necessary and sufficient for the shirk–work incentives at all $x \neq x^*$. At the cutoff, (4.6) also ensures sufficient incentives to justify $\tilde{A}(x^*) = 1$ because drift g is continuous and strictly positive on $[x^*, 1)$ by (4.4) and $\tilde{A}|_{[x^*,1]} = 1$, so Lemma 3 implies that $D^{x^*}(\cdot)$ is continuous on $[x^*, 1)$ and $\lambda D^{x^*}(x^*) = \lambda D^{x^*}(x^*+) \geq c$.

Second, we claim that "full work except at 0" is an equilibrium if and only if

$$\lambda D^0(0+) \geq c. \tag{4.7}$$

By monotonicity of D^0, this condition is necessary and sufficient to induce investment on $(0, 1]$. At $x = 0$, the firm shirks because $D^0(0) = 0$, which is due to the fact that a trajectory that starts at $x_0 = 0$ stays there forever.

Finally, by monotonicity of D^{fs}, full shirk is an equilibrium if and only if

$$\lambda D^{\text{fs}}(1) \leq c. \tag{4.8}$$

Now equilibrium existence follows by continuity of $D^{x^*}(x^*+)$, as established by (P+), which implies that one of the following statements must be the case:

$$\lambda D^{x^*}(x^*+) \begin{cases} = c & \text{for some } x^* \in (0, 1), \\ > c & \text{for all } x^* \in (0, 1), \\ < c & \text{for all } x^* \in (0, 1). \end{cases} \tag{4.9}$$

In the first case, shirk–work with cutoff x^* is an equilibrium because the right inequality of (4.6) is satisfied with equality by assumption and the left inequality is satisfied by monotonicity of $D^{x^*}(\cdot)$. In the second case, "full work except at 0" is an equilibrium because $\lambda D^0(0+) = \lim_{x^*\to 0} \lambda D^{x^*}(x^*+) \geq c$ by (P+), implying (4.7). In the third case, full shirk is an equilibrium because $\lambda D^{\text{fs}}(1) = \lim_{x^*\to 1} D^{x^*}(x^*-) \leq \lim_{x^*\to 1} D^{x^*}(x^*+) \leq c$ by (P−), implying (4.8).

Part (c). Both $x = 0$ and $x = 1$ are absorbing states in a shirk–work equilibrium. Hence, reputational dynamics are not ergodic.

Part (d). If $\lambda \geq \mu_L$, then the direction of drift $g(x) = \lambda(\tilde{A}(x) - x) + \mu_L x(1 - x)$ is determined by believed investment $\tilde{A}(x)$, so any shirk–work cutoff x^* is divergent, property (P−) holds, and property (PD) holds for all $x^* \in (0, 1)$. The assumption $c < \frac{\lambda\mu_L}{(r+\lambda)(r+\mu_L)}$ implies $\lambda D^0(1) > c$. Otherwise, if $\lambda D^0(1) \leq c$, the firm prefers to shirk at $x = 1$, implying $V_L^0(1) = \mathbb{E}^L[\int_0^\infty e^{-rt} 1_{\{h^t=\varnothing\}}\, dt] = \int_0^\infty e^{-(r+\mu_L)t}\, dt = \frac{1}{r+\mu_L}$ and $\lambda D^0(1) = \lambda \int_0^\infty e^{-(r+\lambda)t} \mu_L V_L^0(1)\, dt = \frac{\lambda\mu_L}{(r+\lambda)(r+\mu_L)} > c$, which contradicts the counterfactual assumption $\lambda D^0(1) \leq c$.

We establish equilibrium multiplicity by reconsidering the first two cases in (4.9); our assumptions rule out the third case, $\lambda D^{x^*}(x^*+) < c$ for all $x^* \in (0, 1)$, because $\lim_{x^*\to 1} \lambda D^{x^*}(x^*+) = \lambda D^0(1) > c$ by property (P+). In the first case, with $\lambda D^{x^*}(x^*+) = c$, property (PD) implies $\lambda D^{x^*}(x^*-) < c = \lambda D^{x^*}(x^*+)$. Properties (P+) and (P−) establish that the functions $D^{x^*}(x^*\pm)$ are continuous and monotone, so condition (4.6) holds for all x^{**} in an interval $[x^*, x^* + \varepsilon]$. In the second case, $\lambda D^{x^*}(x^*+) > c$ for all $x^* \in (0, 1)$, we have $\lim_{x^*\to 0} \lambda D^{x^*}(x^*-) = 0 < c \leq \lim_{x^*\to 0} \lambda D^{x^*}(x^*+)$ by property (P−), so any $x^* \in (0, \varepsilon]$ satisfies (4.6) and defines a shirk–work equilibrium.[16] *Q.E.D.*

To understand the equilibrium multiplicity established in Theorem 3(d), note that the assumption $\lambda \geq \mu_L$ implies that believed investment $\tilde{A}$ determines the direction of the drift, so any shirk–work cutoff is divergent. The reputational dynamics, illustrated in Figure 4(b), introduce a discontinuity in the value function at the shirk–work cutoff, which gives rise to the equilibrium multiplicity. Intuitively, market beliefs about investment become self-fulfilling: If the market believes the firm is shirking it faces low future reputation, so dividends and investment incentives are low. If, to the contrary, the market believes the firm is working, then its reputation will rise; this incentivizes investment so as to protect the appreciating reputation.

If $\lambda < \mu_L$, then $x^b > 0$ and divergent equilibria can coexist with a permeable equilibrium. The permeable reputational dynamics, illustrated in Figure 4(a), ensure that the value function is continuous at the cutoff. In equilibrium, the

[16]These are not all the equilibria. For example, full work is an equilibrium for sufficiently small costs; specifically, it can be shown that $c \leq \frac{\lambda^2\mu_L}{(r+2\lambda+\mu_L)^3}$ suffices.

firm must then be indifferent at the cutoff, implying that there can be at most one permeable equilibrium.

Incentives in the shirk region are qualitatively different for divergent and permeable equilibria, as illustrated by the discontinuity of $D^{x^*}(x^*-)$ at $x^* = x^b$ in Figure 5(b). In a divergent equilibrium, incentives are low because the firm's reputation is trapped in $[0, x^*]$. In a permeable equilibrium, incentives are higher because the firm's reputation can drift out of the shirk region and toward $x = 1$. This implies a lower bound on investment incentives in permeable equilibria, so permeable equilibria cannot exist for values of c close to zero, as illustrated in Figure 5(b).

4.3. *Work–Shirk versus Shirk–Work Equilibria*

Investment incentives differ fundamentally between the work–shirk equilibria of perfect good news learning and the shirk–work equilibria of perfect bad news learning. In the former case, investment is rewarded by reputational boosts; the effect of such boosts is transitory because adverse equilibrium beliefs at high reputations bring the reputation down again. In the latter case, investment averts a reputational loss; the effect of such a loss is permanent because adverse equilibrium beliefs at low reputations prevent a recovery. When the rate of quality obsolescence λ is high, the benefit of a reputational boost disappears quickly, while a reputational loss remains permanent. In this sense, incentives under shirk–work beliefs are stronger than those under work–shirk beliefs.

THEOREM 4—Work–Shirk vs. Shirk–Work: (a) *Assume perfect good news learning. There exists λ^g such that for all $\lambda > \lambda^g$, full shirk is the unique equilibrium.*

(b) *Assume perfect bad news learning and $c < \frac{\mu_L}{r+\mu_L}$. There exists λ^b such that for all $\lambda > \lambda^b$ and all $x^* \in (0, 1]$, there exists a shirk–work equilibrium with cutoff x^*.*

PROOF: Part (a). By Theorem 2(d), equilibrium is unique when $\lambda \geq \mu$ and we simply need to show that full shirk is an equilibrium. Investment incentives $\lambda D(x)$ decrease in x by the proof of Theorem 2(a), so it suffices to verify that $\lambda D(0) \leq c$ under full-shirk beliefs.

We first calculate an upper bound for $V_H(1)$ by omitting the investment costs and replacing the firm's expectation over breakthroughs with the most favorable expectation, where quality is always high:

$$V_H(1) = \max_A \mathbb{E}^{A,H}\left[\int_0^\infty e^{-rt}\big(x_t - cA(x_t)\big)\,dt\right] \leq \mathbb{E}^H\left[\int_0^\infty e^{-rt} x_t\,dt\right].$$

To calculate an upper bound for $\mathbb{E}^H[\int_0^\infty e^{-rt} x_t\,dt]$, note that $x_t^\varnothing = x_t(1, \varnothing, \tilde{A}) \leq e^{-\lambda t}$ because the drift $g(x) \leq -\lambda x$ decreases reputation at least at an exponential rate. Then

$$\mathbb{E}^H\left[\int_0^\infty e^{-rt}x_t\,dt\right] = \int_0^\infty e^{-(r+\mu)t}\left(x_t^\varnothing + \mu\mathbb{E}^H\left[\int_0^\infty e^{-rt}x_t\,dt\right]\right)dt$$
$$\leq \int_0^\infty e^{-(r+\mu)t}e^{-\lambda t}\,dt + \frac{\mu}{r+\mu}\mathbb{E}^H\left[\int_0^\infty e^{-rt}x_t\,dt\right]$$

and so

$$\mathbb{E}^H\left[\int_0^\infty e^{-rt}x_t\,dt\right] \leq \frac{r+\mu}{r(r+\mu+\lambda)}.$$

Intuitively, as $\lambda \to \infty$, the negative drift keeps reputation close to 0 at most times, so firm value is close to 0 as well.

By equation (4.2), the value of quality is bounded above by the perpetuity value of the maximal dividends:

$$\lambda D(0) = \lambda \int_0^\infty e^{-(r+\lambda)t}\mu\big[V_H(1) - V_H(0)\big]\,dt$$
$$\leq \frac{\lambda}{r+\lambda}\mu V_H(1) \leq \frac{\lambda}{r+\lambda}\frac{\mu(r+\mu)}{r(r+\mu+\lambda)} \leq \frac{\mu(r+\mu)}{r\lambda}.$$

Thus, for $\lambda > \lambda^g := \mu(r+\mu)/rc$, we have $\lambda D(0) < c$ as required.

Part (b). Assume $\lambda > \mu_L$, fix any $x^* > 0$, and assume the market has shirk–work beliefs with cutoff $x^* \in (0,1]$ and $\tilde{A}(x^*) = 1$ as in the proof of Theorem 3. The shirk region $[0, x^*)$ is absorbing, and a similar calculation as in the proof of part (a) shows that investment incentives in the shirk region are bounded above by $\mu_L/(\lambda - \mu_L)$. Thus, for $\lambda > \mu_L(1+c)/c$ and initial reputation in the shirk region $x_0 \in [0, x^*)$, we have $\lambda D(x_0) < c$ as required.

With initial reputation in the work region $x_0 \in [x^*, 1]$, a high quality firm can avoid breakdowns with certainty by investing. The drift $g(x) > \lambda(1-x)$ implies that $1 - x_t^\varnothing \leq e^{-\lambda t}(1-x_0)$, so $V_H(x_0)$ approaches the first-best perpetuity value

$$V_H(x_0) \geq \int_0^\infty e^{-rt}\big(x_t^\varnothing - c\big)\,dt \geq \int_0^\infty e^{-rt}\big(1 - e^{-\lambda t}(1-x_0) - c\big)\,dt$$
$$= \frac{1-c}{r} - \frac{1-x_0}{r+\lambda}.$$

Using equation (3.4) and $V_H(0) = 0$, and conditioning on the absence of breakdowns $\mathbb{E}^L[V_H(x_t)] = e^{-\mu_L t}V_H(x_t^\varnothing)$, we have

$$\lambda D(x_0) = \lambda\mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\mu_L\big(V_H(x_t) - V_H(0)\big)\,dt\right]$$
$$= \lambda\int_0^\infty e^{-(r+\lambda)t}e^{-\mu_L t}\mu_L V_H\big(x_t^\varnothing\big)\,dt.$$

Since V_H is increasing and $x_t^{\varnothing} \geq x_0$, these two expressions yield a lower bound for investment incentives:

$$\lambda D(x_0) \geq \frac{\lambda \mu_L}{r+\lambda+\mu_L}\left(\frac{1-c}{r} - \frac{1-x_0}{r+\lambda}\right).$$

For high values of λ, the right-hand side approaches $\mu_L(1-c)/r$, which exceeds c by our assumption $c < \mu_L/(r+\mu_L)$. So for sufficiently large λ, working is optimal for all $x_0 \in [x^*, 1]$.[17] *Q.E.D.*

As $\lambda \to \infty$, our reputation game approaches a repeated game where the firm chooses its quality at each instant. Abreu, Milgrom, and Pearce (1991) study a repeated prisoners' dilemma with imperfect monitoring that approaches the same limit game as the frequency of play increases. They find that only bad news signals that indicate defection can sustain cooperation, while good news signals that indicate cooperation are too noisy to deter defections without destroying all surplus by punishments on the equilibrium path. Thus, sustained cooperation depends on the learning process in the same manner as in our model. While the common limit already suggests this analogy, our model highlights an alternative mechanism that distinguishes the role of bad news signals in overcoming moral hazard, namely divergent reputational dynamics.

Theorem 4 has a surprising consequence: Providing more information about the firm's quality may *decrease* equilibrium investment. Specifically, consider a shirk–work equilibrium under perfect bad news learning. Suppose we improve the learning process by introducing an additional perfect good news signal, so that low quality is revealed perfectly with intensity μ_b and high quality is revealed with intensity μ_g. The analysis in Sections 2 and 3.1 extends immediately to this learning process with two Poisson signals; the reputational dividend is given by $\mu_g(V_\theta(1) - V_\theta(x)) + \mu_b(V_\theta(x) - V_\theta(0))$, which equals $(\mu_b - \mu_g)V_\theta(x) + \mu_g V_\theta(1) - \mu_b V_\theta(0)$. When good news is more frequent than bad news, $\mu_g > \mu_b$, reputational dividends are decreasing in reputation. The proof of Theorem 2(a) then shows that the value of quality is also decreasing in reputation and any equilibrium must be work–shirk, full shirk, or full work. If additionally λ is high enough, the proof of Theorem 4(a) extends to this learning process, implying that full shirk is the unique equilibrium.

Thus, for fixed parameter values r, c, and λ there exist shirk–work equilibria under perfect bad news learning, while full shirk is the only equilibrium when a perfect good news signal is additionally available. Intuitively, under perfect bad news learning, a firm with a high reputation works because a breakdown permanently destroys its reputation. Additional good news signals grant the

[17]We saw in the proof of Theorem 3(d) that there exists a shirk–work equilibrium with cutoff x^* for a continuum of values of $x^* \in [a, b]$ if $c < \frac{\lambda}{r+\lambda}\frac{\mu_L}{r+\mu_L}$. Here, we see that every $x^* \in [0, 1]$ defines a shirk–work equilibrium under the additional assumption that λ is sufficiently large, as then $c < \frac{\mu_L}{r+\mu_L}$.

firm a second chance after a breakdown and undermine incentives to work hard in the first place.

5. IMPERFECT LEARNING

In this section, we suppose consumers learn about product quality through imperfect signals with Poisson arrival rates $\mu_H > 0$ and $\mu_L > 0$. This analysis is more involved than the case of perfect Poisson learning because reputational dividends are no longer monotone in reputation, and vanish at $x = 0$ and $x = 1$; this is due to the $x(1 - x)$ dampening factor in the Bayesian updating formula (2.3). Integrals over such nonmonotone future dividends may take a complicated shape.

We first establish the existence of a work–shirk equilibrium and then investigate the possibility of additional, shirk–work–shirk equilibria.

5.1. *Work–Shirk Equilibrium*

Investment incentives across all imperfect Poisson learning processes share three qualitative features. First, investment at the top cannot be sustained in equilibrium. If the firm is believed to be working at the top, the value of quality is zero at $x = 1$ since current dividends are zero and, as the firm's reputation stays at $x = 1$, future dividends are zero as well. Intuitively, a firm that is believed to be working at the top is certain to have a high reputation in the future, undermining incentives to actually invest.[18] Second, for intermediate levels of reputation, dividends and the value of quality are bounded below and the firm invests if the cost is low enough. Third, investment at the bottom can be sustained in equilibrium. If the firm is believed to be working at the bottom, incentives are high because favorable beliefs push the firm's reputation to intermediate levels where dividends are high. The firm thus invests at low levels of reputation not because of the immediate reputational dividends, which are close to zero, but because of the higher future dividends when the firm's reputation is sensitive to actual quality. Hence, the persistence of quality together with the reputational drift imply a fundamental asymmetry between incentives at the top and the bottom.

These three arguments suggest that a work–shirk equilibrium exists for small costs; Theorem 5 confirms this for a class of imperfect Poisson learning processes.

THEOREM 5—Work–Shirk: *Assume either imperfect bad news learning and* $r > \mu_L(\mu_L^2/\mu_H^2 - 1)$ *or imperfect good news learning with* $\lambda < \mu$ *and* $r > 2\mu$.

[18]The same argument applies in case of perfect good news learning, but not in the perfect bad news case where reputation drops to zero at a breakdown.

(a) *There exists* $\bar{c} > 0$ *such that for any cost* $c \in (0, \bar{c})$, *there is a work–shirk equilibrium with cutoff* $x^* \in (0, 1)$.

(b) *Reputational dynamics in such an equilibrium are ergodic*.

For the proof of part (a), see Appendices C–E. Part (b) follows from Lemma 4.

To prove Theorem 5(a), we consider work–shirk beliefs with cutoff x^* and assume that the firm invests according to beliefs, $A = \tilde{A}$. Let $\Pi_{\theta,x^*}(x)$ be the firm's *payoff function*, that is, the firm's discounted expected profits when it follows this potentially suboptimal investment policy, and let $\Delta_{x^*}(x) := \Pi_{H,x^*}(x) - \Pi_{L,x^*}(x)$ be the *payoff of quality*. In Appendix C.1, we show that, just like the value of quality, $\Delta_{x^*}(x)$ determines investment incentives and admits a reputational dividend representation. If the investment incentives conform to the market's belief—work–shirk with cutoff x^*—we have found an equilibrium. Formally, we establish equilibrium existence by solving the one-dimensional fixed point problem $\lambda\Delta_{x^*}(x^*) = c$ and showing that net investment incentives $\lambda\Delta_{x^*}(x) - c$ are single crossing from above in x.

To understand why $\lambda\Delta_{x^*}(x) - c$ is single crossing, consider full-work beliefs. The payoff of quality $\Delta_1(x)$ is strictly positive on $[0, 1)$ and decreasing on some $[1 - \varepsilon, 1]$ with $\lim_{x\to 1} \Delta_1(x) = 0$, as illustrated in Figure 6(a). For small c, there exists x^* such that

$$\lambda\Delta_1(x) \begin{cases} > c & \text{for } x < x^* \Rightarrow \text{work at low reputations,} \\ = c & \text{for } x = x^* \Rightarrow \text{indifference at cutoff } x^*, \\ < c & \text{for } x > x^* \Rightarrow \text{shirk at high reputations.} \end{cases} \tag{5.1}$$

To prove existence, we essentially need to replace Δ_1 on the left-hand side with Δ_{x^*}. This step requires not only that $\Delta_{x^*}(x)$ converges to $\Delta_1(x)$ uniformly in x, but also that $\Delta_{x^*}(x)$ is decreasing at $x = x^*$ so that the firm prefers to shirk above x^*. This argument is technically challenging because it turns out that

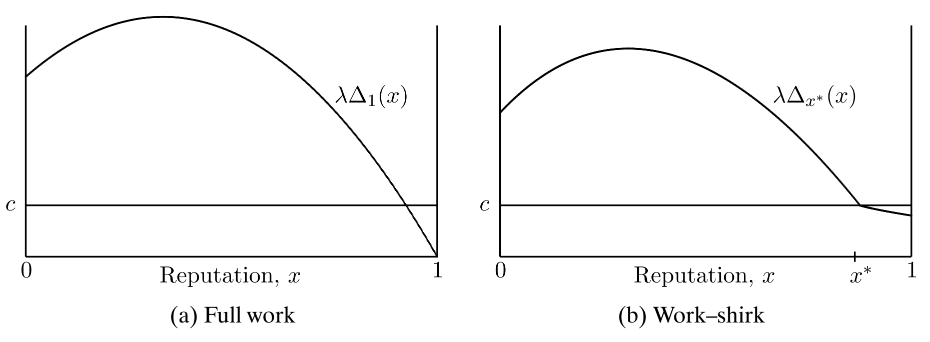


FIGURE 6.—Illustration of the payoff of quality under full work and in a work–shirk equilibrium.

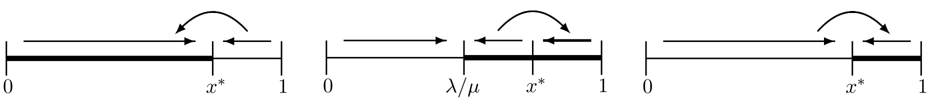


FIGURE 7.—Reputational dynamics under work–shirk beliefs. This figure shows how reputation evolves assuming work–shirk beliefs under good and bad news. The straight arrows indicate the direction of reputational drift, while the curved arrows indicate a typical jump. The support of the resulting long-run distribution is illustrated by the bold segment of the axis.

Δ_{x^*} does not converge to Δ_1 in the C^1-norm.[19] The key step in the proof of Theorem 5(a) (Lemmas 16B and 16G in Appendices D.4 and E.4) establishes $\Delta_{x^*}(x^*) > \Delta_{x^*}(x)$ for $x \in (x^*, 1]$ by arguing that the payoff of quality is determined predominantly by large dividends below x^*, which a firm with initial reputation x^* reaps earlier than a firm with initial reputation $x > x^*$.

We use payoff functions rather than value functions in the proof of Theorem 5(a) because the explicit knowledge of firm investment yields insights into the derivative Π'_θ that we use to establish the single-crossing property for investment incentives; for details, see Appendices C.1 and C.3. In contrast, value functions are convenient to analyze perfect Poisson learning in Section 4 because of their monotonicity properties.

We now turn to a qualitative discussion of the reputational dynamics in this work–shirk equilibrium, illustrated in Figure 7. With bad news learning and a convergent work–shirk cutoff x^* close to 1, the interval $(0, x^*]$ is absorbing. On $(0, x^*)$, reputation is a submartingale that evolves in a pattern of upward drift and downward jumps, eventually reaching the convergent cutoff x^*. At the cutoff, the firm invests at intensity $A(x^*) \in (0, 1)$, and reputation remains constant until the next signal arrives, whereupon reputation drops and the firm resumes work. For initial reputation in the shirk region, $x_0 \in (x^*, 1]$, jumps and negative drift take reputation to $(0, x^*]$ in finite time.

With good news learning, $\mu > \lambda$, and a permeable work–shirk cutoff $x^* \in (\lambda/\mu, 1)$, the interval $(\lambda/\mu, 1)$ is absorbing. Reputational drift is negative both in the shirk region $(x^*, 1]$ and in the work region $(\lambda/\mu, x^*]$ with $\lim_{\varepsilon \to 0} g(\lambda/\mu + \varepsilon) = 0$, so reputation evolves in a pattern of downward drift and upward jumps. For initial reputation $x_0 \in [0, \lambda/\mu]$, jumps and positive drift eventually take reputation to $(\lambda/\mu, 1)$.

With good news learning and $\lambda \geq \mu$—a case not covered by Theorem 5—believed investment alone determines the direction of the reputational drift and every work–shirk cutoff is convergent. Thus, the interval $[x^*, 1)$ is absorbing and reputation evolves in a pattern of upward jumps and subsequent drift down to x^*. This invalidates our argument that $\Delta_{x^*}(x^*) > \Delta_{x^*}(x)$ for $x \in (x^*, 1]$

[19]More specifically, we find that payoff functions and the payoff of quality are flat at a convergent cutoff and nondifferentiable at a permeable cutoff.

because the firm never reaps the large dividends below x^*. Our analysis implies that in this case, with $\lambda \geq \mu$ and c small, there is no work–shirk equilibrium; if an equilibrium exists, it must involve an interval $[\underline{x}, \overline{x}]$ where the firm is indifferent between working and shirking, and chooses $A \in (0, 1)$, as in the strategic experimentation literature with Poisson signals (Keller, Rady, and Cripps (2005)).[20]

5.2. *Other Equilibria*

While Theorem 5 shows that a work–shirk equilibrium exists, slow learning at reputations near 0 and 1 suggests another, *shirk–work–shirk* type of equilibrium with a shirk–work cutoff x_1^* and a work–shirk cutoff $x_2^* > x_1^*$. We next introduce a condition that rules out the existence of such a shirk–work–shirk equilibrium: A learning process satisfies (HOPE) when it is possible for a firm's reputation to increase, even if it is believed to be shirking. That is, there exists an initial reputation x_0, a history of public signals h, and a time t such that even for full-shirk beliefs $\tilde{A} = 0$, we have

$$\text{(HOPE)} \quad x_t(x_0, h, \tilde{A}) > x_0.$$

This condition is satisfied for any good news learning process since the firm's reputation rises upon the arrival of a signal. For bad news learning, it is satisfied if and only if $|\mu| > \lambda$, as then the absence of breakdowns dominates the adverse equilibrium beliefs and reputational drift $g(x) = -\lambda x + |\mu| x(1 - x)$ is positive at low levels of reputation.

THEOREM 6—Work at Bottom: *Fix any imperfect Poisson learning process that satisfies* (HOPE). *For any* $\varepsilon > 0$, *there exists* $c_\varepsilon > 0$ *such that for any* $c < c_\varepsilon$ *and any equilibrium*, *the firm works at all reputation levels* $x \in (0, 1 - \varepsilon)$.

For the proof, see Appendix F.

Theorem 6 states that with (HOPE) and small costs, in equilibrium the firm works at all low and intermediate levels of reputation. For intuition, first note that reputational dividends and the value of quality are bounded below on any interval of intermediate reputation $[\varepsilon, 1 - \varepsilon]$, so for any beliefs, the firm prefers to work in this range if costs are small. If additionally the firm is believed to work on some interval $[x_1^*, \varepsilon]$, these beliefs induce a positive reputational drift and investment incentives in this range are affected by the high dividends above ε. Under (HOPE), a firm with reputation just below x_1^* has a nonzero

[20]Theorem 5 also assumes that the interest rate is not too low. This technical assumption is used to obtain upper bounds for the payoff consequences of reputational trajectories where x_0 is close to 1, but x_t is not. While our proofs rely on this assumption, we conjecture that our results do not.

chance of seeing its reputation increase above x_1^*, so again the high dividends at intermediate reputations put a lower bound on investment incentives. If the cost of investment is sufficiently low, the firm thus prefers to work just below x_1^*.

If the learning process does not satisfy (HOPE), that is, if learning is via bad news and adverse beliefs outweigh learning from the absence of breakdowns, then a firm that is believed to be shirking will see its reputation decrease with certainty. This opens up the possibility of equilibria with a shirk region at the bottom. Specifically, assume that costs c are small and consider shirk–work–shirk beliefs with a shirk–work cutoff $x_1^* \in (0, rc/\lambda)$ and a work–shirk cutoff x_2^* close to 1. A firm with a low reputation is trapped in the shirk region $[0, x_1^*]$ from which it cannot escape because adverse equilibrium beliefs dominate the weak effects of market learning. The firm's value for $x_0 \in [0, x_1^*]$ is thus bounded above by

$$V_\theta(x_0) = \max_A \mathbb{E}^{A,\theta}\left[\int_0^\infty e^{-rt}\big(x_t - cA(x_t)\big)\,dt\right] \le \frac{x_1^*}{r} \le \frac{c}{\lambda}$$

and investment incentives fall short of investment costs, $\lambda D(x_0) \le \lambda V_H(x_0) \le c$. Above the shirk–work cutoff x_1^*, conversely, the firm's reputation drifts up and the high reputational incentives at intermediate levels of reputation incentivize investment, as in the proof of Theorem 6. Divergent reputational drift at the shirk–work cutoff thus creates a discontinuity in the value function that incentivizes investment above the cutoff but not in the shirk region just below.[21]

Economically, investment incentives under shirk–work–shirk beliefs capture the idea that a reputable firm—with reputation close to 1—has low investment incentives and becomes complacent; when it is hit by bad news signals and its reputation drops toward x_1^*, it is put in the "hot seat" where one more breakdown would finish it off. Thus, a firm that fails once fights for its survival, but a firm that fails repeatedly gives up.

To prove the existence of a shirk–work–shirk equilibrium, one needs to show that, given a shirk–work cutoff x_1^* close to 0, there exists a work–shirk cutoff x_2^* close to 1 such that the net investment incentives are single crossing at x_2^*. The existence proof for the work–shirk equilibrium in Theorem 5(a) might serve as an outline for such a proof, but the discontinuity of value functions at the shirk–work cutoff x_1^* (as in the proof of Theorem 3) introduces additional complications. This existence proof remains for future work.

Finally, we would like to address an apparent dissonance between shirk–work equilibria under perfect bad news learning in Theorem 3 and work–shirk

[21]Note that this argument holds for all $x_1^* \in (0, rc/\lambda)$. Thus, as in the case of perfect bad news learning with $\mu_L < \lambda$, Theorem 3(d), there is a continuum of possible values for the shirk–work cutoff x_1^* that are self-fulfilling in the sense that the firm prefers to shirk below x_1^* and work above x_1^*.

equilibria under imperfect bad news learning in Theorem 5. To see the resolution of this dissonance, assume that costs c are low and consider first the investment incentives at high levels of reputation. As imperfect learning approaches perfect learning, that is, in the limit as $\mu_H \to 0$, the jump size $x - j(x)$ and reputational dividends increase for all $x < 1$. Thus, while any equilibrium under imperfect learning must feature shirking at the top, the shirk region vanishes as $\mu_H \to 0$. In particular, the work–shirk equilibria under imperfect learning approximate a full-work equilibrium under perfect learning. Next, consider the investment incentives at low levels of reputation. With imperfect learning and (HOPE), Theorem 6 states that the firm invests on $(0, 1-\varepsilon)$ in any equilibrium. With perfect learning, (HOPE) means that $\mu_L > \lambda$, in which case the firm invests on $(0, 1]$ in any equilibrium.[22] Conversely, if (HOPE) fails, both perfect and imperfect bad news learning allow for a multiplicity of self-fulfilling shirk–work cutoffs x_1^*. In particular, the shirk–work equilibria under perfect learning are consistent with the possibility of shirk–work–shirk equilibria under imperfect learning.

6. CONCLUSION

This paper studies the moral hazard problem of a firm that produces experience goods and controls quality through its investment choice. Investment is incentivized by consumers' learning about product quality, which feeds into the firm's reputation and future revenue.

The key feature that distinguishes our paper from classical models of reputation and repeated games is that we model product quality as a function of past investments rather than current effort. This capital-theoretic model of persistent quality seems realistic: The current state of General Motors is a function of its past hiring policies, investment decisions, and organizational choices, all of which are endogenous and have lasting effects on quality. Similarly, workers' abilities depend on their past human capital investments and career choices. The model can also be interpreted as one of rational inattention with imperfect monitoring, where one agent (the firm) changes her strategy intermittently and another agent (the market) tries to learn about her choice.

The model yields new economic insights: When the market learns quality via breakthroughs of high quality products, a high reputation firm runs down its quality and reputation, while a low reputation firm keeps investing to achieve a breakthrough. Conversely, when the market learns quality via breakdowns of low quality products, a low reputation firm has weak incentives to invest, while a high reputation firm keeps investing to protect its reputation.

There are many interesting ways to extend this model to capture additional important aspects of firm reputation. In the working paper version of this paper (Board and Meyer-ter-Vehn (2010a)), we studied more general imperfect

[22]This assumes $c < \min\{\lim_{x^* \to 0} D^{x^*}(x^*+), D^{x^b}(x^b-)\}$, where $x^b = 1 - \lambda/\mu_L$; see Figure 5(b).

learning processes that may contain a finite number of imperfect Poisson signals and a Brownian signal, where firm quality determines the drift of the Brownian motion. With Brownian noise, (HOPE) is always satisfied, suggesting that equilibrium may be unique. In a companion paper (Board and Meyer-ter-Vehn (2010b)), we suppose the firm faces a cost of remaining in the industry and exits the market when its continuation value drops to zero. We then investigate the investment incentives of a firm that is about to exit, showing that a firm stops investing and coasts into liquidation if it is ignorant of its own quality, while it fights to the bitter end if it does know its own quality. In a recent paper, Dilme (2012) proposed an alternative model of firm reputation where actions have lasting effects due to switching costs.

Beyond firm reputation, we hope that our model will prove useful in other fields. In corporate or international finance, where default signals bad news about a borrower, the shirk–work equilibria generate endogenous credit traps. In political economy, where a scandal is bad news about a politician, the divergent dynamics imply that a politician who is caught cheating will cheat even more, whereas a lucky politician will become more honest. And in personnel economics, our model suggests that in "superstar markets," where agents are judged by their successes, performance tends to be mean-reverting.

APPENDIX A: PROOFS FROM SECTIONS 2 AND 3

A.1. *Admissibility*

Fix admissible beliefs $\tilde{A}$ and an initial reputation x_0. Here we establish the following statements:

(i) There exists a solution to the ODE $\dot{x} = g(x)$, that is, a trajectory $\{x_t^\varnothing\}_{t\geq 0}$ such that $s \mapsto g(x_s^\varnothing)$ is Riemann-integrable and $x_t^\varnothing = x_0 + \int_0^t g(x_s^\varnothing)\,ds$ for all $t \geq 0$.

(ii) The function $t \mapsto \tilde{A}(x_t^\varnothing)$ is right-continuous.

To establish (i), consider first $x_0 \neq x_i^*$. The Picard–Lindelöf theorem implies the existence of a unique local solution $\{x_t^\varnothing\}_{t\in[0,T]}$ to the ODE $\dot{x} = g(x)$, where T is the first time the trajectory hits a cutoff, $x_T = x_i^*$. If reputation starts at a cutoff, $x_0 = x_i^*$, admissibility requires that one of the following statements holds:

(a) $g(x_i^*) = 0$,

(b) $g(x_i^*) > 0$ and $\tilde{A}$ (and therefore g) is right-continuous at x_i^*,

(c) $g(x_i^*) < 0$ and $\tilde{A}$ (and therefore g) is left-continuous at x_i^*.

We consider these cases in turn. In case (a), the constant trajectory $x_t^\varnothing = x_0$ for all $t \geq 0$ solves $\dot{x} = g(x)$. In case (b), drift g is Lipschitz-continuous on $[x_i^*, x_{i+1}^*)$ so there exists a unique local solution $\{x_t^\varnothing\}_{t\in[0,T]}$ in $[x_i^*, x_{i+1}^*)$. In case (c), drift g is Lipschitz-continuous on $(x_{i-1}^*, x_i^*]$ so there exists a unique local solution $\{x_t^\varnothing\}_{t\in[0,T]}$ in $(x_{i-1}^*, x_i^*]$. As in Klein and Rady (2011), these local solutions are not unique in $[0, 1]$, but they are the unique local solutions that are consistent with a discrete-time approximation.[23] Finally, we can concatenate the local solutions to a unique global solution $\{x_t^{\varnothing}\}_{t\geq 0}$ of the ODE $\dot{x} = g(x)$. This establishes (i).[24]

To establish (ii), note first that the trajectory $\{x_t^{\varnothing}\}_{t\geq 0}$ is continuous as a function of time. Thus, $t \mapsto \tilde{A}(x_t^{\varnothing})$ is continuous whenever the trajectory is not at a cutoff, $x_t^{\varnothing} \neq x_i^*$. To see that admissibility implies right continuity at a cutoff, $x_t^{\varnothing} = x_i^*$, we reconsider cases (a), (b), and (c). In case (a), $\{x_{t+\delta}^{\varnothing}\}_{\delta\geq 0}$ is constant, so $\tilde{A}(x_{t+\delta}^{\varnothing}) = \tilde{A}(x_t^{\varnothing})$. In case (b), $\lim_{\delta\to 0} \tilde{A}(x_{t+\delta}^{\varnothing}) = \lim_{\varepsilon\to 0} \tilde{A}(x_t^{\varnothing} + \varepsilon) = \tilde{A}(x_t^{\varnothing})$. In case (c), $\lim_{\delta\to 0} \tilde{A}(x_{t+\delta}^{\varnothing}) = \lim_{\varepsilon\to 0} \tilde{A}(x_t^{\varnothing} - \varepsilon) = \tilde{A}(x_t^{\varnothing})$.

A.2. *Proof of Theorem 1*

LEMMA 5: *For any parameter $\rho > 0$ and any bounded, measurable function $\phi : [0, \infty) \to \mathbb{R}$, the function $\psi(t) = \int_t^{\infty} e^{-\rho(s-t)}\phi(s)\,ds$ is the unique bounded solution to the integral equation*

$$f(t) = \int_t^{t'} \big(\phi(s) - \rho f(s)\big)\,ds + f\big(t'\big) \quad \textit{for all } t' > t. \tag{A.1}$$

PROOF: To see that ψ solves (A.1), we truncate its integral expression at t' and decompose $e^{-\rho(s-t)} = 1 - e^{-\rho(s-t)}(e^{\rho(s-t)} - 1)$ to get

$$\psi(t) = \int_t^{t'} \phi(s)\,ds - \int_t^{t'} e^{-\rho(s-t)}\big(e^{\rho(s-t)} - 1\big)\phi(s)\,ds + e^{-\rho(t'-t)}\psi\big(t'\big). \tag{A.2}$$

We then integrate the second term in (A.2) by parts and rearrange to get

$$\begin{aligned} &\int_t^{t'} e^{-\rho(s-t)}\left(\int_0^{s-t} \rho e^{\rho u}\,du\right)\phi(s)\,ds \\ &\quad = \int_0^{t'-t} \rho \underbrace{\int_{u+t}^{t'} e^{-\rho(s-(t+u))}\phi(s)\,ds}_{=\psi(u+t)-e^{-\rho(t'-(t+u))}\psi(t')}\,du \end{aligned}$$

[23]Indeed, consider $x_0 = x^* \in (0, 1)$ with $\lim_{\varepsilon\to 0} g(x^* - \varepsilon) < 0 < g(x^*) = \lim_{\varepsilon\to 0} g(x^* + \varepsilon)$. In addition to the solution $\{x_t^{\varnothing}\}_{t\in[0,T]}$ in $[x^*, 1]$, there exists a second solution $\{\hat{x}_t^{\varnothing}\}_{t\in[0,\hat{T}]}$ in $[0, x^*]$ because the integral $\hat{x}_t^{\varnothing} - x_0 = \int_0^t g(\hat{x}_s^{\varnothing})\,ds$ does not depend on the initial drift, $g(x_0)$. However, only the first solution, $\{x_t^{\varnothing}\}_{t\in[0,T]}$, is consistent with a discrete-time approximation.

[24]This proof parallels Appendix B in Klein and Rady (2011). For example, a cutoff $x_i^* \in (0, 1)$ with $\lim_{\varepsilon\to 0}(x_i^* + \varepsilon) = g(x_i^*) > 0$ corresponds to their "equivalence classes of transitions" $(-1, 1, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$, where the three numbers indicate the direction of the drift to the left of the cutoff, at the cutoff, and to the right of the cutoff.

$$= \int_0^{t'-t} \rho\psi(u+t)\,du - e^{-\rho(t'-t)} \int_0^{t'-t} \rho e^{\rho u}\,du\psi(t')$$

$$= \int_t^{t'} \rho\psi(s)\,ds - \left(1 - e^{-\rho(t'-t)}\right)\psi(t').$$

Substituting back into (A.2) shows that ψ solves (A.1).

To see that the integral equation (A.1) has a unique bounded solution, assume counterfactually that there exist two solutions, f and $\hat{f}$. Then the difference $\xi = f - \tilde{f}$ satisfies the homogeneous integral equation $\xi(t) = -\rho \int_t^{t'} \xi(s)\,ds + \xi(t')$ for all $t \geq 0$. Thus, ξ satisfies the ODE $\xi'(t) = \rho\xi(t)$, implying $\xi(t) = \alpha e^{\rho t}$, which is bounded if and only if $\alpha = 0$. *Q.E.D.*

PROOF OF THEOREM 1: We truncate the integral expression of $V_\theta(x_t^\varnothing)$ at the first technology shock or signal:

$$V_\theta(x_t^\varnothing) = \int_t^\infty e^{-(r+\lambda+\mu_\theta)(s-t)} \times \underbrace{\left[x_s^\varnothing - cA(x_s^\varnothing) + \lambda\left(V_L(x_s^\varnothing) + A(x_s^\varnothing)D(x_s^\varnothing)\right) + \mu_\theta V_\theta\left(j(x_s^\varnothing)\right)\right]}_{=:\phi(s)}\,ds.$$

Next we apply Lemma 5 to $\psi(t) := V_\theta(x_t^\varnothing)$ and $\rho := r + \lambda + \mu_\theta$ to get

$$V_\theta(x_t^\varnothing) - V_\theta(x_{t'}^\varnothing) = \int_t^{t'} \left[x_s^\varnothing - cA(x_s^\varnothing) + \lambda\left(V_L(x_s^\varnothing) + A(x_s^\varnothing)D(x_s^\varnothing)\right) + \mu_\theta V_\theta\left(j(x_s^\varnothing)\right) - (r+\lambda+\mu_\theta)V_\theta(x_s^\varnothing)\right]ds. \tag{A.3}$$

Taking differences $D = V_H - V_L$ and subtracting/adding $\mu_L(V_H(j(x_s^\varnothing)) - V_H(x_s^\varnothing))$ yields

$$\begin{aligned} D(x_t^\varnothing) - D(x_{t'}^\varnothing) &= \int_t^{t'} \left[\mu_H\left(V_H\left(j(x_s^\varnothing)\right) - V_H(x_s^\varnothing)\right) - \mu_L\left(V_L\left(j(x_s^\varnothing)\right) - V_L(x_s^\varnothing)\right) - (r+\lambda)D(x_s^\varnothing)\right]ds \\ &= \int_t^{t'} \left[\underbrace{\mu\left(V_L\left(j(x_s^\varnothing)\right) - V_L(x_s^\varnothing)\right) + \mu_L D\left(j(x_s^\varnothing)\right)}_{=:\hat{\phi}(s)} - (r+\lambda+\mu_L)D(x_s^\varnothing)\right]ds. \end{aligned}$$

Then $\hat{\psi}(t) := D(x_t^\varnothing)$ is a bounded solution of the integral equation $f(t) = \int_t^{t'} (\hat{\phi}(s) - \hat{\rho} f(s))\,ds + f(t')$ with $\hat{\rho} := r + \lambda + \mu_L$. Thus, Lemma 5 implies

$\hat{\psi}(t) = \int_t^\infty e^{-\hat{\rho}(s-t)}\hat{\phi}(s)\,ds$, that is

$$\begin{aligned} &D(x_t^\varnothing) \\ &\quad = \int_t^\infty e^{-(r+\lambda+\mu_L)(s-t)}\big[\mu\big(V_L\big(j\big(x_s^\varnothing\big)\big) - V_L\big(x_s^\varnothing\big)\big) + \mu_L D\big(j\big(x_s^\varnothing\big)\big)\big]\,ds \\ &\quad = \mathbb{E}^L\bigg[\int_t^\infty e^{-(r+\lambda)(s-t)}\mu\big(V_L\big(j\big(x_s^\varnothing\big)\big) - V_L\big(x_s^\varnothing\big)\big)\,ds\bigg], \end{aligned}$$

finishing the proof of Theorem 1. *Q.E.D.*

REMARK: Dividing the integral equation (A.3) by $t' - t$, taking the limit $t' \to t$, and noting that $\frac{d}{dt}V_\theta(x_t^\varnothing) = g(x_t^\varnothing)V_\theta'(x_t^\varnothing)$ yields the heuristic differential-difference equations (3.1) and (3.2) in the main text.

A.3. *Proof of Lemma 4 (Continued)*

We complete the proof of Lemma 4 by showing that $\mathbb{E}^{\tilde{A},x_0}[T(x_0, x^*)]$ is bounded for all x_0. To do so, we verify that there exist $\tau > 0, \alpha \in (0,1)$, such that $\Pr^{\tilde{A},x_0}[T(x_0, x^*) \leq \tau] \geq \alpha$ for all x_0. Then $\Pr^{\tilde{A},x_0}[T(x_0, x^*) > n\tau] \leq (1-\alpha)^n$, so that $T(x_0, x^*)/\tau$ is first-order stochastically dominated by a geometric random variable with parameter α. Since the mean of such a variable is $1/\alpha$, we have $\mathbb{E}^{\tilde{A},x_0}[T(x_0, x^*)] \leq \tau/\alpha$.

To determine τ and α, we begin by considering the simplest case, in which $\lambda > |\mu|$ and $x_0 \neq x^*$. In this case, reputational drift is strictly positive in the work region, $g(x) = (\lambda - \mu x)(1-x)$, and strictly negative in the shirk region, $g(x) = -(\lambda + \mu(1-x))x$. Absent a signal, reputation thus drifts toward x^* and reaches it at or before time $\tau = 1/\inf_{x \neq x^*}|g(x)| < \infty$. The probability of no signal arriving in $[0, \tau]$ is bounded below by $\alpha = \exp(-\max\{\mu_L, \mu_H\}\tau)$.

When $\lambda \leq |\mu|$ or $x_0 = x^*$, we focus on the good news case, $\mu > 0$ (the bad news case is similar). For $x_0 \in (x^*, 1]$, the proof is as above: Reputational drift in the shirk region, $g(x) = -(\lambda + \mu(1-x))x$, is strictly negative, so absent a signal, reputation drifts downward and reaches x^* before $\tau_1 = 1/\inf_{x > x^*}|g(x)| < \infty$. The probability of no signal in $[0, \tau_1]$ is bounded below by $\alpha_1 = \exp(-\mu_H \tau_1)$.

For $x_0 \in [0, x^*]$, the proof is more complicated because drift in the work region need not be positive, so reputation may first need to jump above x^* before drifting down to x^*. To formalize this idea, we establish that for any $\tau_2 > 0$, there exists $\alpha_2 > 0$ such that $\Pr^{\tilde{A},x_0}(x_t > x^* \text{ for some } t \leq \tau_2) \geq \alpha_2$. This follows since the drift g around $x = 0$ is strictly positive, so $x_{\tau_2/2}$ is bounded away from 0. With a finite number ν of jumps, we thus have $j^\nu(x_{\tau_2}) > x^*$, where the probability of ν or more jumps in $[\tau_2/2, \tau_2]$ is bounded below by

$\alpha_2 = (\mu_L \tau_2/2)^\nu e^{-\mu_L \tau_2/2}/\nu!$. Therefore,

$$\begin{aligned}&\Pr^{\tilde{A},x_0}\big(T\big(x_0, x^*\big) \le \tau_1 + \tau_2\big)\\ &\quad \ge \Pr^{\tilde{A},x_0}\big(x_t > x^* \text{ for some } t \le \tau_2\big) \inf_{x>x^*}\big[\Pr^{\tilde{A},x_0}\big(T\big(x, x^*\big) \le \tau_1\big)\big] \ge \alpha_2\alpha_1,\end{aligned}$$

proving the result for $\tau = \tau_1 + \tau_2$ and $\alpha = \alpha_2\alpha_1$.

APPENDIX B: PERFECT LEARNING

B.1. *Perfect Good News: Continuity in the Proof of Theorem 2*

Let $V_{\theta,x^*}(x)$ be firm value under work–shirk beliefs when $x^* \in (0, 1)$, firm value under full-shirk beliefs when $x^* = 0$, and firm value under full-work beliefs when $x^* = 1$. In this appendix, we show that $V_{\theta,x^*}(x^*)$—and hence $D_{x^*}(x^*) = V_{H,x^*}(x^*) - V_{L,x^*}(x^*)$—is continuous in x^*.

Consider two hypothetical firms: The first firm faces work–shirk beliefs with cutoff x^* and has initial reputation $x_0 = x^*$. Denote its reputation by $x_t = x_t(x^*, h, \tilde{A}) = x_t(x^*, h, x^*)$ and its optimal investment by $a = \{a_t\}_{t\ge0}$. The second firm faces work–shirk beliefs with cutoff $\hat{x}^*$ and has initial reputation $x_0 = \hat{x}^*$. Denote its reputation by $\hat{x}_t = x_t(\hat{x}^*, h, \hat{x}^*)$ and its optimal investment by $\hat{a} = \{\hat{a}_t\}_{t\ge0}$.

We wish to show that the second firm's profits are close to the first firm's when $\hat{x}^*$ is close to x^*. To do this, we suppose the first firm mimics the second firm's investment strategy, as in the proof of Lemma 2.

CLAIM: *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\hat{x}^* \in [x^* - \delta, x^* + \delta]$ and any history h, we have*

$$\int_0^\infty e^{-rt}|\hat{x}_t - x_t|\,dt \le \varepsilon. \tag{B.1}$$

Together with revealed preference, this claim implies

$$\begin{aligned}V_{\theta,x^*}\big(x^*\big) &= \mathbb{E}^{a,\theta}\bigg[\int_0^\infty e^{-rt}(x_t - ca_t)\,dt\bigg] \ge \mathbb{E}^{\hat{a},\theta}\bigg[\int_0^\infty e^{-rt}(x_t - c\hat{a}_t)\,dt\bigg]\\ &\ge \mathbb{E}^{\hat{a},\theta}\bigg[\int_0^\infty e^{-rt}(\hat{x}_t - c\hat{a}_t)\,dt\bigg] - \varepsilon = V_{\theta,\hat{x}^*}\big(\hat{x}^*\big) - \varepsilon.\end{aligned}$$

If the second firm mimics the first firm, we obtain a converse expression. Together, these two equations imply that when $\hat{x}^*$ is close to x^*,

$$V_{\theta,\hat{x}^*}\big(\hat{x}^*\big) + \varepsilon \ge V_{\theta,x^*}\big(x^*\big) \ge V_{\theta,\hat{x}^*}\big(\hat{x}^*\big) - \varepsilon.$$

Hence $V_{\theta,x^*}(x^*)$ is continuous in x^*, implying that $D_{x^*}(x^*) = V_{H,x^*}(x^*) - V_{L,x^*}(x^*)$ is also continuous in x^*, as required.

PROOF OF THE CLAIM: We consider three cases. Recall that $x^g = \min\{\lambda/\mu, 1\}$ and that a work–shirk cutoff x^* is convergent if $x^* \leq x^g$ and permeable if $x^* > x^g$. First suppose $x^* \in [0, x^g]$, so that $g(x) = 0$ for $x = x^*$. Fix $\varepsilon > 0$ and consider any $\hat{x}^* \in [x^*, \min\{x^* + r\varepsilon, 1\}]$.[25] Before the first signal arrival, we have $x_t = x^*$ because x^* is convergent, and have $\hat{x}_t \in [x^*, \hat{x}^*]$ because the drift of $\hat{x}_t$ is positive below x^* and negative above $\hat{x}^*$. At a signal, both trajectories $\{x_t\}_{t\geq 0}$ and $\{\hat{x}_t\}_{t\geq 0}$ jump to $x = 1$ and then coincide until they have drifted down to $\hat{x}^*$. From there, $\{x_t\}_{t\geq 0}$ drifts down further to x^*, while $\{\hat{x}_t\}_{t\geq 0}$ is contained in $[x^*, \hat{x}^*]$ until the next signal hits. To summarize, for any time t and history h, we have $|\hat{x}_t - x_t| \leq r\varepsilon$, yielding (B.1).

Second, suppose $x^* \in (x^g, 1)$ so that $g(x) < 0$ for $x = x^*$. Here, both trajectories drift below their respective work–shirk cutoffs and the distance $|\hat{x}_t - x_t|$ may increase over time. To address this issue, we convert reputation into log-likelihood ratios $L(x) = \log(x/(1-x))$, denoting $\ell_t = L(x_t)$ and $\hat{\ell}_t = L(\hat{x}_t)$.[26] Fix ε and choose $\delta > 0$ so that for every $\hat{x}^* \in [x^*, x^* + \delta]$, we have

$$\hat{\ell}_0 - \ell_0 = L(\hat{x}^*) - L(x^*) \leq r\varepsilon.$$

In ℓ-space, in the work region $[L(\lambda/\mu), L(x^*)]$, reputational drift $\lambda(1 + e^{-\ell}) - \mu$ is decreasing, so the distance $|\hat{\ell}_t - \ell_t|$ shrinks in the absence of a signal. When a signal arrives at time t_1, both trajectories jump to $\hat{\ell}_{t_1} = \ell_{t_1} = \infty$ and then coincide until they have drifted down to $L(\hat{x}^*)$ at time $\hat{T}$. At that time, due to the different beliefs, $\{\ell_t\}_{t\geq 0}$ drifts down faster than $\{\hat{\ell}_t\}_{t\geq 0}$, until $\{\ell_t\}_{t\geq 0}$ reaches $L(x^*)$ at time T. The negative drift $\lambda(1 + e^{-\ell}) - \mu$ on $[L(x^*), L(\hat{x}^*)]$ implies $\hat{\ell}_T \in (L(x^*), L(\hat{x}^*))$ and after time T, the increment $|\hat{\ell}_t - \ell_t|$ starts to shrink again by the argument above. To summarize, for any time t and history h, we have $|\hat{\ell}_t - \ell_t| \leq r\varepsilon$. To translate this into x-space, we use the mean value theorem and $1/L'(x) = x(1-x) \leq 1$ to get

$$|\hat{x}_t - x_t| \leq \max_{x\in(0,1)} \left(\frac{1}{L'(x)}\right) |\hat{\ell}_t - \ell_t| \leq r\varepsilon \quad \text{for all } t,$$

yielding (B.1).

Third, suppose that $x^* = 1$. In this case, $\hat{x}_t \geq \hat{x}_t^{\varnothing} = \hat{x}^* + \int_{s=0}^{t} g(\hat{x}_s^{\varnothing})\, ds$, where $g(x) = (\lambda - \mu x)(1 - x)$. As $\hat{x}^* \to 1$, then $\hat{x}_t \to 1$ for all t and (B.1) is satisfied. This finishes the proof of the claim. *Q.E.D.*

B.2. *Perfect Good News: Uniqueness in the Proof of Theorem 2(d)*

In this appendix, we derive equation (4.3). Fix a work–shirk equilibrium with convergent cutoff x^*, suppose $x_0 = 1$, and let $x_t^{\varnothing} = x_t(1, \varnothing, x^*)$. As reputational

[25]We focus on $\hat{x}^* \geq x^*$ to fix ideas. This formally only establishes right continuity. The proof of left continuity is analogous.

[26]See Appendix C.2 for details on the evolution of reputation in log-likelihood ratio space.

drift is strictly negative in the shirk region $(x^*, 1]$, $g(x) = -(\lambda + \mu(1-x))x$, reputation in the absence of signals $\{x_t^\varnothing\}_{t\geq 0}$ drifts from $x_0 = 1$ to x^* and remains there forever. In equilibrium, the firm weakly prefers to shirk on $[x^*, 1]$, so we calculate the value function based on the assumption that it shirks. Conditional on low quality, reputational dynamics are deterministic and firm value is given by

$$V_L(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-rt} x_{t+s}^\varnothing \, dt. \tag{B.2}$$

With a high quality product, dynamics are more complicated because the reputation jumps to $x = 1$ at a breakthrough and quality disappears at a technology shock,

$$V_H(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \left[x_{t+s}^\varnothing + \lambda V_L(x_{t+s}^\varnothing) + \mu V_H(1)\right] dt. \tag{B.3}$$

We rewrite the integral of the second, $\lambda V_L(x_{t+s}^\varnothing)$, term by changing the order of integration,

$$\begin{aligned}\int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_L(x_{t+s}^\varnothing)\, dt &= \lambda \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \left(\int_{u=t}^{\infty} e^{-r(u-t)} x_{u+s}^\varnothing \, du\right) dt \\ &= \lambda \int_{u=0}^{\infty} e^{-ru} x_{u+s}^\varnothing \left(\int_{t=0}^{u} e^{-(\lambda+\mu)t}\, dt\right) du \\ &= \frac{\lambda}{\lambda+\mu} \int_{u=0}^{\infty} x_{u+s}^\varnothing e^{-ru}\left[1 - e^{-(\lambda+\mu)u}\right] du,\end{aligned}$$

and substitute back into (B.3) to obtain

$$V_H(x_s^\varnothing) = \int_{t=0}^{\infty} e^{-rt} x_{t+s}^\varnothing \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}\right] dt + \frac{\mu}{r+\lambda+\mu} V_H(1). \tag{B.4}$$

Evaluating (B.4) at $x_s^\varnothing = 1$ and rearranging, we obtain

$$V_H(1) = \frac{r+\lambda+\mu}{r+\lambda} \int_{t=0}^{\infty} x_t^\varnothing e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}\right] dt.$$

The value of quality is the difference between the value functions (B.4) and (B.2):

$$\begin{aligned} D(x_s^\varnothing) = {}& \frac{\mu}{r+\lambda} \int_{t=0}^{\infty} x_t^\varnothing e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}\right] dt \\ & - \frac{\mu}{\lambda+\mu} \int_{t=0}^{\infty} x_{t+s}^\varnothing e^{-rt} \left[1 - e^{-(\lambda+\mu)t}\right] dt. \end{aligned} \tag{B.5}$$

When $x_s^{\varnothing} = x^*$ so that $x_{s+t}^{\varnothing} = x^*$ for all $t \geq 0$, elementary calculations show that

$$D(x^*) = \frac{\mu}{r+\lambda} \int_{t=0}^{\infty} (x_t^{\varnothing} - x^*) e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \right] dt,$$

which is equal to (4.3), as required.

Intuitively, quality at x^* is valuable because of the possibility that reputation jumps from x^* to $x = 1$. The probability of this event is captured by the factor $\mu/(r+\lambda)$, while the terms in brackets capture the possibilities of technology shocks and breakthroughs as $\{x_t^{\varnothing}\}_{t \geq 0}$ descends from $x_0 = 1$ to x^*.

B.3. *Perfect Bad News: Properties of $D^{x^*}(x^*\pm)$ in the Proof of Theorem 3*

This appendix proves the following properties of the value of quality $D^{x^*}(\cdot)$ in the neighborhood of a shirk–work cutoff x^*:[27]

(P+) Properties of $D^{x^*}(x^*+)$: The function $x^* \mapsto D^{x^*}(x^*+)$ is continuous and strictly increasing on $[0, 1)$, with $\lim_{x^* \to 1} D^{x^*}(x^*+) = D^0(1)$.

(P−) Properties of $D^{x^*}(x^*-)$: $\lim_{x^* \to 1} D^{x^*}(x^*-) = D^{\text{fs}}(1)$. If $\lambda \geq \mu_L$, then the function $x^* \mapsto D^{x^*}(x^*-)$ is continuous and strictly increasing on $(0, 1)$, with $\lim_{x^* \to 0} D^{x^*}(x^*-) = 0$.

(PD) Discontinuity of $D^{x^*}(x)$ at $x = x^*$: At any divergent cutoff $x^* \in (x^b, 1)$, we have $D^{x^*}(x^*+) > D^{x^*}(x^*-)$.

To show property (PD), note first that absent a signal, the shirk region $[0, x^*)$ and the work region $[x^*, 1]$ are both absorbing, because reputational drift is strictly negative on (x^b, x^*) and strictly positive on $[x^*, 1)$. This discontinuity of g at x^* generates a discontinuity of future reputation as a function of initial reputation; that is, $\lim_{\varepsilon \to 0} x_t(x^* - \varepsilon, \varnothing, x^*) < x^* < \lim_{\varepsilon \to 0} x_t(x^* + \varepsilon, \varnothing, x^*)$ for any $t > 0$.[28] Then using (4.5) and $V_L(0) = 0$, (PD) follows from

$$D^{x^*}(x_0) = \int_0^{\infty} e^{-(r+\lambda)t} \mu_L V_L^{x^*}\big(x_t(x_0, \varnothing, x^*)\big)\, dt$$

and the strict monotonicity of $V_L^{x^*}(\cdot)$.

To show (P+), we first establish that the "shirk–work beliefs with cutoff x^*" and "full work except at 0" give rise to the same value of quality above the cutoff; that is, $D^{x^*}(x_0) = D^0(x_0)$ for all $x^* \in (0, 1)$ and $x_0 > x^*$. To see this, note that the trajectories $\{x_t(x_0, h, x^*)\}_{t \geq 0}$ and $\{x_t(x_0, h, 0)\}_{t \geq 0}$ are contained in $\{0\} \cup [x^*, 1]$ by the positive drift on $[x^*, 1]$ before a breakdown, and the absence of drift at $\{0\}$ after a breakdown. Therefore, these trajectories, the

[27] Recall that $x^b = \max\{1 - \lambda/\mu_L, 0\}$ is the stationary level of reputation when the firm is believed to be shirking.

[28] Here, the third argument x^* in $x_t(x_0, h, x^*)$ represents the beliefs "shirk–work with cutoff x^*." Similarly we write $x_t(x_0, h, 0)$ when beliefs are "full work except at 0" and write $x_t(x_0, h, \text{fs})$ when beliefs are "full shirk."

firm value, and the value of quality do not depend on $\tilde{A}|_{(0,x^*)}$. This implies that the right-sided limits of the value of quality coincide: $D^{x^*}(x^*+) = D^0(x^*+)$ for all $x^* \in [0,1)$. In turn, D^0 is strictly increasing and continuous on $(0,1)$; the latter follows by Lemma 3 because drift g is strictly positive and continuous on $[0,1)$. Therefore, $D^{x^*}(x^*+) = D^0(x^*+) = D^0(x^*)$ is continuous and strictly increasing in x^* on $[0,1)$.

To establish the limit as $x^* \to 1$, it suffices to show that $V_\theta^0(x)$ is continuous at $x = 1$; then $\lim_{x^*\to 1} D^{x^*}(x^*+) = \lim_{x^*\to 1} D^0(x^*+) = D^0(1)$. To establish this continuity, consider a low firm with initial reputation $x_0 = 1-\varepsilon$ and a high firm with initial reputation $\hat{x}_0 = 1$. For any history, the distance between the trajectories remains bounded by $\hat{x}_t - x_t \le \varepsilon$ since before a breakdown, $x_t^\varnothing \ge x_0 = 1-\varepsilon$ and $\hat{x}_t^\varnothing = 1$, and after a breakdown, $x_t = \hat{x}_t = 0$. When the low firm mimics the optimal investment of the high firm, $\hat{a}_t = A(\hat{x}_{t-})$, it faces the same probability distribution over histories, so

$$\begin{aligned} V_\theta^0(x_0) &= \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}(x_t - ca_t)\,dt\right] \ge \mathbb{E}^{\hat{a},\theta}\left[\int_0^\infty e^{-rt}(x_t - c\hat{a}_t)\,dt\right] \\ &\ge \mathbb{E}^{\hat{a},\theta}\left[\int_0^\infty e^{-rt}(\hat{x}_t - \varepsilon - c\hat{a}_t)\,dt\right] = V_\theta^0(\hat{x}_0) - \frac{\varepsilon}{r}. \end{aligned}$$

Together with monotonicity of V_θ^0, this implies $V_\theta^0(1-\varepsilon) \in [V_\theta^0(1) - \varepsilon/r, V_\theta^0(1)]$, implying that V_θ^0 is continuous at $x = 1$.

The proof of (P−) parallels the proof of (P+). Specifically, we can show for any divergent cutoff $x^* \ge x^b = \max\{1 - \lambda/\mu, 0\}$ that "shirk–work with cutoff x^*" and "full shirk" give rise to the same value of quality below the cutoff; that is, $D^{x^*}(x_0) = D^{\text{fs}}(x_0)$ for all $x^* \in (0,1)$ and $x_0 < x^*$. To see this, note that trajectories $\{x_t(x_0, h, x^*)\}_{t\ge 0}$ and $\{x_t(x_0, h, \text{fs})\}_{t\ge 0}$ are contained in $[0, x^*)$ by the negative drift in the shirk region. Therefore, these trajectories, the firm value, and the value of quality do not depend on $\tilde{A}|_{[x^*,1]}$. This implies that the left-sided limits of the value of quality coincide: $D^{x^*}(x^*-) = D^{\text{fs}}(x^*-)$ for all $x^* \in [x^b, 1)$. In turn, D^{fs} is strictly increasing and continuous on $(0,1]$; the latter follows by Lemma 3 because drift g is strictly negative and continuous on $(0,1]$. Therefore, $D^{x^*}(x^*-) = D^{\text{fs}}(x^*-) = D^{\text{fs}}(x^*)$ is continuous and strictly increasing in x^* on $[x^b, 1)$. This implies $\lim_{x^*\to 1} D^{x^*}(x^*-) = D^{\text{fs}}(1)$.

Now assume $\lambda \ge \mu_L$, so that $x^b = 0$. To establish the limit as $x^* \to 0$, it suffices to show that $\lim_{x\to 0} V_\theta^{\text{fs}}(x) = 0$. This follows because reputation only ever declines under full-shirk beliefs, that is, $x_t(x_0, h, \text{fs}) \le x_0$ for any x_0, h, and t, so $V_\theta^{\text{fs}}(x) \in [0, x/r]$ converges to zero as $x \to 0$. Then $\lim_{x^*\to 0} D^{x^*}(x^*-) = \lim_{x^*\to 0} D^{\text{fs}}(x^*-) = 0$.

APPENDIX C: PRELIMINARY RESULTS FOR IMPERFECT LEARNING

Appendices C–E prove Theorem 5(a). This appendix establishes preliminary results, Appendix D proves Theorem 5(a) for bad news learning, and Appendix E proves Theorem 5(a) for good news learning.

The preliminary results in this appendix lay the groundwork for the proof of Theorem 5(a) by shifting the focus from value functions $V_\theta(x)$ to payoff functions $\Pi_\theta(x)$ in Section C.1, performing a change of variables by writing reputation as the log-likelihood ratio of high quality $\ell = L(x) = \log(x/(1-x))$ in Section C.2, and laying out the proof strategy for Theorem 5(a) in Section C.3.

C.1. *Payoff Functions*

In this section, we shift the focus from the firm's value function $V_\theta(x)$, which is based on the firm's optimal response to market beliefs about investment, to its payoff function $\Pi_\theta(x)$, which assumes that the firm follows the investment strategy believed by the market.

Fix admissible beliefs $\tilde{A}$, and recall that we can write time t reputation $x_t = x_t(x_0, h, \tilde{A})$ as a function of initial reputation, history, and reputational drift induced by beliefs. The firm's value and payoff functions are then defined as

$$V_\theta(x) = \max_A \mathbb{E}^{A,\theta}\left[\int_0^\infty e^{-rt}\big(x_t - cA(x_t)\big)\,dt\right],$$
$$\Pi_\theta(x) = \mathbb{E}^{A,\theta}\left[\int_0^\infty e^{-rt}\big(x_t - cA(x_t)\big)\,dt\right], \quad \text{where} \quad A = \tilde{A}.$$

In equilibrium these functions coincide, but to establish equilibrium existence, we need to analyze these functions for any beliefs. For the perfect learning cases of Section 4, we use value functions because of their monotonicity properties. In the proof of Theorem 5(a), we use payoff functions because the explicit knowledge of investment helps us to analyze payoff functions and their derivatives.

Analogous to the value of quality, we define the *payoff of quality* as $\Delta(x) = \Pi_H(x) - \Pi_L(x)$.

LEMMA 6: *Fix admissible beliefs $\tilde{A}$. Then $\langle A, \tilde{A}\rangle$ is a Markov perfect equilibrium if and only if $A = \tilde{A}$ and*

$$A(x) = \begin{cases} 1 & \text{if } \lambda\Delta(x) > c, \\ 0 & \text{if } \lambda\Delta(x) < c, \end{cases} \quad \text{for all } x \in [0,1]. \tag{C.1}$$

PROOF: The "only if" part follows from Lemma 1. The "if" part follows by standard verification arguments, for example, Davis (1993, Theorem 42.8).[29] *Q.E.D.*

To evaluate the payoff of quality Δ and analyze optimal investment (C.1), we express Δ as the expectation over future reputational dividends:

$$\Delta(x_0) = \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\mu\big(\Pi_H\big(j(x_t)\big) - \Pi_H(x_t)\big)\,dt\right], \tag{C.2}$$

$$= \mathbb{E}^H\left[\int_0^\infty e^{-(r+\lambda)t}\mu\big(\Pi_L\big(j(x_t)\big) - \Pi_L(x_t)\big)\,dt\right]. \tag{C.3}$$

These formulae obtain because the proof of Theorem 1 does not rely on optimality, but only requires that high and low quality firms follow the same investment strategy.

In contrast to value functions $V_\theta(x)$, payoff functions $\Pi_\theta(x)$ need not increase in x, hence the payoff of quality need not be positive. However, Lemmas 9B and 9G establish that for work–shirk beliefs with cutoff x^*, the value of quality at the cutoff satisfies $\Delta(x^*) > 0$; if additionally we have $\lambda\Delta(x^*) = c$, then payoff functions $\Pi_\theta(x)$ are increasing in x.

Like value functions, payoff functions $\Pi_\theta(x)$ are not generally continuous in x but discontinuities can arise only when drift g changes sign or approaches zero at x. More specifically, we have the following analogue of Lemma 3.

LEMMA 7: *Fix admissible beliefs $\tilde{A}$, an interval $[\underline{x}, \overline{x}]$, and $\varepsilon > 0$ such that either $g(x) > \varepsilon$ for all $x \in [\underline{x}, \overline{x})$ or $g(x) < -\varepsilon$ for all $x \in (\underline{x}, \overline{x}]$. Then the restriction of Π_θ to $[\underline{x}, \overline{x}]$ is continuous.*

PROOF: The argument that the function $t \mapsto V_\theta(x_t^\varnothing)$ is continuous in the proof of Lemma 1 does not rely on the optimality of firm investment and implies that the function $t \mapsto \Pi_\theta(x_t^\varnothing)$ is also continuous. Thus, the proof of Lemma 3 for value functions also applies to the payoff functions considered here. *Q.E.D.*

Lemma 7 has a useful implication: If drift g is continuous and nonzero at x, then Π_θ is continuous at x.

[29] Davis (1993) studied a more general class of piecewise deterministic processes. In contrast to the standard assumptions in Davis, the vector field of our process is not continuous everywhere; specifically, reputational drift is discontinuous at the work–shirk cutoff x^*. However, Davis' results still apply because our assumption of admissible beliefs guarantees well defined reputational trajectories.

C.2. *Log-Likelihood-Ratio Transformation*

For most of the proofs in Appendices D and E, we represent reputation not by the probability of high quality $x = \Pr(\theta = H)$, but by its log-likelihood ratio $\ell = L(x) = \log(x/(1-x)) \in \mathbb{R} \cup \{-\infty, \infty\}$. The relevant transformation functions are

$$(C.4) \qquad X(\ell) = \frac{e^\ell}{1+e^\ell}, \quad X'(\ell) = \frac{e^\ell}{(1+e^\ell)^2} = x(1-x), \quad X''(\ell) = \frac{e^\ell(1-e^\ell)}{(1+e^\ell)^3} = x(1-x)(1-2x).$$

With abuse of notation, we write $\Pi_\theta(\ell)$ for the firm's payoff function in ℓ-space and equivalently for all other functions of reputation.

The advantage of this transformation is that Bayesian learning is linear in ℓ-space. At a signal, reputation jumps from ℓ_{t-} to $\ell_t = j(\ell_{t-})$, where $j(\ell) := \ell + \log(\mu_H/\mu_L)$. Absent a signal, the reputational drift as a function of ℓ is governed by

$$\frac{d\ell}{dt} = \frac{dx/dt}{X'(\ell)} = \frac{\lambda(\tilde{A}(x) - x) - \mu x(1-x)}{x(1-x)},$$

so, with abuse of notation, equation (2.2) becomes

$$(C.5) \qquad \dot{\ell} = g(\ell) = \begin{cases} \dfrac{\lambda}{X(\ell)} - \mu = \lambda\big(1+e^{-\ell}\big) - \mu & \text{if } \tilde{A}(\ell) = 1, \\ -\dfrac{\lambda}{1-X(\ell)} - \mu = -\lambda\big(1+e^{\ell}\big) - \mu & \text{if } \tilde{A}(\ell) = 0. \end{cases}$$

For work–shirk beliefs with cutoff ℓ^*, drift $g(\ell)$ is illustrated in Figures 8 and 9. We will make extensive use of the fact that g is decreasing.

In analogy to $x_t^\varnothing$ and $x_t = x_t(x_0, h, x^*)$, we write $\ell_t^\varnothing$ for the log-likelihood ratio of reputation in the absence of signals at time t, and write $\ell_t = \ell_t(\ell_0, h, \ell^*)$ after history h with starting value ℓ_0 and work–shirk beliefs with cutoff ℓ^*.

C.3. *Proof Strategy*

The proof of Theorem 5(a) relies on finding a work–shirk cutoff ℓ^* so that the firm is indifferent at ℓ^* and then analyzing the resulting investment incentives to show that the firm prefers to work below the cutoff and shirk above the cutoff.

Lemma 11 shows that for sufficiently small costs, there exists a large work–shirk cutoff with indifference at the cutoff. Formally, for any $\ell > 0$, there exists

$\overline{c}(\ell) > 0$ such that for all $c < \overline{c}(\ell)$, there exists $\ell^* \in (\ell, \infty)$ with $\lambda\Delta_{\ell^*,c}(\ell^*) = c$;[30] this follows from the intermediate value theorem. Lemmas 14–16 show that for sufficiently high work–shirk cutoffs and low costs with indifference at the cutoff, investment incentives are single crossing: That is, there exist $\ell^\Delta > 0$ and $\overline{\overline{c}} > 0$ such that for any $\ell^* > \ell^\Delta$ and $c < \overline{\overline{c}}$ with $\lambda\Delta_{\ell^*,c}(\ell^*) = c$, the function $\lambda\Delta_{\ell^*,c}(\cdot) - c$ crosses 0 once and from above on $[-\infty, \infty]$. Now for any $c < \min\{\overline{c}(\ell^\Delta), \overline{\overline{c}}\}$, there exists $\ell^* > \ell^\Delta$ such that

$$\lambda\Delta_{\ell^*,c}(\ell) \begin{cases} > c & \text{for } \ell \in \big[-\infty, \ell^*\big) \text{ by Lemmas 14 and 15} \\ & \text{(work below cutoff)}, \\ = c & \text{for } \ell = \ell^* \text{ by Lemma 11} \\ & \text{(indifference at cutoff)}, \\ < c & \text{for } \ell \in \big(\ell^*, \infty\big] \text{ by Lemma 16} \\ & \text{(shirk above cutoff)}. \end{cases} \tag{C.6}$$

Thus, given work–shirk beliefs with cutoff ℓ^*, investment $A = \tilde{A}$ satisfies the optimality condition (C.1) and, therefore, constitutes a Markov perfect equilibrium.

Lemmas 14–16 constitute the core of the proof. The actual proofs are fairly intricate, but the basic idea can be easily summarized: Lemma 14 shows that Δ is decreasing on $[\ell^\Delta, \ell^*]$. To prove this, we take the derivative of (C.2) to obtain

$$\Delta'(\ell_0) = \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\mu\big(\Pi'_H\big(j(\ell_t)\big) - \Pi'_H(\ell_t)\big)\,dt\right]$$

and show in Lemma 13 that $\mu(\Pi'_H(j(\ell_t)) - \Pi'_H(\ell_t))$ is negative; the proof of Lemma 13 relies on the fact that $X'(\ell)$ and $\Pi'(\ell)$ behave like $e^{-\ell}$ for large values of ℓ.

Lemma 15 shows that Δ is uniformly bounded below on $[-\infty, \ell^\Delta]$ for all $\ell^* > \ell^\Delta$. Thus, for small c, the firm prefers to work at those low levels of reputation. Finally, Lemma 16 shows that $\Delta(\ell^*) > \Delta(\ell)$ for all $\ell \in (\ell^*, \infty]$. To prove this, we express the reputational dividend as an integral over the marginal payoff of reputation $\mu(\Pi_H(j(\ell_t)) - \Pi_H(\ell_t)) = \mu\int_{\ell_t}^{j(\ell_t)} \Pi'_H(\ell)\,d\ell$. Lemma 12 shows that $\Pi'_H(\ell)$ is much higher below the cutoff than above the cutoff. Thus, reputational dividends are much higher below the cutoff than above and a firm with initial reputation ℓ^* reaps these high dividends earlier than a firm starting at $\ell \in (\ell^*, \infty]$.

[30]Notational convention: We write $\Delta_{\ell^*,c}(\ell)$ and $\Pi_{\theta,\ell^*,c}(\ell)$ for payoff of quality and firm payoff in the parts of the proof where we treat the work–shirk cutoff ℓ^* and the investment cost c as variable (Sections D.2 and E.2), but retain the lighter notation $\Delta(\ell)$ and $\Pi_\theta(\ell)$ in the other parts where these are fixed.

APPENDIX D: BAD NEWS

We separate the analysis for bad news learning and good news learning to avoid case differentiations that would break the flow of the analysis. In this appendix, we restrict attention to bad news learning, and in Appendix E, we adapt these arguments to good news learning.

Some of the intermediate results rely on the assumption that

$$r > \mu_L\left(\mu_L^2/\mu_H^2 - 1\right). \tag{D.1}$$

Also it is convenient to define

$$\gamma := \left|\log(\mu_H/\mu_L)\right| > 0,$$

so that $j(\ell) = \ell - \gamma$.

D.1. *Bad News—Marginal Payoff of Reputation*

In this section, we derive an integral expression for the marginal payoff of reputation, $\Pi'_\theta(\ell)$; this expression, (D.8), provides the basis for the upper and lower bounds calculated in Sections D.3 and D.4.

As a preliminary step, we study how future reputation ℓ_t depends on initial reputation ℓ_0. Let

$$\ell^b = L\left(x^b\right) = \begin{cases} \log\left(|\mu|/\lambda - 1\right) & \text{if } |\mu| > \lambda, \\ -\infty & \text{if } |\mu| \leq \lambda. \end{cases}$$

In what follows, we restrict attention to $\ell^* > \ell^b$, so that reputational drift is strictly positive below the cutoff and strictly negative above the cutoff,

$$\begin{aligned} \underline{g} &:= \lim_{\varepsilon \to 0} g\left(\ell^* - \varepsilon\right) = \lambda\left(1 + e^{-\ell^*}\right) + |\mu| > 0, \\ \overline{g} &:= \lim_{\varepsilon \to 0} g\left(\ell^* + \varepsilon\right) = -\lambda\left(1 + e^{\ell^*}\right) + |\mu| < 0, \end{aligned} \tag{D.2}$$

as illustrated in Figure 8.

For work–shirk beliefs with cutoff ℓ^* and initial reputation ℓ_0, we define the *time-to-cutoff* as the first time the process $\{\ell_s(\ell_0, h, \ell^*)\}_{s\geq 0}$ hits the cutoff ℓ^*,

$$T = T\left(\ell_0, \ell^*\right) := \min\left\{s \geq 0 : \ell_s\left(\ell_0, h, \ell^*\right) = \ell^*\right\}. \tag{D.3}$$

In the following lemma, we focus on work–shirk beliefs with convergent cutoff $\ell^* > \ell^b$, initial reputation $\ell_0 \in R$, history $h = (t_1, t_2, \ldots)$, and time t such that

$$\ell_{s-\delta} \neq \ell^* \quad \text{for all small } \delta > 0 \quad \text{implies} \quad \ell_{s-} \neq \ell^* \quad \text{for } s = t \text{ and any } s \in \{t_1, t_2, \ldots\}. \tag{D.4}$$

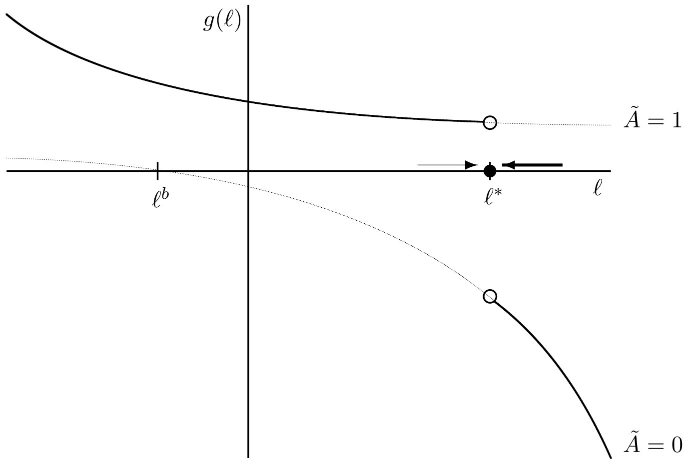


FIGURE 8.—Reputational drift in log-likelihood-ratio space: bad news. The top curve shows $g(\ell)$ given $\tilde{A}=1$; this curve asymptotes to $|\mu|+\lambda$. The bottom curve shows $g(\ell)$ given $\tilde{A}=0$; this curve asymptotes to $|\mu|-\lambda$ and is negative for $\ell>\ell^b$. We focus on work–shirk beliefs with high cutoff ℓ^*. For such beliefs, there is a weak positive drift $\underline{g}$ immediately below ℓ^* and a strong negative drift $\overline{g}$ immediately above ℓ^*.

Intuitively, this means that the trajectory $\{\ell_s\}_{s\geq 0}$ does not drift into the cutoff ℓ^* at time t or any signal arrival time t_i. This condition ensures that the left- and right-sided derivatives of $\ell_t(\ell_0, h, \ell^*)$ coincide. It is generic in the sense that for any ℓ^*, ℓ_0, t, it is satisfied for almost all histories h.

LEMMA 8B: *Fix any admissible work–shirk beliefs with cutoff $\ell^*>\ell^b$, initial reputation $\ell_0\in\mathbb{R}$, history $h=(t_1,t_2,\dots)$, and time $t\neq T$ that satisfy* (D.4). *Then $\ell_t=\ell_t(\ell_0,h,\ell^*)$ is twice partially differentiable in ℓ_0. If $t<T$, the derivatives are given by*

$$\frac{\partial\ell_t}{\partial\ell_0}=\exp\left(\int_0^t g'(\ell_s)\,ds\right), \tag{D.5}$$

$$\frac{\partial^2\ell_t}{\partial\ell_0^2}=\frac{\partial\ell_t}{\partial\ell_0}\int_0^t g''(\ell_s)\frac{\partial\ell_s}{\partial\ell_0}\,ds. \tag{D.6}$$

If $t>T$, the derivatives are zero. As $g'(\ell)<0$, we have $\partial\ell_t/\partial\ell_0<1$.

Intuitively, the decreasing reputational drift g diminishes the reputational increment $\ell'_s-\ell_s$ at rate $g'(\ell_s)<0$. This rate equals $-\infty$ when $\ell_s=\ell^*$, as in Figure 8, so the increment disappears.

PROOF OF LEMMA 8B: Fix a work–shirk cutoff $\ell^*>\ell^b$, a history $h=(t_1,t_2,\dots)$, and $\varepsilon>0$. Consider the reputational trajectories $\{\ell_s\}_{s\geq 0}$ and $\{\ell_s^\varepsilon\}_{s\geq 0}$

that originate at ℓ_0 and $\ell_0^\varepsilon = \ell_0 + \varepsilon$, that is, $\ell_s = \ell_s(\ell_0, h, \ell^*)$ and $\ell_s^\varepsilon = \ell_s(\ell_0^\varepsilon, h, \ell^*)$. The distance $\ell_s^\varepsilon - \ell_s$ is continuous in s: between signal arrivals, its time derivative equals $g(\ell_s^\varepsilon) - g(\ell_s)$, and at a signal arrival time t_i, both trajectories jump down by γ, leaving the distance unchanged.

We first establish that $\ell_s \neq \ell_s^\varepsilon$ for all $s \in [0, T)$. Otherwise, let $\tau = \min\{s \in (0, T) : \ell_s = \ell_s^\varepsilon\}$; the minimum is attained because $\ell_s^\varepsilon - \ell_s$ is continuous in s. Assume first that τ is not a signal arrival time. Then there exists $\delta > 0$ such that no signal arrives in $[\tau - \delta, \tau]$, and both $\{\ell_s\}_{s\in[\tau-\delta,\tau]}$ and $\{\ell_s^\varepsilon\}_{s\in[\tau-\delta,\tau]}$ are governed by the same ODE $\dot{\ell} = g(\ell)$ with the same final condition $\ell_\tau = \ell_\tau^\varepsilon = \ell^*$; thus, we have $\ell_{\tau-\delta} = \ell_{\tau-\delta}^\varepsilon$, contrary to the minimality of τ. If τ *is* a signal arrival time, then $\ell_{\tau-} = \ell_\tau + \gamma = \ell_\tau^\varepsilon + \gamma = \ell_{\tau-}^\varepsilon$ and the same argument implies $\ell_{\tau-\delta} = \ell_{\tau-\delta}^\varepsilon$, contrary again to the minimality of τ. This establishes $\ell_s \neq \ell_s^\varepsilon$, and by continuity $\ell_s < \ell_s^\varepsilon$, for all $s \in [0, T)$.

For $t \in [0, \min\{T, t_1\})$ and $s \leq t$, $\log(\ell_s^\varepsilon - \ell_s)$ is differentiable with derivative

$$\frac{d}{ds}\log\big(\ell_s^\varepsilon - \ell_s\big) = \frac{\frac{d}{ds}(\ell_s^\varepsilon - \ell_s)}{\ell_s^\varepsilon - \ell_s} = \frac{g(\ell_s^\varepsilon) - g(\ell_s)}{\ell_s^\varepsilon - \ell_s}.$$

Integrating, delogging, and taking the limit $\varepsilon \to 0$ yields

$$\lim_{\varepsilon\to 0}\frac{\ell_t^\varepsilon - \ell_t}{\ell_0^\varepsilon - \ell_0} = \lim_{\varepsilon\to 0}\exp\left(\int_0^t \frac{g(\ell_s^\varepsilon) - g(\ell_s)}{\ell_s^\varepsilon - \ell_s}\,ds\right). \tag{D.7}$$

The distance $\ell_s^\varepsilon - \ell_s$ is continuous in s also at signal arrival times t_i, so (D.7) holds for all $t \in [0, T)$. The same form of argument applies for the evolution of the distance $\ell_s - \ell_s^{-\varepsilon}$, where $\ell_s^{-\varepsilon} = \ell_s(\ell_0 - \varepsilon, h, \ell^*)$.

To derive (D.5) from (D.7), we need to take the limit inside the integral. To do so, note first that g is decreasing by (C.5) (see also Figure 8), so the integrand in (D.7) is negative and $\ell_s^\varepsilon - \ell_s \leq \ell_0^\varepsilon - \ell_0 = \varepsilon$. Moreover, by assumption (D.4), the trajectory $\{\ell_s\}_{s\in[0,t]}$ is bounded away from ℓ^*, so the integrand in (D.7) is bounded uniformly for all $s \in [0, t]$ and converges pointwise to $g'(\ell_s)$. Thus, the bounded convergence theorem implies (D.5).

The formula for the second derivative follows from

$$\begin{aligned}\frac{\partial^2\ell_t/\partial\ell_0^2}{\partial\ell_t/\partial\ell_0} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\log\frac{\partial\ell_t}{\partial\ell_0}\big(\ell_0^\varepsilon\big) - \log\frac{\partial\ell_t}{\partial\ell_0}(\ell_0)\right)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_0^t g'\big(\ell_s^\varepsilon\big) - g'(\ell_s)\,ds\right).\end{aligned}$$

Next we fix $t > T$ and argue that the convergent drift at the cutoff ℓ^* eliminates the increment $\ell_s^\varepsilon - \ell_s$ when the trajectory $\{\ell_s\}_{s\geq 0}$ hits the cutoff. The analysis for $s < T$ together with the continuity of $\ell_s^\varepsilon - \ell_s$ in s implies that $\ell_T^{-\varepsilon} \in [\ell^* - \varepsilon, \ell^*]$ and $\ell_T^\varepsilon \in [\ell^*, \ell^* + \varepsilon]$. Let ε be sufficiently small that no signal arrives in $(T, T^{\pm\varepsilon}]$, where $T^{\pm\varepsilon} := T + \varepsilon/\min\{\underline{g}, |\overline{g}|\}$ and the drift around the cutoff $\underline{g}, |\overline{g}|$ was defined in (D.2). Then all trajectories starting at $\ell \in [\ell_0 - \varepsilon, \ell_0 + \varepsilon]$ drift into the cutoff by time $T^{\pm\varepsilon}$, that is, $\ell_{T^{\pm\varepsilon}}(\ell, h, \ell^*) = \ell^*$. For $t > T^{\pm\varepsilon}$, ℓ_t depends on ℓ_0 only via $\ell_{T^{\pm\varepsilon}}$, so $\ell_t(\cdot, h, \ell^*)$ is constant on $[\ell_0 - \varepsilon, \ell_0 + \varepsilon]$, implying $\partial\ell_t/\partial\ell_0 = 0$. *Q.E.D.*

Next, we use these facts about reputational dynamics to investigate the marginal payoff of reputation.[31]

LEMMA 9B: *Fix admissible work–shirk beliefs with cutoff* $\ell^* > \ell^b$.

(a) *Payoff functions* $\Pi_\theta(\ell)$ *are continuous on* $\mathbb{R}$ *and* $\lim_{\ell\to\pm\infty} \Pi_\theta(\ell) = \Pi_\theta(\pm\infty)$.

(b) *The payoff of quality at the cutoff is strictly positive*: $\Delta(\ell^*) > 0$.

(c) *If the firm is indifferent at the cutoff* ℓ^*, *that is*, $\lambda\Delta(\ell^*) = c$, *payoff functions are differentiable in reputation*, *with derivative*

$$\Pi'_\theta(\ell_0) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{T(\ell_0,\ell^*)} e^{-rt}\frac{\partial\ell_t}{\partial\ell_0} X'(\ell_t)\,dt\right], \tag{D.8}$$

where $\partial\ell_t/\partial\ell_0$ *is given by* (D.5) *for almost all histories and* $T(\ell_0, \ell^*)$ *is defined in* (D.3).

Lemma 9B(c) has the flavor of the envelope theorem. In general, initial reputation ℓ_0 affects firm payoff directly through its effect on revenue $X(\ell_t(\ell_0, h, \tilde{A}))$ and indirectly via the firm's investment $a_t = A(\ell_t(\ell_0, h, \tilde{A}))$. If the firm is indifferent at the cutoff, then the indirect effect is zero: A low reputation firm works more than a high reputation firm when their reputation levels are on different sides of ℓ^*. The low reputation firm gains $\Delta(\ell^*)$ when a technology shock hits in that time. In expectation, this gain is exactly offset by the extra cost borne by the firm. The marginal payoff of reputation $\Pi'_\theta(\ell)$ is thus determined solely by the direct effect, which depends on the "durability" of the reputational increment $\ell^\varepsilon_t - \ell_t$.

PROOF OF LEMMA 9B: Part (a). In x-space, this follows immediately from Lemma 7: drift g is positive and bounded away from zero on $[0, x^*)$, and negative and bounded away from zero on $(x^*, 1]$.

Part (c). Fix initial reputations $\ell < \ell^\varepsilon = \ell + \varepsilon \in \mathbb{R}$ of a "low" and a "high" firm. Let $\ell_t = \ell_t(\ell, h, \ell^*)$ and $\ell^\varepsilon_t = \ell_t(\ell^\varepsilon, h, \ell^*)$ be the time-t reputations of the low and high firms, and let $a = \{a_t\}_{t\geq 0}$ and $a^\varepsilon = \{a^\varepsilon_t\}_{t\geq 0}$ be their investment

[31] Recall from Section C.3 that for general beliefs, $\Pi_\theta(\ell)$ need not be monotone in either θ or ℓ.

processes, where $a_t = A(\ell_t)$ and $a_t^\varepsilon = A(\ell_t^\varepsilon)$. Now let

$$\Pi_\theta(\ell^\varepsilon, \ell) := \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}\big(X(\ell_t^\varepsilon) - ca_t\big)\,dt\right]$$

be the payoff to the high firm from mimicking the low firm. By definition $\Pi_\theta(\ell, \ell) = \Pi_\theta(\ell)$. We then decompose the incremental payoff of reputation as

$$\Pi_\theta(\ell^\varepsilon) - \Pi_\theta(\ell) = \underbrace{\big[\Pi_\theta(\ell^\varepsilon, \ell^\varepsilon) - \Pi_\theta(\ell^\varepsilon, \ell)\big]}_{\text{indirect effect}} + \underbrace{\big[\Pi_\theta(\ell^\varepsilon, \ell) - \Pi_\theta(\ell, \ell)\big]}_{\text{direct effect}}. \tag{D.9}$$

Indirect effect: We truncate the integral expressions for $\Pi_\theta(\ell^\varepsilon, \ell^\varepsilon)$ and $\Pi_\theta(\ell^\varepsilon, \ell)$ at the first technology shock,

$$\begin{aligned}
&\Pi_\theta(\ell^\varepsilon, \ell^\varepsilon)\\
&\quad = \mathbb{E}^\theta\left[\int_0^\infty e^{-(r+\lambda)t}\big[X(\ell_t^\varepsilon) - ca_t^\varepsilon + \lambda\big(a_t^\varepsilon\Delta(\ell_t^\varepsilon, \ell_t^\varepsilon) + \Pi_L(\ell_t^\varepsilon, \ell_t^\varepsilon)\big)\big]\,dt\right],\\
&\Pi_\theta(\ell^\varepsilon, \ell)\\
&\quad = \mathbb{E}^\theta\left[\int_0^\infty e^{-(r+\lambda)t}\big[X(\ell_t^\varepsilon) - ca_t + \lambda\big(a_t\Delta(\ell_t^\varepsilon, \ell_t) + \Pi_L(\ell_t^\varepsilon, \ell_t)\big)\big]\,dt\right].
\end{aligned}$$

Taking the difference, adding and subtracting $\lambda a_t\Delta(\ell_t^\varepsilon, \ell_t^\varepsilon)$, and reversing the truncation, we get

$$\begin{aligned}
&\Pi_\theta(\ell^\varepsilon, \ell^\varepsilon) - \Pi_\theta(\ell^\varepsilon, \ell)\\
&\quad = \mathbb{E}^\theta\left[\int_0^\infty e^{-(r+\lambda)t}\big[(a_t^\varepsilon - a_t)\big(\lambda\Delta(\ell_t^\varepsilon, \ell_t^\varepsilon) - c\big)\right.\\
&\qquad\left. + \lambda\big(a_t\big(\Delta(\ell_t^\varepsilon, \ell_t^\varepsilon) - \Delta(\ell_t^\varepsilon, \ell_t)\big) + \Pi_L(\ell_t^\varepsilon, \ell_t^\varepsilon) - \Pi_L(\ell_t^\varepsilon, \ell_t)\big)\big]\,dt\right]\\
&\quad = \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}(a_t^\varepsilon - a_t)\big(\lambda\Delta(\ell_t^\varepsilon, \ell_t^\varepsilon) - c\big)\,dt\right].
\end{aligned} \tag{D.10}$$

Intuitively, whenever the low firm works and the high firm shirks, so that $a_t^\varepsilon - a_t = -1$, the upside of the high firm's investment plan is the cost savings c, while the downside is the expected opportunity cost of missing a technology shock, $\lambda\Delta(\ell_t^\varepsilon)$.

We next establish an upper bound for the indirect effect, (D.10). Recall from Lemma 8B that $\ell_t^\varepsilon - \ell_t$ is decreasing in t, and the rate of decrease $-\frac{d}{dt}(\ell_t^\varepsilon - \ell_t)$ is at least $\underline{g} = \lim_{\varepsilon\to 0} g(\ell^* - \varepsilon)$ whenever $\ell_t < \ell^* \leq \ell_t^\varepsilon$ and at least $|\overline{g}| =$

$\lim_{\varepsilon\to 0}|g(\ell^*+\varepsilon)|$ whenever $\ell_t \le \ell^* < \ell_t^\varepsilon$. Thus, we obtain

$$\begin{aligned}(\text{D.11})\quad &\left|\Pi_\theta\left(\ell^\varepsilon,\ell^\varepsilon\right)-\Pi_\theta\left(\ell^\varepsilon,\ell\right)\right| \\ &\le \mathbb{E}^{a,\theta}\left[\int_{\{t:\ell_t<\ell^*\le\ell_t^\varepsilon \text{ or } \ell_t\le\ell^*<\ell_t^\varepsilon\}} e^{-rt}\left|\lambda\Delta\left(\ell_t^\varepsilon\right)-c\right|dt\right] \\ &\le \frac{\varepsilon}{\min\{\underline{g},|\overline{g}|\}}\max_{\ell'\in[\ell^*,\ell^*+\varepsilon]}\left|\lambda\Delta\left(\ell'\right)-c\right|.\end{aligned}$$

Recall that we assume $\lambda\Delta(\ell^*)=c$ for this part. By part (a), Δ is continuous, so $\lim_{\varepsilon\to 0}(\max_{\ell'\in[\ell^*,\ell^*+\varepsilon]}|\lambda\Delta(\ell')-c|)=0$ and the upper bound of the indirect effect, (D.11), vanishes faster than ε, that is,

$$\lim_{\varepsilon\to 0}\frac{|\Pi_\theta(\ell^\varepsilon,\ell^\varepsilon)-\Pi_\theta(\ell^\varepsilon,\ell)|}{\varepsilon}=0,$$

as in the envelope theorem.

Direct effect: Dividing the direct effect by ε and taking the limit $\varepsilon\to 0$, we get

$$(\text{D.12})\quad \lim_{\varepsilon\to 0}\frac{\Pi_\theta(\ell^\varepsilon,\ell)-\Pi_\theta(\ell,\ell)}{\varepsilon}=\lim_{\varepsilon\to 0}\mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}\frac{X(\ell_t^\varepsilon)-X(\ell_t)}{\varepsilon}\,dt\right].$$

By the proof of Lemma 8B, we have $\ell_t^\varepsilon-\ell_t\le\varepsilon$. Thus, the integrand $(X(\ell_t^\varepsilon)-X(\ell_t))/\varepsilon$ is bounded uniformly with limit $X'(\ell_t)\partial\ell_t/\partial\ell_0$ for almost all h and t, so the bounded convergence theorem applies. Finally, we truncate the integral at $T(\ell_0,\ell^*)$ because $\partial\ell_t/\partial\ell_0=0$ for $t>T(\ell_0,\ell^*)$ by Lemma 8B. This establishes (D.8) for the right-sided derivative of Π at ℓ. Analogous arguments show that the left-sided derivative is given by the same expression.

Part (b). Assume by contradiction that the payoff of quality is nonpositive at the cutoff, $\Delta(\ell^*)\le 0$. By part (a), Δ is continuous, implying $\lambda\Delta(\ell_t)<c$ for ℓ_t close to the cutoff ℓ^*. Thus, for small ε, equation (D.10) implies that the indirect effect $\Pi_\theta(\ell^\varepsilon,\ell^\varepsilon)-\Pi_\theta(\ell^\varepsilon,\ell)$ is positive. Intuitively, this is because the high firm works less than the low firm, which is profitable since $\lambda\Delta(\ell^*)\le 0<c$. The direct effect $\Pi_\theta(\ell^\varepsilon,\ell)-\Pi_\theta(\ell,\ell)$ is positive by construction, so payoff Π_θ is increasing in reputation. This implies that reputational dividends are strictly positive and, by equations (C.2) and (C.3), $\Delta(\ell^*)$ is strictly positive as well. *Q.E.D.*

D.2. *Bad News—Indifference at the Cutoff*

We now show that for small costs $c>0$, there exist work–shirk beliefs with cutoff ℓ^* that make the firm indifferent between working and shirking at ℓ^*. To emphasize the dependence of the firm's payoff on c and ℓ^*, we write $\Pi_{\theta,\ell^*,c}(\ell)$ and $\Delta_{\ell^*,c}(\ell)$ in this section. We say that the pair ℓ^*, c satisfies *indifference at the cutoff* if

$$\text{(D.13)}\quad \lambda\Delta_{\ell^*,c}(\ell^*) = c.$$

Lemma 11B shows that for small costs c, there exist work–shirk beliefs with indifference at the cutoff ℓ^*. The key step in the argument is Lemma 10B, which shows that the payoff at the cutoff is continuous.

LEMMA 10B: *For any θ and $c > 0$, the function $\ell^* \mapsto \Pi_{\theta,\ell^*,c}(\ell^*)$ is continuous on (ℓ^b, ∞) and the limit $\lim_{\ell^*\to\infty} \Pi_{\theta,\ell^*,c}(\ell^*) = \frac{1-c}{r}$ is independent of θ.*

PROOF: Consider a "low" firm with work–shirk cutoff ℓ^* and initial reputation ℓ, and a "high" firm with cutoff $\ell^* + \varepsilon$ and initial reputation $\ell^\varepsilon \in [\ell, \ell + \varepsilon]$, where $\varepsilon \in (0, \gamma/2)$. We write $\ell_t^\varepsilon = \ell_t(\ell^\varepsilon, h, \ell^* + \varepsilon)$ and $\ell_t = \ell_t(\ell, h, \ell^*)$ for the reputations of the high and low firms at time t, and write $a_t^\varepsilon = A^\varepsilon(\ell_t^\varepsilon)$, $a_t = A(\ell_t)$ for their respective investment levels. We first show that reputation and investment of these two firms stay close in any history h, and then show that $\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^\varepsilon) - \Pi_{\theta,\ell^*,c}(\ell)$ is of order ε.

Reputation stays close: We now show that for all $t \geq 0$,

$$\text{(D.14)}\quad \ell_t^\varepsilon - \ell_t \in [0, \varepsilon].$$

To understand (D.14), note that it holds for $t = 0$ and that the distance $\ell_t^\varepsilon - \ell_t$ is Lipschitz-continuous in t: between signals, the rate of change equals the difference of reputational drift, and at a signal, both trajectories jump down by the same amount γ. Moreover, $\frac{d}{dt}(\ell_t^\varepsilon - \ell_t) \geq 0$ whenever $\ell_t^\varepsilon - \ell_t = 0$ because the high firm is believed to invest weakly more, and $\frac{d}{dt}(\ell_t^\varepsilon - \ell_t) \leq 0$ whenever $\ell_t^\varepsilon - \ell_t = \varepsilon$ because drift g is decreasing in ℓ for fixed $\tilde{A}$, as illustrated in Figure 8. This establishes (D.14).

Investment stays close: We next show that $|a_t^\varepsilon - a_t|$ is small for most t. More precisely, we show that there exist constants $\kappa_1, \kappa_2, \kappa_3 > 0$ such that for every history h and every $\varepsilon \in (0, \gamma/2)$, there exists a set $\mathcal{T} = \mathcal{T}(\varepsilon, h) \subseteq [0, \infty)$ such that

$$\text{(D.15)}\quad |a_t^\varepsilon - a_t| \leq \kappa_1\varepsilon \quad \text{for} \quad t \notin \mathcal{T}$$

and

$$\text{(D.16)}\quad t \in \mathcal{T} \quad \Rightarrow \quad [t + \kappa_2\varepsilon, t + \kappa_3] \cap \mathcal{T} = \varnothing.$$

That is, investment is close for $t \notin \mathcal{T}$, and $\mathcal{T}$ is included in a union of small intervals $[t', t' + \kappa_2\varepsilon]$ with distance between intervals at least $\kappa_3 - \varepsilon\kappa_2$. Jointly, (D.15) and (D.16) imply

$$\begin{aligned}
&\int_0^\infty e^{-(r+\lambda)t}\left|a_t^\varepsilon - a_t\right| dt \\
&\quad = \int_{t\notin\mathcal{T}} e^{-(r+\lambda)t}\left|a_t^\varepsilon - a_t\right| dt + \int_{t\in\mathcal{T}} e^{-(r+\lambda)t}\left|a_t^\varepsilon - a_t\right| dt \\
&\quad \le \int_0^\infty e^{-(r+\lambda)t}\kappa_1\varepsilon\, dt + \sum_{n=0}^\infty e^{-(r+\lambda)n\kappa_3}\kappa_2\varepsilon \\
&\quad = \left(\frac{\kappa_1}{r+\lambda} + \frac{\kappa_2}{1-e^{-(r+\lambda)\kappa_3}}\right)\varepsilon =: \kappa\varepsilon.
\end{aligned}$$

To define the set $\mathcal{T}$, let $\kappa_1 := |\mu|/\lambda + 1$, $\kappa_2 := 1/\min\{\underline{g}, |\overline{g}|\}$ and $\kappa_3 := \gamma/(2g(\ell^* - \gamma))$ and define $\mathcal{T} := \{t \in [0,\infty) : \ell^* \le \ell_t, \ell_t^\varepsilon \le \ell^* + \varepsilon$ with at least one strict inequality$\}$. Then (D.14) implies for any $t \notin \mathcal{T}$ that either $a_t^\varepsilon = a_t = 1$ or $a_t^\varepsilon = a_t = 0$ or $\ell_t = \ell^*$ and $\ell_t^\varepsilon = \ell^* + \varepsilon$; in the latter case, admissibility implies

$$\begin{aligned}
&\left|a_t^\varepsilon - a_t\right| \\
&\quad = \left|A^\varepsilon(\ell^* + \varepsilon) - A(\ell^*)\right| \\
&\quad = \left|X(\ell^*+\varepsilon)\left(1 + \frac{\mu}{\lambda}\left(1 - X(\ell^*+\varepsilon)\right)\right) - X(\ell^*)\left(1 + \frac{\mu}{\lambda}\left(1 - X(\ell^*)\right)\right)\right| \\
&\quad \le \kappa_1\varepsilon.
\end{aligned}$$

This establishes (D.15).

To show (D.16), fix $t \in \mathcal{T}$ and let $t_1 \ge t$ be the first signal arrival time. If $\ell^* < \ell_t$, then ℓ_t drifts down at rate $|g(\ell_t)| \ge |\overline{g}|$; if $\ell_t^\varepsilon < \ell^* + \varepsilon$, then ℓ_t^ε drifts up at rate $g(\ell_t^\varepsilon) \ge \underline{g}$. Consider first any $t' \in [t + \kappa_2\varepsilon, t_1)$ before the first signal arrival: $\kappa_2 = 1/\min\{\underline{g}, |\overline{g}|\}$ implies $\ell_{t'} = \ell^*$ and $\ell_{t'}^\varepsilon = \ell^* + \varepsilon$, and so $t' \notin \mathcal{T}$. Consider next any $t' \in [t_1, t_1 + \kappa_3]$ after the first arrival: $\ell_{t_1} \le \ell^* - \gamma/2$ and so $\kappa_3 = \gamma/(2g(\ell^* - \gamma))$ implies $\ell_{t'} < \ell^*$, and so $t' \notin \mathcal{T}$. Thus, we have shown (D.16).

Payoff stays close: To compare the payoffs of the high and low firms, $\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^* + \varepsilon)$ and $\Pi_{\theta,\ell^*,c}(\ell^*)$, we first need to address the issue that their different investment strategies $\{a_t^\varepsilon\}_{t\ge 0}$ and $\{a_t\}_{t\ge 0}$ induce different probability measures over histories h. Rather than analyzing the payoff effect of the different probability measures directly, we truncate the cash-flow expansion at the first technology shock and capture the effect of the different investment strategies through the continuation payoffs:

$$\begin{aligned}
&\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^\varepsilon) \\
&\quad = \mathbb{E}^\theta\left[\int_0^\infty e^{-(r+\lambda)t}\right. \\
&\qquad \left. \times \left(X(\ell_t^\varepsilon) - a_t^\varepsilon c + \lambda\left(a_t^\varepsilon \Pi_{H,\ell^*+\varepsilon,c}(\ell_t^\varepsilon) + (1 - a_t^\varepsilon)\Pi_{L,\ell^*+\varepsilon,c}(\ell_t^\varepsilon)\right)\right) dt\right],
\end{aligned}$$

$$\begin{aligned}
&\Pi_{\theta,\ell^*,c}(\ell) \\
&\quad = \mathbb{E}^\theta\Big[\int_0^\infty e^{-(r+\lambda)t} \\
&\qquad \times \big(X(\ell_t) - a_t c + \lambda\big(a_t \Pi_{H,\ell^*,c}(\ell_t) + (1-a_t)\Pi_{L,\ell^*,c}(\ell_t)\big)\big)\,dt\Big].
\end{aligned}$$

We take the difference of these expressions, and then subtract and add $a_t \Pi_{H,\ell^*+\varepsilon,c}(\ell_t^\varepsilon) + (1-a_t)\Pi_{L,\ell^*+\varepsilon,c}(\ell_t^\varepsilon)$ to obtain

$$\begin{aligned}
&\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^\varepsilon) - \Pi_{\theta,\ell^*,c}(\ell) \\
&\quad = \mathbb{E}^\theta\Big[\int_0^\infty e^{-(r+\lambda)t}\Big[X(\ell_t^\varepsilon) - X(\ell_t) - c(a_t^\varepsilon - a_t) \\
&\qquad + \lambda(a_t^\varepsilon - a_t)\big(\Pi_{H,\ell^*+\varepsilon,c}(\ell_t^\varepsilon) - \Pi_{L,\ell^*+\varepsilon,c}(\ell_t^\varepsilon)\big) \\
&\qquad + \lambda\big(a_t\big(\Pi_{H,\ell^*+\varepsilon,c}(\ell_t^\varepsilon) - \Pi_{H,\ell^*,c}(\ell_t)\big) \\
&\qquad + (1-a_t)\big(\Pi_{L,\ell^*+\varepsilon,c}(\ell_t^\varepsilon) - \Pi_{L,\ell^*,c}(\ell_t)\big)\big)\Big]\,dt\Big].
\end{aligned}$$

The first term captures the cash-flow difference between the two firms for the same histories. The second term captures the difference in continuation payoffs when a technology shock hits and the firms' investment differs. This difference in payoffs is bounded above by $(1+c)/r$, while the difference in investment $a_t^\varepsilon - a_t$ is bounded by (D.15). The third term captures the difference in continuation payoffs when a technology shock hits and the firms' investment coincides.

By (D.14), reputation levels at time t remain within ε of each other, and the difference in continuation payoffs is bounded above by

$$\beta = \sup_{\theta,\ell^*,\ell,\ell^\varepsilon}\big\{\big|\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^\varepsilon) - \Pi_{\theta,\ell^*,c}(\ell)\big| : \ell^\varepsilon - \ell \in [0,\varepsilon]\big\}.$$

This implies

$$\begin{aligned}
\beta &\le \mathbb{E}^\theta\Big[\int_0^\infty e^{-(r+\lambda)t} \\
&\qquad \times \Big(\big|X(\ell_t^\varepsilon) - X(\ell_t)\big| + c\big|a_t^\varepsilon - a_t\big| + \lambda\frac{1+c}{r}\big|a_t^\varepsilon - a_t\big| + \lambda\beta\Big)\,dt\Big] \\
&\le \frac{\varepsilon + c\kappa\varepsilon + \lambda\dfrac{1+c}{r}\kappa\varepsilon}{r+\lambda} + \frac{\lambda}{r+\lambda}\beta
\end{aligned}$$

and hence $\beta \leq \frac{1}{r}(1 + c\kappa + \lambda\frac{1+c}{r}\kappa)\varepsilon$. As $|\Pi_{\theta,\ell^*+\varepsilon,c}(\ell^* + \varepsilon) - \Pi_{\theta,\ell^*,c}(\ell^*)| \leq \beta$, this implies that the function $\ell^* \mapsto \Pi_{\theta,\ell^*,c}(\ell^*)$ is Lipschitz-continuous.

Limit as $\ell^* \to \infty$: For the limit as $\ell^* \to \infty$, revenue $X(\ell_t(\ell^*, h, \ell^*))$ converges to 1 pointwise for every h and t. Investment $A(\ell_t(\ell^*, h, \ell^*))$ converges to 1 as well because $\ell_t \in (-\infty, \ell^*]$ for all t and h, and $A(\ell) = 1$ for $\ell < \ell^*$ and $A(\ell^*) = X(\ell^*)(1 - \frac{|\mu|}{\lambda}(1 - X(\ell^*)))$, which converges to 1 as $\ell^* \to \infty$. Thus, flow profits $X(\ell_t) - cA(\ell_t)$ converge pointwise to $1 - c$. *Q.E.D.*

LEMMA 11B: *For every* $\ell > \ell^b$, *there exists* $\overline{c}(\ell) > 0$ *such that for all* $c < \overline{c}(\ell)$, *there exists* $\ell^* \in (\ell, \infty)$ *such that* ℓ^* *and* c *satisfy indifference at the cutoff*, (D.13), *that is*, $\lambda\Delta_{\ell^*,c}(\ell^*) = c$.

PROOF: Fix $\ell > \ell^b$. Using Lemma 10B, we want to apply the intermediate value theorem to the continuous function $\ell^* \mapsto \Delta_{\ell^*,c}(\ell^*)$ on $[\ell, \infty]$. We thus need to establish the boundary conditions

$$\text{(D.17)} \qquad \lambda\Delta_{\ell,c}(\ell) > c \quad \text{and} \quad \lim_{\ell' \to \infty} \lambda\Delta_{\ell',c}(\ell') < c.$$

To show the second inequality for any $c > 0$, note that $\Pi_{\theta,\infty,c}(\infty) = \frac{1-c}{r}$ is independent of quality by Lemma 10B, hence $\lambda\Delta_{\infty,c}(\infty) = 0$. For the first inequality, fix $\ell \in \mathbb{R}$ and consider $\Delta_{\ell,c'}(\ell)$ as a function of $c' \in [0, \lambda/(r + \lambda)]$. By Lemma 9B(b), we have $\Delta_{\ell,c'}(\ell) > 0$ for all c'. Since $\Delta_{\ell,c'}(\ell)$is continuous in c', it obtains a strictly positive minimum at some c''. Now let $\overline{c}(\ell) = \lambda\Delta_{\ell,c''}(\ell)$. For any $c \in (0, \overline{c}(\ell))$, the first inequality in (D.17) now follows by $\lambda\Delta_{\ell,c}(\ell) \geq \lambda\Delta_{\ell,c''}(\ell) = \overline{c}(\ell) > c$. *Q.E.D.*

D.3. *Bad News—Work Below Cutoff*

The main results of this section, Lemmas 14B and 15B, show that there exists a threshold ℓ^Δ such that investment incentives are bounded below on $[-\infty, \ell^\Delta]$ and decreasing on $[\ell^\Delta, \ell^*]$ whenever the work–shirk cutoff $\ell^* > \ell^\Delta$ is such that the firm is indifferent at the cutoff. In Lemmas 12B and 13B, we establish upper and lower bounds for the marginal payoff of reputation as well as for reputational dividends.

LEMMA 12B: *Assume* (D.1). *There exist* ℓ^Π *and* $k_1, k_2, k_3 > 0$[32,33] *such that for any* $c > 0$, $\ell^* > \ell^\Pi$ *with indifference at the cutoff*, (D.13), *the marginal payoff*

[32] In the proof of this and the subsequent lemmas, we calculate these parameters explicitly only if the resulting expressions are sufficiently simple. This is the case for some but not all k_i; we omit explicit expressions for the parameters ℓ^x.

[33] We use the notation k_i for constants that are used across lemmas and use κ_i for constants that are only used within a lemma.

of reputation satisfies[34]

(D.18) $\Pi'_\theta(\ell) \geq k_1 e^{-\ell}$ *for all* $\ell \in [0, \ell^* - \gamma/2]$,

(D.19) $\Pi'_\theta(\ell) \leq k_2 e^{-\ell}$ *for all* $\ell \in (-\infty, \ell^*]$,

(D.20) $\Pi'_\theta(\ell) \leq k_3 e^{-2\ell^*}$ *for all* $\ell \in [\ell^*, \infty)$.

Equations (D.18) and (D.20) state that for large work–shirk cutoffs ℓ^*, marginal reputation is much more valuable below ℓ^* than above ℓ^*. Intuitively, incremental reputation disappears at $T = T(\ell_0, \ell^*) = \min\{t : \ell_t = \ell^*\}$, so it is less valuable for $\ell_0 \in (\ell^*, \infty)$, where reputational drift is high and the time-to-cutoff is close to 0. The utility of Lemma 12B is in showing later that reputational dividends are larger at the cutoff than above the cutoff. This is Lemma 13B, which is important to prove that the value of quality is single crossing, Lemma 16B.

PROOF OF LEMMA 12B: Recall from equation (D.8) that

$$\Pi'_\theta(\ell_0) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{T} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\,dt\right].$$

The idea of the proof is that the drift on $[0, \ell^*]$ is no larger than $g(0)$ (compare Figure 8), so that for $\ell_0 \in [0, \ell^* - \gamma/2]$, the time-to-cutoff is bounded below by $T(\ell_0, \ell^*) \geq (\ell^* - \ell_0)/g(0) \geq \gamma/(2g(0)) =: \tau$. In contrast, above the cutoff, the drift is unbounded and hence the time-to-cutoff vanishes as ℓ^* grows large.

Lower bound (D.18): First, for $\ell_0 \in [0, \ell^* - \gamma/2]$ and $t \leq \tau$, we claim that

$$X'(\ell_t^\varnothing) = \frac{e^{\ell_t^\varnothing}}{(1+\ell_t^\varnothing)^2} \geq \frac{1}{4}e^{-\ell_t^\varnothing} \geq \frac{1}{4}e^{-(\ell_0+\gamma/2)}.$$

The first inequality follows from $\ell_t^\varnothing \geq 0$ and the second inequality follows from $\ell_t^\varnothing = \ell_0 + \int_0^t g(\ell_s^\varnothing)\,ds \leq \ell_0 + tg(0) \leq \ell_0 + \tau g(0) = \ell_0 + \gamma/2$.

Next we establish

$$\begin{aligned}\Pi'_\theta(\ell_0) &= \mathbb{E}^{A,\theta}\left[\int_{t=0}^{T(\ell_0,\ell^*)} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\right] \\ &\geq \int_{t=0}^{\tau} e^{-(r+\lambda+\mu_\theta)t}\frac{\partial \ell_t^\varnothing}{\partial \ell_0}X'(\ell_t^\varnothing)\,dt \\ &\geq \tau e^{-(r+\lambda+\mu_\theta)\tau}e^{-\lambda\tau}\frac{e^{-\gamma/2}}{4}e^{-\ell_0} =: k_1 e^{-\ell_0}.\end{aligned}$$

[34]The proof of this lemma shows that the marginal value of reputation vanishes at the cutoff, $\Pi'_\theta(\ell^*) = 0$. Thus a lower bound $\Pi'_\theta(\ell) \geq k_1 e^{-\ell}$ requires bounding ℓ away from ℓ^*. For the purpose of this lemma, fixing any $\alpha > 0$ and requiring $\ell \leq \ell^* - \alpha$ would suffice, but for future results, it will be important that $\alpha \in (0, \gamma)$; $\alpha = \gamma/2$ is a convenient choice.

The second line truncates the integral expansion at $\tau < T(\ell_0, \ell^*)$ and discards any histories with a signal arrival or a technology shock. The third line applies the lower bounds $\partial\ell_t^{\varnothing}/\partial\ell_0 = \exp(\int_0^t g'(\ell_s^{\varnothing})\, ds) = \exp(-\int_0^t \lambda e^{-\ell_s^{\varnothing}}\, ds) \geq \exp(-\lambda\tau)$ (using equations (D.5) and (C.5), $e^{-\ell_s^{\varnothing}} < 1$, and $t \leq \tau$) and $X'(\ell_t^{\varnothing}) \geq e^{-(\ell_0+\gamma/2)}/4$ for $t \leq \tau$.

Upper bound (D.19): When reputation starts at $\ell_0 \in (-\infty, \ell^*]$ and ν signals have arrived by time $t > 0$, then $\ell_t > \ell_0 - \nu\gamma$ because reputational drift is positive on $(-\infty, \ell^*]$. We now show that

$$
\begin{aligned}
\Pi'_\theta(\ell_0) &= \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{\partial\ell_t}{\partial\ell_0}X'(\ell_t)\, dt\right] \\
&\leq \mathbb{E}^{L}\left[\int_0^{\infty} e^{-rt}e^{-\ell_t}\, dt\right] \\
&\leq \int_0^{\infty} e^{-rt}\sum_{\nu=0}^{\infty}\left[\frac{(\mu_L t)^\nu}{\nu!}e^{-\mu_L t}e^{-(\ell_0-\nu\gamma)}\right] dt \\
&= e^{-\ell_0}\int_0^{\infty} e^{-(r+\mu_L)t}\sum_{\nu=0}^{\infty}\frac{(\mu_L t e^\gamma)^\nu}{\nu!}\, dt \\
&= e^{-\ell_0}\int_0^{\infty} e^{-(r+\mu_L)t}e^{\mu_L(\mu_L/\mu_H)t}\, dt \\
&= \frac{1}{r-\mu_L(\mu_L/\mu_H-1)}e^{-\ell_0} =: k_2 e^{-\ell_0}. \qquad \text{(D.21)}
\end{aligned}
$$

The second line uses $\partial\ell_t/\partial\ell_0 < 1$, $X'(\ell) \leq e^{-\ell}$ and applies the probability measure $\mathbb{E}^L$ that maximizes the probability of low realizations for ℓ_t. The third line uses $\ell_t > \ell_0 - \nu\gamma$ and the definition of the Poisson distribution $\Pr^\beta(X = \nu) = \frac{\beta^\nu}{\nu!}e^{-\beta}$ over the number of signal arrivals ν with parameter $\beta = \mu_L t$. The fourth line rearranges terms. The fifth line uses $e^\gamma = \mu_L/\mu_H$ and the definition of the Poisson distribution with parameter $\mu_L t e^\gamma = \mu_L(\mu_L/\mu_H)t$. The sixth line uses that $r > \mu_L(\mu_L^2/\mu_H^2 - 1) > \mu_L(\mu_L/\mu_H - 1)$ by assumption (D.1).

Upper bound (D.20): If $\ell_0 > \ell^*$ and ν signals have arrived by time t, the positive drift below ℓ^* implies $\ell_t > \ell^* - \nu\gamma$. Then the equations leading to (D.21) imply

$$
\Pi'_\theta(\ell_0) \leq k_2 e^{-\ell^*} \quad \text{for any } \ell_0 > \ell^*. \qquad \text{(D.22)}
$$

For $\ell_0 > \ell^*$, let $T^{\varnothing} = T^{\varnothing}(\ell_0, \ell^*) = \min\{t : \ell_t^{\varnothing} = \ell^*\}$ be the time-to-cutoff in the absence of signals; we next derive an upper bound for $T^{\varnothing}$. Drift in the shirk region in x-space, $-X(\ell)(\lambda - |\mu|(1 - X(\ell)))$, is continuous and converges to

$-\lambda$ as $\ell \to \infty$, so there exists ℓ^{Π} such that $-X(\ell)(\lambda - |\mu|(1 - X(\ell))) \leq -\lambda/2$ for all $\ell \geq \ell^* \geq \ell^{\Pi}$, that is, reputation drifts down at a rate greater than $-\lambda/2$ in x-space. As the size of the shirk region is bounded above by $1 - X(\ell^*) = 1/(1 + e^{\ell^*}) \leq e^{-\ell^*}$, the time-to-cutoff is bounded above by

$$\text{(D.23)} \quad \sup_{\ell_0 > \ell^*} T^{\varnothing}(\ell_0, \ell^*) \leq 2e^{-\ell^*}/\lambda.$$

Then we show

$$\begin{aligned}
&\Pi'_\theta(\ell_0) \\
&\quad = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{T(\ell_0,\ell^*)} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\,dt\right] \\
&\quad = \int_{t=0}^{T^{\varnothing}(\ell_0,\ell^*)} e^{-(r+\lambda+\mu_\theta)t}\frac{\partial \ell_t^{\varnothing}}{\partial \ell_0}\big(X'(\ell_t^{\varnothing}) + \lambda \Pi'_L(\ell_t^{\varnothing}) + \mu_\theta \Pi'_\theta(\ell_t^{\varnothing} - \gamma)\big)\,dt \\
&\quad \leq T^{\varnothing}(\ell_0, \ell^*)\Big[\max_{\ell \in (\ell^*, \ell_0)}\big(X'(\ell) + \lambda \Pi'_L(\ell) + \mu_\theta \Pi'_\theta(\ell - \gamma)\big)\Big] \\
&\quad \leq \frac{2}{\lambda} e^{-\ell^*}\big(e^{-\ell^*} + \lambda k_2 e^{-\ell^*} + \mu_\theta k_2 e^{-(\ell^* - \gamma)}\big) =: k_3 e^{-2\ell^*}.
\end{aligned}$$

The second line truncates the integral expression of $\Pi'_\theta(\ell_0)$ when a signal arrives or a technology shock hits. The third line uses $\partial \ell_t^{\varnothing}/\partial \ell_0 < 1$, $e^{-(r+\lambda+\mu_\theta)t} < 1$, and $\ell_t^{\varnothing} \in (\ell^*, \ell_0)$ to obtain an upper bound for the integrand. The fourth line uses the upper bounds (D.23), $X'(\ell) < e^{-\ell} < e^{-\ell^*}$, (D.22), and (D.21). *Q.E.D.*

We now consider the reputational dividend

$$\text{(D.24)} \quad \Gamma_H(\ell) := \mu\big(\Pi_H(j(\ell)) - \Pi_H(\ell)\big) = |\mu|\big(\Pi_H(\ell) - \Pi_H(\ell - \gamma)\big).$$

The next lemma shows that there exists an interval $[\ell^{\Gamma}, \ell^*]$ on which Γ_H is decreasing. This is important to show (in Lemma 14B) that there exists an interval $[\ell^{\Delta}, \ell^*]$ on which the payoff of quality Δ is decreasing.

LEMMA 13B: *Assume* (D.1). *There exist* ℓ^{Γ} *and* $k_4 > 0$ *such that for any* $c > 0$, $\ell^* > \ell^{\Gamma}$ *with indifference at the cutoff*, (D.13), *we have*

$$\text{(D.25)} \quad \Gamma'_H(\ell) \leq -k_4 e^{-\ell} \quad \textit{for all } \ell \in [\ell^{\Gamma}, \ell^*),$$

$$\text{(D.26)} \quad \Gamma_H(\ell^*) > \Gamma_H(\ell) \quad \textit{for all } \ell \in (\ell^*, \infty).$$

Thus, $\Gamma_H(\cdot) - \Gamma_H(\ell^*)$ *is strictly single crossing from above on* $[\ell^{\Gamma}, \infty)$.

PROOF: To prove (D.25), note first that the derivative of (D.24) is given by $\Gamma'_H(\ell) = |\mu|(\Pi'_H(\ell) - \Pi'_H(\ell - \gamma))$. The key step in this proof is to show that there exist $\kappa_1, \kappa_2 > 0$ such that for all $\alpha \in [0, \gamma]$,

$$\Pi'_H(\ell) - \Pi'_H(\ell - \alpha) \le -\kappa_1 \alpha e^{-\ell} \quad \text{for all } \ell \in \left[\ell^\Gamma - 3\gamma/2, \ell^* - \gamma/2\right], \tag{D.27}$$

$$\Pi'_H(\ell) - \Pi'_H(\ell - \alpha) \le \kappa_2 \alpha e^{-2\ell} \quad \text{for all } \ell \in \left[\ell^* - \gamma/2, \ell^*\right). \tag{D.28}$$

Inequalities (D.27) and (D.28) imply (D.25): For $\ell \in [\ell^\Gamma, \ell^* - \gamma/2]$, we just need to set $k_4 = |\mu|\kappa_1\gamma$. For $\ell \in [\ell^* - \gamma/2, \ell^*)$, we have $\ell - \gamma \ge \ell^* - 3\gamma/2 > \ell^\Gamma - 3\gamma/2$, so

$$\begin{aligned} \Gamma'_H(\ell) &= |\mu|\left(\Pi'_H(\ell) - \Pi'_H\left(\ell^* - \gamma/2\right) + \Pi'_H\left(\ell^* - \gamma/2\right) - \Pi'_H(\ell - \gamma)\right) \\ &\le |\mu| \frac{\gamma}{2}\left(\kappa_2 e^{-2\ell} - \kappa_1 e^{-(\ell^* - \gamma/2)}\right), \end{aligned}$$

and the negative term dominates the positive term for all $\ell^* > \ell^\Gamma$ and sufficiently large ℓ^Γ.

Establishing (D.27) and (D.28): Fix $\ell \in [\ell^\Gamma - 3\gamma/2, \ell^*)$, $\alpha \in (0, \gamma]$, and consider the trajectories $\{\ell_t\}_{t\ge0}$ and $\{\ell_t^{-\alpha}\}_{t\ge0}$ defined by $\ell_t = \ell_t(\ell, h, \ell^*)$ and $\ell_t^{-\alpha} = \ell_t(\ell - \alpha, h, \ell^*)$ with times-to-cutoff $T = T(\ell, \ell^*)$ and $T^\alpha = T(\ell - \alpha, \ell^*)$. As $\ell_t^{-\alpha} \le \ell_t$ for all times t and histories h, we have $T \le T^\alpha$. The formula for the marginal payoff of reputation, (D.8), then allows us to write[35]

$$\begin{aligned} \Pi'_H(\ell) - \Pi'_H(\ell - \alpha) = {} & \mathbb{E}^H\left[\int_{t=0}^{T} e^{-rt}\left(\frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t) - \frac{\partial \ell_t^{-\alpha}}{\partial \ell_0} X'\left(\ell_t^{-\alpha}\right)\right) dt\right] \\ & - \mathbb{E}^H\left[\int_{T}^{T^\alpha} e^{-rt} \frac{\partial \ell_t^{-\alpha}}{\partial \ell_0} X'\left(\ell_t^{-\alpha}\right) dt\right]. \end{aligned} \tag{D.29}$$

As we are looking for an upper bound on $\Pi'_H(\ell) - \Pi'_H(\ell - \alpha)$, we can drop the last term. To analyze the integrand of the first term, note that for all h and $t < T$, the function $\beta \mapsto (\partial \ell_t^\beta / \partial \ell_0) X'(\ell_t^\beta)$ is differentiable on $[-\alpha, 0]$,[36] so we express the integrand as an integral:

$$\frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t) - \frac{\partial \ell_t^{-\alpha}}{\partial \ell_0} X'\left(\ell_t^{-\alpha}\right) = \int_{-\alpha}^{0} \left(\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2} X'\left(\ell_t^\beta\right) + \left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 X''\left(\ell_t^\beta\right)\right) d\beta.$$

[35] As initial quality is high and the firm invests until the cutoff time, $\theta_0 = H$ and $A(\ell_t) = 1$ for $t < T$, firm quality stays high until the cutoff time, allowing us to replace the probability measure $\mathbb{E}^{A,H}$ by $\mathbb{E}^H$.

[36] Here we write $\ell_t^\beta = \ell_t(\ell + \beta, h, \ell^*)$ for all $\beta \in [-\alpha, 0]$; below we also use the notation $\ell_t^{\beta,\varnothing} = \ell_t(\ell + \beta, \varnothing, \ell^*)$.

Decomposing the integrand of this expression into a sum of three terms, we substitute back into (D.29),

$$\Pi'_H(\ell) - \Pi'_H(\ell-\alpha)$$

$$\le \mathbb{E}^H\left[\int_{t=0}^T e^{-rt}\int_{-\alpha}^0 \left(\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2} X'(\ell_t^\beta)\right) d\beta\, dt\right] \tag{D.30}$$

$$+ \mathbb{E}^H\left[\int_{t=0}^T e^{-rt}\int_{-\alpha}^0 \left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{X''(\ell_t^\beta), 0\}\, d\beta\, dt\right] \tag{D.31}$$

$$- \mathbb{E}^H\left[\int_{t=0}^T e^{-rt}\int_{-\alpha}^0 \left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{-X''(\ell_t^\beta), 0\}\, d\beta\, dt\right], \tag{D.32}$$

and proceed by showing that the negative term (D.32) is bounded below by some $\kappa_3 \alpha e^{-\ell_0}$ when $\ell_0 \in [\ell^\Gamma - 3\gamma/2, \ell^* - \gamma/2]$, while terms (D.30) and (D.31) are bounded above by some $\kappa_4 \alpha e^{-2\ell_0}$, $\kappa_5 \alpha e^{-2\ell_0}$ when $\ell_0 < \ell^*$. Thus, for ℓ^Γ sufficiently large and $\ell_0 \in [\ell^\Gamma - 3\gamma/2, \ell^* - \gamma/2]$, the negative term (D.32) dominates the terms (D.30) and (D.31), implying (D.27); (D.28) follows with $\kappa_2 = \kappa_4 + \kappa_5$.

Negative term (D.32): Fix $\ell_0 \in [\ell^\Gamma - 3\gamma/2, \ell^* - \gamma/2]$. The proof of the lower bound in (D.18) shows that $\tau := \gamma/(2g(0))$ is a lower bound for the time-to-cutoff $T = T(\ell_0, \ell^*)$, subject to $\ell^\Gamma \ge 3\gamma/2$. By (C.4), we have $X''(\ell) = \frac{e^\ell(1-e^\ell)}{(1+e^\ell)^3}$ so that $\lim_{\ell\to\infty} X''(\ell) e^\ell = -1$, and thus $-X''(\ell) \ge e^{-\ell}/2$ for all $\ell \ge \ell^\Gamma - 5\gamma/2$, for sufficiently large ℓ^Γ. This allows us to show

$$\mathbb{E}^H\left[\int_{t=0}^T e^{-rt}\int_{-\alpha}^0 \left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{-X''(\ell_t^\beta), 0\}\, d\beta\, dt\right]$$
$$\ge \int_{t=0}^\tau e^{-(r+\mu_H)t}\int_{-\alpha}^0 \left(\frac{\partial \ell_t^{\beta,\varnothing}}{\partial \ell_0}\right)^2 \left(-X''(\ell_t^{\beta,\varnothing})\right) dt$$
$$\ge \tau e^{-(r+\mu_H)\tau} \alpha e^{-2\lambda\tau} \frac{e^{-(\ell_0+\gamma/2)}}{2} =: \kappa_3 \alpha e^{-\ell_0}.$$

The first line truncates the integral at τ and discards any histories with a signal arrival. The second line applies the lower bounds $\partial \ell_t^{\beta,\varnothing}/\partial \ell_0 = \exp(\int_0^t g'(\ell_s^{\beta,\varnothing})\, ds) = \exp(-\int_0^t \lambda e^{-\ell_s^{\beta,\varnothing}}\, ds) \ge \exp(-\lambda\tau)$ (using equations (D.5) and (C.5), $e^{-\ell_s^{\beta,\varnothing}} < 1$, and $t \le \tau$) and $-X''(\ell_t^{\beta,\varnothing}) \ge e^{-\ell_t^{\beta,\varnothing}}/2 \ge e^{-\ell_t^{\varnothing}}/2 \ge e^{-(\ell_0+g(0)\tau)}/2 = e^{-(\ell_0+\gamma/2)}/2$ for all $t \le \tau$.

First smaller term (D.30): Equation (D.6), together with $\partial \ell_t^\beta/\partial \ell_0 < 1$, $g''(\ell) = \lambda e^{-\ell}$, and $\ell_s^\beta \ge \ell_s - \gamma$, implies an upper bound on the second derivative,

$$\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2} = \frac{\partial \ell_t^\beta}{\partial \ell_0}\int_{s=0}^t \frac{\partial \ell_s^\beta}{\partial \ell_0} g''(\ell_s^\beta)\, ds \le \int_{s=0}^t \lambda e^{-\ell_s^\beta}\, ds \le \lambda t e^{-\min_{s\le t}\{\ell_s - \gamma\}}.$$

This allows us to show

$$\begin{aligned}
&\mathbb{E}^H\left[\int_{t=0}^{T} e^{-rt}\int_{-\alpha}^{0}\left(\frac{\partial^2\ell_t^\beta}{\partial\ell_0^2}X'\left(\ell_t^\beta\right)\right)d\beta\, dt\right]\\
&\quad\le \mathbb{E}^H\left[\int_{t=0}^{T} e^{-rt}\alpha\lambda t e^{-\min_{s\le t}\{\ell_s-\gamma\}}e^{-(\ell_t-\gamma)}\,dt\right]\\
&\quad\le \alpha\lambda\int_{t=0}^{\infty} te^{-rt}\sum_{\nu=0}^{\infty}\left[\frac{(\mu_H t)^\nu}{\nu!}e^{-\mu_H t}e^{-2(\ell_0-\nu\gamma-\gamma)}\right]dt\\
&\quad= \alpha\lambda e^{-2(\ell_0-\gamma)}\int_{t=0}^{\infty} te^{-(r+\mu_H)t}\sum_{\nu=0}^{\infty}\left[\frac{(\mu_H te^{2\gamma})^\nu}{\nu!}\right]dt\\
&\quad= \alpha\lambda e^{-2(\ell_0-\gamma)}\int_{t=0}^{\infty} te^{-(r+\mu_H)t}e^{\mu_H(\mu_L^2/\mu_H^2)t}\,dt\\
&\quad= \frac{\alpha\lambda e^{2\gamma}e^{-2\ell_0}}{(r-\mu_H(\mu_L^2/\mu_H^2-1))^2} =: \kappa_4\alpha e^{-2\ell_0}.
\end{aligned}$$

The first line uses $X'(\ell_t^\beta)\le e^{-\ell_t^\beta}\le e^{-(\ell_t-\gamma)}$ and $\partial^2\ell_t/\partial^2\ell_0^\beta\le\lambda te^{-\min_{s\le t}\{\ell_s-\gamma\}}$. The second line uses that both ℓ_t and $\min_{s\le t}\{\ell_s\}$ are bounded below by $\ell_0-\nu\gamma$ if ν signals have arrived by time t, and the definition of the Poisson distribution with parameter $\mu_H t$. The third line rearranges. The fourth line uses $e^\gamma=\mu_L/\mu_H$ and the definition of the Poisson distribution with parameter $\mu_H(\mu_L^2/\mu_H^2)t$. The fifth line uses $r>\mu_L(\mu_L^2/\mu_H^2-1)>\mu_H(\mu_L^2/\mu_H^2-1)$ by assumption (D.1) and the fact that $\int_0^\infty te^{-\alpha t}\,dt=1/\alpha^2$.

Second smaller term (D.31): We next show that

$$\begin{aligned}
&\mathbb{E}^H\left[\int_{t=0}^{T} e^{-rt}\int_{-\alpha}^{0}\left(\frac{\partial\ell_t^\beta}{\partial\ell_0}\right)^2\max\left\{X''\left(\ell_t^\beta\right),0\right\}d\beta\, dt\right]\\
&\quad\le \mathbb{E}^H\left[\int_0^{T} e^{-rt}\alpha e^{-2(\ell_t-\gamma)}\,dt\right]\\
&\quad\le \alpha\int_0^{\infty} e^{-rt}\sum_{\nu=0}^{\infty}\left[\frac{(\mu_H t)^\nu}{\nu!}e^{-\mu_H t}e^{-2(\ell_0-\nu\gamma-\gamma)}\right]dt\\
&\quad= \alpha e^{-2(\ell_0-\gamma)}\int_0^{\infty} e^{-(r+\mu_H)t}e^{\mu_H(\mu_L^2/\mu_H^2)t}\,dt\\
&\quad= \frac{\alpha e^{2\gamma}e^{-2\ell_0}}{r-\mu_H(\mu_L^2/\mu_H^2-1)} =: \kappa_5\alpha e^{-2\ell_0}.
\end{aligned}$$

The first line uses $\partial\ell_t^\beta/\partial\ell_0 < 1$, $X''(\ell) = \frac{e^\ell - e^{2\ell}}{(1+e^\ell)^3} \le e^{-2\ell}$, and $\ell_t^\beta \ge \ell_t - \gamma$. The second line uses the definition of the Poisson distribution with parameter $\mu_H t$ and the lower bound on reputation $\ell_t \ge \ell_0 - \nu\gamma$. The third line uses $e^\gamma = \mu_L/\mu_H$ and the definition of the Poisson distribution with parameter $\mu_H(\mu_L^2/\mu_H^2)t$. The fourth line uses that $r > \mu_L(\mu_L^2/\mu_H^2 - 1) > \mu_H(\mu_L^2/\mu_H^2 - 1)$ by assumption (D.1).

We have thus established the desired upper bounds for (D.30) and (D.31) and the lower bound for (D.32), implying (D.27) and (D.28), and thus (D.25).

Dividends above the cutoff (D.26): For any $\ell > \ell^*$, we now show

$$
\begin{aligned}
&\Gamma_H(\ell^*) - \Gamma_H(\ell) \\
&\quad = |\mu|\left(\int_{\ell^*-\gamma}^{\ell^*} \Pi_H'(\ell')\,d\ell' - \int_{\ell-\gamma}^{\ell} \Pi_H'(\ell')\,d\ell'\right) \\
&\quad = |\mu|\left(\int_{\ell^*-\gamma}^{\min\{\ell^*,\ell-\gamma\}} \Pi_H'(\ell')\,d\ell' - \int_{\max\{\ell^*,\ell-\gamma\}}^{\ell} \Pi_H'(\ell')\,d\ell'\right) \\
&\quad > |\mu|\big(\min\{\ell-\ell^*, \gamma/2\}k_1 e^{-\ell^*} - \min\{\ell-\ell^*, \gamma\}k_3 e^{-2\ell^*}\big).
\end{aligned}
$$

The last line uses the bounds (D.18) and (D.20) from Lemma 12B. This term is strictly positive for all $\ell^* \ge \ell^\Gamma$ and sufficiently large ℓ^Γ. *Q.E.D.*

Lemma 14B shows that investment incentives are decreasing on an interval $\ell \in [\ell^\Delta, \ell^*]$, establishing that the firm prefers to work in this range, as required by (C.6). The idea of the proof is to express the investment incentives in terms of decreasing future reputational dividends (as shown in Lemma 13B).

LEMMA 14B: *Assume* (D.1). *There exists ℓ^Δ such that for any $c > 0$, $\ell^* > \ell^\Delta$ with indifference at the cutoff*, (D.13), *we have $\lambda\Delta'(\ell) < 0$ for $\ell \in [\ell^\Delta, \ell^*)$.*

PROOF: By (C.2), we write the payoff of quality as an integral over future reputational dividends,

$$\Delta(\ell_0) = \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\Gamma_H(\ell_t)\,dt\right].$$

Next, we differentiate and decompose the integral into its positive and negative contributions:

$$\Delta'(\ell_0) = \mathbb{E}^L\left[\int_{t=0}^{T} e^{-(r+\lambda)t}\frac{\partial\ell_t}{\partial\ell_0}\Gamma_H'(\ell_t)\,dt\right] \tag{D.33}$$

$$= \mathbb{E}^L\left[\int_{t=0}^{T} e^{-(r+\lambda)t}\frac{\partial\ell_t}{\partial\ell_0}\max\{\Gamma_H'(\ell_t), 0\}\,dt\right] \tag{D.34}$$

$$- \mathbb{E}^L\left[\int_{t=0}^{T} e^{-(r+\lambda)t}\frac{\partial\ell_t}{\partial\ell_0}\max\{-\Gamma_H'(\ell_t), 0\}\,dt\right]. \tag{D.35}$$

Recall from Lemma 13B that ℓ^Γ is such that $\Gamma_H'(\ell) \le -k_4 e^{-\ell}$ for all $\ell^* > \ell^\Gamma$ and all $\ell \in [\ell^\Gamma, \ell^*)$. Now choose $\ell^\Delta \ge 3\max\{\ell^\Gamma, 0\} + 3\gamma/2$. We first prove

$$\text{(D.36)} \quad \Delta'(\ell_0) < 0 \quad \text{for} \quad \ell_0 \in [\ell^\Delta - 3\gamma/2, \ell^* - \gamma/2]$$

by establishing that the negative term (D.35) is bounded below by some $\kappa e^{-\ell_0}$, while (D.34) is of order $O(e^{-(4/3)\ell_0})$. Then (D.33) is negative for all $\ell_0 > \ell^\Delta$ and sufficiently large ℓ^Δ.

Negative term (D.35): Recall from the proof of the lower bound in (D.18) that $\tau := \gamma/(2g(0))$ is a lower bound for the time-to-cutoff $T = T(\ell_0, \ell^*)$ when $\ell_0 \in [0, \ell^* - \gamma/2]$. Then we can show

$$\begin{aligned}
&\mathbb{E}^L\left[\int_{t=0}^{T} e^{-(r+\lambda)t} \frac{\partial \ell_t}{\partial \ell_0} \max\{-\Gamma_H'(\ell_t), 0\}\, dt\right] \\
&\quad \ge \int_{t=0}^{\tau} e^{-(r+\lambda+\mu_L)t} \frac{\partial \ell_t^\varnothing}{\partial \ell_0} \left(-\Gamma_H'(\ell_t^\varnothing)\right) dt \\
&\quad \ge \tau e^{-(r+\lambda+\mu_L)\tau} e^{-\lambda\tau} k_4 e^{-(\ell_0+\gamma/2)} =: \kappa e^{-\ell_0}.
\end{aligned}$$

The first line truncates the integral at τ and discards any histories with a signal arrival. The second line uses the bounds $\partial \ell_t^\varnothing / \partial \ell_0 = \exp(\int_0^\tau g'(\ell_s^\varnothing)\, ds) = \exp(-\int_0^\tau \lambda e^{-\ell_s^\varnothing}\, ds) \ge \exp(-\lambda\tau)$ and $-\Gamma_H'(\ell_t^\varnothing) \ge k_4 e^{-\ell_t^\varnothing} \ge k_4 e^{-(\ell_0+\gamma/2)}$ from Lemma 13B.

Smaller positive term (D.34): We first establish that for any $\ell < \ell^*$,

$$\text{(D.37)} \quad \max\{\Gamma_H'(\ell), 0\} \le \frac{|\mu|}{r} e^{-2(\ell-\ell^\Gamma)}.$$

For $\ell \in (\ell^\Gamma, \ell^*)$, this follows because $\Gamma_H'(\ell) < 0$ (Lemma 13B). For $\ell \le \ell^\Gamma$, (D.37) follows because $\Gamma_H'(\ell) = |\mu|(\Pi_H'(\ell) - \Pi_H'(\ell - \gamma)) \le \frac{|\mu|}{r} \le \frac{|\mu|}{r} e^{-2(\ell-\ell^\Gamma)}$, where the first inequality follows by

$$\Pi_H'(\ell) = \mathbb{E}^{A,\theta}\left[\int_0^\infty e^{-rt} \frac{d\ell_t}{d\ell_0} X'(\ell_t)\, dt\right] \le \int_0^\infty e^{-rt}\, dt = \frac{1}{r},$$

where we used $\partial \ell_t / \partial \ell_0 < 1$ and $X'(\ell) < 1$.

Then when $\ell_0 \in [\ell^\Delta - 3\gamma/2, \ell^* - \gamma/2]$, we show that

$$\begin{aligned}
&\mathbb{E}^L\left[\int_{t=0}^{T} e^{-(r+\lambda)t} \frac{\partial \ell_t}{\partial \ell_0} \max\{\Gamma_H'(\ell_t), 0\}\, dt\right] \\
&\quad \le \frac{|\mu|}{r} \mathbb{E}^L\left[\int_{t=0}^{\infty} e^{-(r+\lambda)t} e^{-2(\ell_t-\ell^\Gamma)}\, dt\right] \\
&\quad \le \frac{|\mu|}{r} \int_{t=0}^{\infty} e^{-(r+\lambda)t} e^{-\mu_L t} \sum_{v=0}^{\infty} \frac{(\mu_L t)^\nu}{\nu!} e^{-2(\ell_0-\nu\gamma-\ell^\Gamma)}\, dt
\end{aligned}$$

$$= \frac{|\mu|}{r} e^{-2(\ell_0-\ell^\Gamma)} \int_{t=0}^{\infty} e^{-(r+\lambda)t} e^{-\mu_L t} \sum_{v=0}^{\infty} \frac{(\mu_L^3 t/\mu_H^2)^\nu}{\nu!}\, dt$$

$$\leq \frac{|\mu| e^{-(4/3)\ell_0}}{r(r+\lambda-\mu_L(\mu_L^2/\mu_H^2-1))} = O\big(e^{-(4/3)\ell_0}\big).$$

The first line uses $\partial\ell_t/\partial\ell_0 < 1$ and (D.37). The second line uses $\ell_t \geq \ell_0 - \nu\gamma$ and the definition of the Poisson distribution with parameter $\mu_L t$. The third line rearranges using $e^{2\nu\gamma} = (\mu_L/\mu_H)^{2\nu}$. The fourth line uses the definition of the Poisson distribution with parameter $(\mu_L^3/\mu_H^2)t$, assumption (D.1), and $\ell_0 > \ell^\Delta - 3\gamma/2 \geq 3\ell^\Gamma$ so that $2(\ell_0 - \ell^\Gamma) \geq 4\ell_0/3$.

This finishes the proof that $\Delta'(\ell_0) < 0$ for all $\ell_0 \in [\ell^\Delta - 3\gamma/2, \ell^* - \gamma/2]$.

Establishing $\Delta'(\ell_0) < 0$ *for* $\ell_0 \in [\ell^* - \gamma/2, \ell^*)$: We truncate (D.33) when either a signal arrives or $\ell_t^\varnothing$ reaches the cutoff at time $T^\varnothing = T^\varnothing(\ell_0, \ell^*) = \min\{t : \ell_t^\varnothing = \ell^*\}$:

$$\Delta'(\ell_0) = \int_0^{T^\varnothing} e^{-(r+\lambda+\mu_L)t} \frac{\partial \ell_t}{\partial \ell_0} \big[\Gamma_H'\big(\ell_t^\varnothing\big) + \mu_L \Delta'\big(\ell_t^\varnothing - \gamma\big)\big]\, dt.$$

By Lemma 13B, the flow payoff $\Gamma_H'(\ell_t^\varnothing)$ is negative, and the continuation term $\Delta'(\ell_t^\varnothing - \gamma)$ is negative by (D.36) because $\ell_t^\varnothing - \gamma \in [\ell^\Delta - 3\gamma/2, \ell^* - \gamma/2]$. *Q.E.D.*

Finally, we show that investment incentives for low levels of reputation $\ell \in [-\infty, \ell^\Delta]$ are bounded below, so if costs are sufficiently low, then the firm prefers to invest in this range as required by (C.6).

LEMMA 15B: *Assume* (D.1) *and fix* ℓ^Δ *as in Lemma* 14B. *There exists* $\overline{\overline{c}} > 0$ *such that for all* $\ell^* > \ell^\Delta$, $c < \overline{\overline{c}}$ *with indifference at the cutoff*, (D.13), *we have* $\lambda\Delta(\ell) > c$ *for all* $\ell \in [-\infty, \ell^\Delta]$.

PROOF: We show that there exists $\kappa > 0$ such that for all $\ell^* > \ell^\Delta$ and c with indifference at the cutoff, we have

$$\Delta(\ell) > \kappa e^{-\ell^\Delta} \quad \text{for all } \ell \in \big[-\infty, \ell^\Delta\big].$$

We then set $\overline{\overline{c}} = \lambda\kappa e^{-\ell^\Delta}$ and the lemma is proven.

Consider first $\ell_0 \in [\gamma, \ell^\Delta]$. We show that

$$\Delta(\ell_0) = \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\Gamma_H(\ell_t)\, dt\right]$$

$$\geq \int_0^\infty e^{-(r+\lambda+\mu_L)t}\Gamma_H\big(\ell_t^\varnothing\big)\, dt$$

$$\geq \int_0^1 e^{-(r+\lambda+\mu_L)t}|\mu|\left(\int_{\ell_t^\varnothing-\gamma}^{\ell_t^\varnothing} \Pi_H'(\ell)\,d\ell\right)dt$$
$$\geq e^{-(r+\lambda+\mu_L)}|\mu|\frac{\gamma}{2}k_1 e^{-\ell_1^\varnothing}$$
$$\geq e^{-(r+\lambda+\mu_L)}|\mu|\frac{\gamma}{2}k_1 e^{-(\ell^\Delta+g(\ell^\Delta))}.$$

The second line discards all histories with a signal arrival. The third line truncates the integral at $t=1$ and writes the reputational dividend as an integral over the marginal payoff of reputation. The fourth line applies the lower bound $\Pi_H'(\ell)\geq k_1e^{-\ell}$ from Lemma 12B to all $\ell\in[\ell_t^\varnothing-\gamma,\ell_t^\varnothing-\gamma/2]\subseteq[0,\ell^*-\gamma/2]$. The fifth line uses $\ell_t^\varnothing\leq\ell^\Delta+g(\ell^\Delta)$ for $t\in[0,1]$.

For $\ell_0\in[-\infty,\gamma]$ we have to amend the above argument, allowing $\{\ell_t^\varnothing\}_{t\geq0}$ to drift above γ, so that the bound $\Pi_H'(\ell)\geq k_1e^{-\ell}$ from Lemma 12B applies. Thus, let $T^\varnothing:=\min\{t:\ell_t^\varnothing\geq\gamma\}$. In x-space, it is easy to see that $T^\varnothing\leq\frac{x(\gamma)}{\lambda(1-x(\gamma))}=e^\gamma/\lambda$, because reputational drift in the work region, $\lambda(1-x)+|\mu|x(1-x)$, is bounded below by $\lambda(1-X(\gamma))$ on $[0,X(\gamma)]$. Then the same logic as above, focusing on times $t\in[T^\varnothing,T^\varnothing+1]$ rather than $t\in[0,1]$, shows that $\Delta(\ell_0)$ is bounded below by the constant $e^{-(r+\lambda+\mu_L)(T^\varnothing+1)}|\mu|\frac{\gamma}{2}k_1e^{-(\gamma+g(\gamma))}$. *Q.E.D.*

D.4. *Bad News—Shirk Above Cutoff*

Lemma 16B shows that firms with high reputations $\ell\in[\ell^*,\infty]$ shirk, as required by (C.6). The idea of the proof is to write the payoff of quality as a short stream of dividends and a continuation payoff, and then show that both terms are higher when the firm's initial reputation is at the cutoff rather than above the cutoff.

LEMMA 16B: *Assume* (D.1) *and fix* ℓ^Δ *as in Lemma* 14B. *For any* $c>0$, $\ell^*>\ell^\Delta+\gamma$ *with indifference at the cutoff*, (D.13), *we have* $\lambda\Delta(\ell^*)>\lambda\Delta(\ell)$ *for all* $\ell\in(\ell^*,\infty]$.

PROOF: Assume to the contrary that the set $\{\ell\in(\ell^*,\infty]:\Delta(\ell^*)\leq\Delta(\ell)\}$ is nonempty. If its infimum is strictly greater than ℓ^*, let ℓ_0 be this infimum; otherwise, let ℓ_0 be any element of $\{\ell\in(\ell^*,\ell^*+\gamma):\Delta(\ell^*)\leq\Delta(\ell)\}$.

To obtain a contradiction, we compare $\Delta(\ell^*)$ and $\Delta(\ell_0)$ by terminating their respective dividend expansion when a signal arrives or $\ell_t^\varnothing$ reaches the cutoff at time $T^\varnothing=T^\varnothing(\ell_0,\ell^*)=\min\{t:\ell_t^\varnothing=\ell^*\}$:

$$\Delta(\ell^*)=\int_0^{T^\varnothing} e^{-(r+\lambda+\mu_L)t}\big(\Gamma_H(\ell^*)+\mu_L\Delta(\ell^*-\gamma)\big)\,dt$$
$$+e^{-(r+\lambda+\mu_L)T^\varnothing}\Delta(\ell^*),$$

$$\Delta(\ell_0) = \int_0^{T^\varnothing} e^{-(r+\lambda+\mu_L)t}\big(\Gamma_H\big(\ell_t^\varnothing\big) + \mu_L\Delta\big(\ell_t^\varnothing - \gamma\big)\big)\,dt$$
$$+e^{-(r+\lambda+\mu_L)T^\varnothing}\Delta\big(\ell^*\big).$$

First note that $\ell_t^\varnothing > \ell^*$ for $t < T^\varnothing$. Thus, equation (D.26) implies $\Gamma_H(\ell^*) > \Gamma_H(\ell_t^\varnothing)$. Next, if $\ell_t^\varnothing - \gamma \leq \ell^*$, then $\Delta(\ell^* - \gamma) > \Delta(\ell_t^\varnothing - \gamma)$ follows by Lemma 14B. Finally, if $\ell_t^\varnothing - \gamma > \ell^*$, then $\Delta(\ell^* - \gamma) > \Delta(\ell^*) > \Delta(\ell_t^\varnothing - \gamma)$, where the first inequality follows by Lemma 14B and the second follows by the choice of ℓ_0, because $\ell^* < \ell_0 - \gamma$ implies that $\ell_0 = \inf\{\ell \in (\ell^*, \infty] : \Delta(\ell^*) \leq \Delta(\ell)\}$. *Q.E.D.*

This finishes the existence proof for a work–shirk equilibrium under imperfect bad news learning. We have found a cutoff ℓ^* such that the firm (a) prefers to invest below ℓ^* (Lemmas 14B and 15B), (b) is indifferent at ℓ^* (Lemma 11B), and (c) prefers to disinvest above the cutoff (Lemma 16B).

APPENDIX E: GOOD NEWS

We now show how to adapt the proof of Theorem 5(a) to the case of good news learning under the assumption $\mu > \lambda$. The proof strategy and the sequence of lemmas is identical to the case of bad news learning, but some of the more nuanced arguments are specific to the learning process. The proofs in this appendix are not self-contained, but are based on the proofs in the bad news case presented in Appendix D.

First, note that reputation jumps up at the arrival of a good news signal, so now we have $j(\ell) = \ell + \gamma$, where $\gamma = \log(\mu_H/\mu_L)$. Second, some of the intermediate results assume

(E.1) $\qquad r > 2\mu.$

E.1. *Good News—Marginal Payoff of Reputation*

Recall that we assume $\mu > \lambda$. This implies that for high ℓ, reputational drift is negative even if the firm is believed to be working, since $g(\ell) = \lambda(1 + e^{-\ell}) - \mu$. Let

$$\ell^g = L\big(x^g\big) = \log\frac{\lambda}{\mu - \lambda}.$$

We restrict attention to $\ell^* \in (\ell^g, \infty)$ in what follows. The drift in the work region $g(\ell) = \lambda(1 + e^{-\ell}) - \mu$ is negative in (ℓ^g, ℓ^*) and vanishes at ℓ^g. Denote the left- and right-sided limits of reputational drift at the cutoff by

$$\underline{g} := \lim_{\varepsilon\to 0} g\big(\ell^* - \varepsilon\big) = \lambda\big(1 + e^{-\ell^*}\big) - \mu < 0,$$
$$\overline{g} := \lim_{\varepsilon\to 0} g\big(\ell^* + \varepsilon\big) = -\lambda\big(1 + e^{\ell^*}\big) - \mu < 0,$$

as shown in Figure 9.

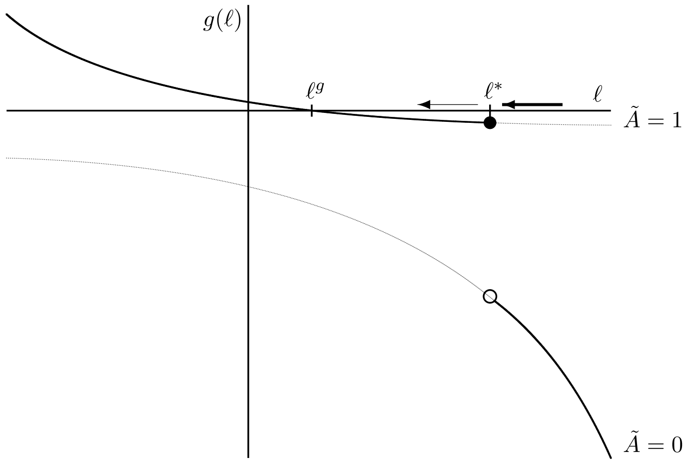


FIGURE 9.—Reputational drift in log-likelihood-ratio space: good news. The top curve shows $g(\ell)$ given $\tilde{A}=1$; this curve asymptotes to $-\mu+\lambda$, and is negative for $\ell > \ell^g$. The bottom curve shows $g(\ell)$ given $\tilde{A}=0$; this curve asymptotes to $-\mu-\lambda$. We focus on work–shirk equilibria for high ℓ^*, which means there is a small negative drift $\underline{g}$ immediately below ℓ^* and a large negative drift $\overline{g}$ immediately above ℓ^*.

In the following lemma, we focus on work–shirk beliefs with permeable cutoff $\ell^* > \ell^g$, initial reputation $\ell_0 \in \mathbb{R}$, history $h = (t_1, t_2, \dots)$, and time t that satisfy

$$\ell_{s-} \neq \ell^* \quad \text{and} \quad \ell_s \neq \ell^* \quad \text{for } s = t \text{ and any } s \in \{t_1, t_2, \dots\}. \tag{E.2}$$

Intuitively, this means that the trajectory $\{\ell_s\}_{s\geq 0}$ does not drift or jump to the cutoff ℓ^* at time t or at any signal arrival time t_i. This condition ensures that left- and right-sided derivatives of $\ell_t(\ell_0, h, \ell^*)$ coincide. It is generic in the sense that for any ℓ^*, ℓ_0, t, it is satisfied for almost all histories h.

Also recall the notion of the time-to-cutoff $T = T(\ell_0, \ell^*)$, defined in equation (D.3), and let $\#\{T_i < t : \ell_{T_i} = \ell^*\}$ be the number of times reputation passes through the cutoff before time t.

LEMMA 8G: *Fix admissible work–shirk beliefs with permeable cutoff $\ell^* > \ell^g$, initial reputation $\ell_0 \in \mathbb{R}$, history h, and time $t \neq T$ that satisfy (E.2). If $t < T$, then $\ell_t(\ell_0, h, \ell^*)$ is twice differentiable in ℓ_0 and the derivatives are given by*

$$\frac{\partial \ell_t}{\partial \ell_0} = \exp\left(\int_0^t g'(\ell_s)\, ds\right), \tag{E.3}$$

$$\frac{\partial^2 \ell_t}{\partial \ell_0^2} = \frac{\partial \ell_t}{\partial \ell_0}\left(\int_0^t g''(\ell_s)\frac{\partial \ell_s}{\partial \ell_0}\, ds\right). \tag{E.4}$$

If $t > T$, then $\ell_t(\ell_0, h, \ell^*)$ is differentiable in ℓ_0 and the derivative is given by

$$\frac{\partial \ell_t}{\partial \ell_0} = \exp\left(\int_0^t g'(\ell_s)\,ds\right)(\underline{g}/\overline{g})^{\#\{T_i<t:\ell_{T_i}=\ell^*\}}. \tag{E.5}$$

As $g'(\ell)$ is negative and $\underline{g}/\overline{g} < 1$, we have $\partial \ell_t/\partial \ell_0 < 1$.

Intuitively, whenever reputation passes through the cutoff, the reputational increment $\ell_t^\varepsilon - \ell_t$ decreases by a factor $\underline{g}/\overline{g}$, because, when $\ell_t < \ell^* < \ell_t^\varepsilon$, the trajectory $\{\ell_t\}_{t\geq 0}$ decreases at rate $\underline{g}$ while $\{\ell_t^\varepsilon\}_{t\geq 0}$ decreases at rate $\overline{g}$. For high values of ℓ^*, the factor $\underline{g}/\overline{g} < \frac{\mu-\lambda}{\lambda}e^{-\ell^*}$ is close to 0.

PROOF OF LEMMA 8G: For $t < T$, the trajectory $\{\ell_t\}_{t\geq 0}$ is bounded away from the cutoff ℓ^* by assumption (E.2), so formulas (E.3) and (E.4) follow as in the bad news case.

For $t > T$, formula (E.3) is incorrect because it ignores the fact that $g'(\ell_T) = g'(\ell^*) = \lim_{\varepsilon\to 0}(\overline{g} - \underline{g})/\varepsilon = -\infty$. To address this issue and derive (E.5), assume that $\{\ell_s\}_{s\in[0,t]}$ hits the cutoff ℓ^* once at time $T < t$, and define the left- and right-translated trajectories $\{\ell_s^{\pm\varepsilon}\}_{s\in[0,t]}$ by $\ell_s^{\pm\varepsilon} = \ell_s(\ell_0 \pm \varepsilon, h, \ell^*)$. Then, for small $\varepsilon > 0$, assumption (E.2) and negative drift at ℓ^* imply that $\{\ell_s^{-\varepsilon}\}_{s\in[0,t]}$ hits ℓ^* just before T and that $\{\ell_s^\varepsilon\}_{s\in[0,t]}$ hits ℓ^* just after T, with times-to-cutoff $T^{-\varepsilon} = T(\ell_0^{-\varepsilon}) \in (T - \varepsilon/|\underline{g}|, T)$ and $T^\varepsilon \in (T, T + \varepsilon/|\overline{g}|)$.

Next, we separate the time interval $[0, t]$ into three subintervals $[0, T]$, $[T, T^\varepsilon]$, and $[T^\varepsilon, t]$:

$$\begin{aligned} &\log\left(\frac{\ell_t^\varepsilon - \ell_t}{\ell_0^\varepsilon - \ell_0}\right) \\ &\quad = \log\left(\frac{\ell_T^\varepsilon - \ell_T}{\ell_0^\varepsilon - \ell_0}\right) + \log\left(\frac{\ell_{T^\varepsilon}^\varepsilon - \ell_{T^\varepsilon}}{\ell_T^\varepsilon - \ell_T}\right) + \log\left(\frac{\ell_t^\varepsilon - \ell_t}{\ell_{T^\varepsilon}^\varepsilon - \ell_{T^\varepsilon}}\right) \\ &\quad = \int_0^T \frac{g(\ell_s^\varepsilon) - g(\ell_s)}{\ell_s^\varepsilon - \ell_s}\,ds + \log\left(\frac{\ell_{T^\varepsilon}^\varepsilon - \ell_{T^\varepsilon}}{\ell_T^\varepsilon - \ell_T}\right) + \int_{T^\varepsilon}^t \frac{g(\ell_s^\varepsilon) - g(\ell_s)}{\ell_s^\varepsilon - \ell_s}\,ds. \end{aligned} \tag{E.6}$$

To analyze the middle term, we use a Taylor expansion of ℓ_t around $t = T$, $\ell_{T^\varepsilon} = \ell_T + \underline{g}(T^\varepsilon - T) + O(\varepsilon^2)$; together with $\ell_{T^\varepsilon}^\varepsilon = \ell_T = \ell^*$, this implies $\ell_{T^\varepsilon}^\varepsilon - \ell_{T^\varepsilon} = \ell_T - \ell_{T^\varepsilon} = \underline{g}(T - T^\varepsilon) + O(\varepsilon^2)$. Similarly, $\ell_T^\varepsilon - \ell_T = \ell_T^\varepsilon - \ell_{T^\varepsilon}^\varepsilon = \overline{g}(T - T^\varepsilon) + O(\varepsilon^2)$. Taking the exponential of (E.6), in the limit as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\frac{\ell_t^\varepsilon - \ell_t}{\ell_0^\varepsilon - \ell_0} = \exp\left(\int_0^t g'(\ell_s)\,ds\right)(\underline{g}/\overline{g}).$$

The left-sided limit coincides with this term by applying analogous arguments to $\ell_T - \ell_T^{-\varepsilon}$ and $\ell_{T^{-\varepsilon}} - \ell_{T^{-\varepsilon}}^{-\varepsilon}$. Generally, when there are multiple times $T_i \in [0, t]$ with $\ell_{T_i} = \ell^*$, (E.5) follows by the same arguments. *Q.E.D.*

LEMMA 9G: *Fix admissible work–shirk beliefs with cutoff* $\ell^* > \ell^g$.

(a) *Payoff functions* $\Pi_\theta(\ell)$ *are continuous on* $\mathbb{R} \cup \{-\infty, \infty\}$.

(b) *Payoff of quality at the cutoff is strictly positive*: $\Delta(\ell^*) > 0$.

(c) *If the firm is indifferent at the cutoff* ℓ^*, *that is*, $\lambda\Delta(\ell^*) = c$, *payoff functions are differentiable with derivative*

$$\Pi'_\theta(\ell) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt} \frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\, dt\right] > 0 \quad \textit{for all } \ell \in \mathbb{R}, \tag{E.7}$$

where $\partial \ell_t / \partial \ell_0$ *is given by* (E.3) *for almost all histories*.

PROOF: The proof is identical to the proof of Lemma 9B, with one exception: The denominator in (D.11) now equals $|\overline{g}|$ rather than $\min\{\underline{g}, |\overline{g}|\}$ because drift is negative around the cutoff and the trajectory $\{\ell_t^\varepsilon\}_{t\geq 0}$ takes at most time $\varepsilon/|\overline{g}|$ to drift out of the interval $[\ell^*, \ell^* + \varepsilon]$. *Q.E.D.*

It is useful to separate (E.7) into two terms

$$\Pi'_\theta(\ell) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{T} e^{-rt} \frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\, dt + \int_{t=T}^{\infty} e^{-rt} \frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\, dt\right],$$

where $T = T(\ell_0, \ell^*) = \min\{t : \ell_t = \ell^*\}$ is the first time the trajectory passes through the cutoff. In contrast to the bad news case, reputational increments do not disappear at time T and the second term does not vanish. However, it is of order $O(e^{-\ell^*})$ because $\partial \ell_t / \partial \ell_0 \leq \underline{g}/\overline{g} < \frac{\mu-\lambda}{\lambda} e^{-\ell^*}$ for $t > T$, as established in (E.3).

E.2. *Good News—Indifference at Cutoff*

LEMMA 10G: *For any* θ *and* $c > 0$, *the function* $\ell^* \mapsto \Pi_{\theta,\ell^*,c}(\ell^*)$ *is continuous on* (ℓ^g, ∞) *and the limit* $\lim_{\ell^*\to\infty} \Pi_{\theta,\ell^*,c}(\ell^*) = \frac{1-c}{r}$ *does not depend on* θ.

PROOF: The proof is identical to the proof of Lemma 10B with three exceptions: First, we can define $\kappa_1 = 0$ because the trajectories spend no time at their respective permeable work–shirk cutoffs. Second, we need to define $\kappa_2 = 1/|\overline{g}|$ because the trajectory $\{\ell_t^\varepsilon\}_{t\geq 0}$ takes at most time $\varepsilon/|\overline{g}|$ to drift out of the interval $[\ell^*, \ell^* + \varepsilon]$. Third, to see $\lim_{\ell^*\to\infty} A(\ell_t(\ell^*, h, \ell^*)) = 1$ at the end of the proof, fix any t and let $t_i < t$ be the last arrival of a signal before t. By (D.23), the time-to-cutoff in the absence of signals $T^\varnothing(\ell_0, \ell^*)$ is bounded above by $2e^{-\ell^*}/\lambda$ for any $\ell_0 \geq \ell^*$. Thus, for any $\ell^* \geq -\log(\lambda(t - t_i)/2)$, we have $\ell_t(\ell^*, h, \ell^*) < \ell^*$ and $A(\ell_t(\ell^*, h, \ell^*)) = 1$. *Q.E.D.*

LEMMA 11G: *For every* $\ell > \ell^g$, *there exists* $\overline{c} = \overline{c}(\ell) > 0$ *such that for all* $c < \overline{c}(\ell)$, *there exists* $\ell^* > \ell$ *such that* ℓ^* *and* c *satisfy indifference at the cutoff*, *that is*,

$$\lambda\Delta_{\ell^*,c}\big(\ell^*\big) = c. \tag{E.8}$$

The proof is the same as the proof of Lemma 11B.

E.3. *Good News—Work Below Cutoff*

The main results of this section, Lemmas 14G and 15G, show that there exists a threshold ℓ^Δ such that investment incentives are bounded below on $[-\infty, \ell^\Delta]$ and decreasing on $[\ell^\Delta, \ell^*]$ whenever the work–shirk cutoff ℓ^* is greater than ℓ^Δ and the firm is indifferent at the cutoff. As auxiliary results, we establish upper and lower bounds for the marginal payoff of reputation and reputational dividends in Lemmas 12G and 13G.

LEMMA 12G: *Assume* (E.1). *There exist* ℓ^Π *and* $k_1, k_2, k_3 > 0$ *such that for any* $c > 0$, $\ell^* > \ell^\Pi$ *with indifference at the cutoff*, (E.8), *the marginal payoff of reputation satisfies*

$$\Pi'_\theta(\ell) \geq k_1 e^{-\ell} \quad \text{for all } \ell \in \big[\ell^\Pi, \ell^*\big), \tag{E.9}$$

$$\Pi'_\theta(\ell) \leq k_2 e^{-\ell} \quad \text{for all } \ell \in \big(-\infty, \ell^*\big), \tag{E.10}$$

$$\Pi'_\theta(\ell) \leq k_3 e^{-2\ell^*} \quad \text{for all } \ell \in \big(\ell^*, \infty\big). \tag{E.11}$$

That is, for large ℓ^*, *marginal reputation is much more valuable below the cutoff than above the cutoff.*[37]

Loosely speaking, incremental reputation $\ell_t^\varepsilon - \ell_t$ is less "durable" above ℓ^* because the increment shrinks by a factor $(\underline{g}/\overline{g})$ when reputation ℓ_t hits the cutoff ℓ^*, as discussed after the statement of Lemma 8G.

PROOF OF LEMMA 12G: *Lower bound (E.9)*: Recall that $g(\ell) \in (-\mu, 0)$ for $\ell \in (\ell^g, \ell^*)$. Fix $\tau = 1$, say, and assume $\ell^\Pi > \max\{\mu\tau, \ell^g\}$ so that $\ell_t^\varnothing \in (0, \ell_0)$ for all $\ell_0 \in [\ell^\Pi, \ell^*)$ and $t \in [0, \tau]$. This ensures the lower bound $X'(\ell_t^\varnothing) > X'(\ell_0) = e^{\ell_0}/(1 + e^{\ell_0})^2 \geq e^{-\ell_0}/4$. It also ensures

$$\frac{\partial \ell_t^\varnothing}{\partial \ell_0} = \exp\left(\int_0^t g'\big(\ell_s^\varnothing\big)\, ds\right) = \exp\left(-\int_0^t \lambda e^{-\ell_s^\varnothing}\, ds\right) \geq e^{-\lambda\tau} \tag{E.12}$$

[37]The proof of this lemma shows that Π_θ is not differentiable at ℓ^* but admits left- and right-sided derivatives, and the right-sided derivative $\Pi'_\theta(\ell^*+) := \lim_{\varepsilon\to 0}(\Pi_\theta(\ell^* + \varepsilon) - \Pi_\theta(\ell^*))/\varepsilon$ is bounded above by $\Pi'_\theta(\ell^*+) \leq k_3 e^{-2\ell^*}$.

for all $\ell_0 \in [\ell^{\Pi}, \ell^*)$ and $t \in [0, \tau]$, where the first equality in (E.12) uses (E.3); this formula applies because the trajectory $\{\ell_t^{\varnothing}\}_{t\in[0,\tau]}$ does not hit the cutoff ℓ^*. Then we can show

$$\Pi'_\theta(\ell_0) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\right] \geq \int_{t=0}^{\tau} e^{-(r+\mu_H)t}\frac{\partial \ell_t^{\varnothing}}{\partial \ell_0}X'\big(\ell_t^{\varnothing}\big)\,dt$$

$$\geq \tau e^{-(r+\mu_H)\tau}e^{-\lambda\tau}\frac{e^{-\ell_0}}{4} =: k_1 e^{-\ell_0}.$$

The first inequality keeps only histories without signal arrivals, which have probability at least $e^{-\mu_H t}$ at time t, and truncates the integral at τ. The second inequality uses $X'(\ell_t^{\varnothing}) \geq e^{-\ell_0}/4$ and (E.12).

Upper bound (E.10): For $\ell_0 < \ell^*$, we now show that

$$\Pi'_\theta(\ell_0) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\right] \leq \int_{0}^{\infty} e^{-rt}e^{-(\ell_0-\mu t)}\,dt \tag{E.13}$$

$$= \frac{e^{-\ell_0}}{r-\mu} =: k_2 e^{-\ell_0}.$$

The inequality uses $\partial\ell_t/\partial\ell_0 < 1$, $X'(\ell) \leq e^{-\ell}$, and $\ell_t \geq \ell_0 - \mu t$. The subsequent equality uses assumption (E.1).

Upper bound in (E.11): For $\ell_0 > \ell^*$, we have $\ell_t \geq \ell^* - \mu t$ and just as in (E.13), we get

$$\Pi'_\theta(\ell_0) = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\right] \leq \int_{0}^{\infty} e^{-rt}e^{-(\ell^*-\mu t)}\,dt \tag{E.14}$$

$$= \frac{e^{-\ell^*}}{r-\mu} = k_2 e^{-\ell^*}.$$

To derive the upper bound, (E.11), let $T^{\varnothing} = T^{\varnothing}(\ell_0, \ell^*) = \min\{t : \ell_t^{\varnothing} = \ell^*\}$ be the time-to-cutoff in the absence of signals. As in the proof of Lemma 12B, equation (D.23), we have $T^{\varnothing}(\ell_0, \ell^*) \leq 2e^{-\ell^*}/\lambda$ for all $\ell_0 > \ell^* > 0$. Finally, we show

$$\begin{aligned}
&\Pi'_\theta(\ell_0) \\
&\quad = \mathbb{E}^{A,\theta}\left[\int_{t=0}^{\infty} e^{-rt}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\right] \\
&\quad = \int_{t=0}^{T^{\varnothing}} e^{-(r+\lambda+\mu_\theta)t}\frac{\partial \ell_t^{\varnothing}}{\partial \ell_0}\big(X'\big(\ell_t^{\varnothing}\big) + \lambda\Pi'_L\big(\ell_t^{\varnothing}\big) + \mu_\theta\Pi'_\theta\big(\ell_t^{\varnothing} + \gamma\big)\big)\,dt \\
&\qquad + e^{-(r+\lambda+\mu_\theta)T^{\varnothing}}\mathbb{E}^{A,\theta}\left[\int_{t=T^{\varnothing}}^{\infty} e^{-r(t-T^{\varnothing})}\frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\,dt\,\middle|\,h^{T^{\varnothing}} = \varnothing\right]
\end{aligned}$$

$$\le T^{\varnothing}\Big[\max_{t\le T^{\varnothing}}\big(X'\big(\ell_t^{\varnothing}\big)+\lambda\Pi_L'\big(\ell_t^{\varnothing}\big)+\mu_\theta\Pi_\theta'\big(\ell_t^{\varnothing}+\gamma\big)\big)\Big]$$
$$+(\underline{g}/\overline{g})\int_{t=T^{\varnothing}}^{\infty}e^{-r(t-T^{\varnothing})}e^{-(\ell^*-\mu(t-T^{\varnothing}))}\,dt$$
$$\le\frac{2e^{-\ell^*}}{\lambda}\big(e^{-\ell^*}+k_2\big(\lambda e^{-\ell^*}+\mu_\theta e^{-\ell^*}\big)\big)+\frac{\mu-\lambda}{\lambda}e^{-\ell^*}\frac{1}{r-\mu}e^{-\ell^*}$$
$$=: k_3e^{-2\ell^*}.$$

The second equality truncates the integral expression of $\Pi_\theta'(\ell_0)$ at the minimum of the first signal arrival, the first technology shock, and $T^{\varnothing}$. The first inequality uses

$$\partial\ell_t^{\varnothing}/\partial\ell_0<\begin{cases}1 & \text{for any } t\in\big(0,T^{\varnothing}\big),\\ \underline{g}/\overline{g} & \text{for any } t>T^{\varnothing},\end{cases}$$

and $X'(\ell_t)\le e^{-\ell_t}\le e^{-(\ell^*-\mu(t-T^{\varnothing}))}$, which follows from $\ell_t>\ell^*-\mu(t-T^{\varnothing})$. The second inequality additionally uses $T^{\varnothing}\le 2e^{-\ell^*}/\lambda$, the upper bound (E.14), $\underline{g}/\overline{g}\le(\mu-\lambda)e^{-\ell^*}/\lambda$, and assumption (E.1). *Q.E.D.*

The next lemma shows that the reputational dividend

$$\Gamma_H(\ell):=\mu\big(\Pi_H\big(j(\ell)\big)-\Pi_H(\ell)\big)=\mu\big(\Pi_H(\ell+\gamma)-\Pi_H(\ell)\big)$$

is decreasing on an interval $[\ell^\Gamma,\ell^*]$ below the cutoff.

LEMMA 13G: *Assume* (E.1). *There exist* ℓ^Γ *and* $k_4,k_5>0$ *such that for any* $c>0$, $\ell^*>\ell^\Gamma$ *with indifference at the cutoff*, (E.8), *the dividend is decreasing in the region below the cutoff*

$$\Gamma_H'(\ell)\le -k_4e^{-\ell}\quad \textit{for all } \ell\in\big[\ell^\Gamma,\ell^*\big)\setminus\big\{\ell^*-\gamma\big\}. \tag{E.15}$$

Moreover, the dividend is small above the cutoff, that is, there exists $k_5>0$ *such that*

$$\Gamma_H(\ell)\le k_5e^{-2\ell^*}\quad \textit{for all } \ell\in\big(\ell^*,\infty\big). \tag{E.16}$$

PROOF: We first address the easy cases, $\ell\in(\ell^*,\infty)$ and $\ell\in(\ell^*-\gamma,\ell^*)$: For $\ell\in(\ell^*,\infty)$, we know $\Pi_H'(\ell)\le k_3e^{-2\ell^*}$ from (E.11), so the upper bound (E.16) follows from

$$\Gamma_H(\ell)=\mu\big(\Pi_H(\ell+\gamma)-\Pi_H(\ell)\big)=\mu\int_{\ell}^{\ell+\gamma}\Pi_H'\big(\ell'\big)\,d\ell'$$
$$\le\mu\gamma\max_{\ell'>\ell^*}\big\{\Pi_H'\big(\ell'\big)\big\}\le\mu\gamma k_3e^{-2\ell^*}=: k_5e^{-2\ell^*}.$$

For $\ell \in (\ell^* - \gamma, \ell^*)$, the upper bound (E.15) follows from (E.11) and (E.9) by

$$\begin{aligned}\Gamma'_H(\ell) &= \mu\big(\Pi'_H(\ell+\gamma) - \Pi'_H(\ell)\big) \le \mu k_3 e^{-2\ell^*} - \mu k_1 e^{-\ell} \\ &= -\mu\big(k_1 - k_3 e^{-(2\ell^*-\ell)}\big)e^{-\ell},\end{aligned}$$

and we can disregard the term $k_3 e^{-(2\ell^*-\ell)}$ for $\ell^* > \ell^\Gamma$ and sufficiently large ℓ^Γ.

Now we turn to the main part of the proof, showing the bound (E.15) for $\ell \in [\ell^\Gamma, \ell^* - \gamma)$. Let $\ell_t^\beta = \ell_t(\ell_0 + \beta, h, \ell^*)$ for any $\beta \in [0, \gamma]$ and define $T^\geq := \min\{t : \ell_t^\gamma \geq \ell^*\}$. Truncating the integral expansions of $\Pi'_H(\ell+\gamma)$ and $\Pi'_H(\ell)$ gives[38]

$$\begin{aligned}\text{(E.17)}\quad & \Pi'_H(\ell+\gamma) - \Pi'_H(\ell) \\ &= \mathbb{E}^H\bigg[\int_{t=0}^{T^\geq} e^{-rt}\bigg(\frac{\partial \ell_t^\gamma}{\partial \ell_0}X'(\ell_t^\gamma) - \frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t)\bigg)dt \\ &\quad + e^{-rT^\geq}\frac{\partial \ell_{T^\geq}^\gamma}{\partial \ell_0}\Pi'_H(\ell_{T^\geq}^\gamma) - e^{-rT^\geq}\frac{\partial \ell_{T^\geq}}{\partial \ell_0}\Pi'_H(\ell_{T^\geq})\bigg].\end{aligned}$$

As we are looking for an upper bound on $\Pi'_H(\ell+\gamma) - \Pi'_H(\ell)$, we can drop the last, negative term. Moreover, $\ell_{T^\geq}^\gamma > \ell^*$ almost surely, so (E.11) implies that the second-to-last term is bounded above by $k_3 e^{-2\ell^*}$. To analyze the integrand of the first term, note that for all h and all the function, $\beta \mapsto (\partial \ell_t^\beta / \partial \ell_0) X'(\ell_t^\beta)$ is differentiable at all $\beta \in [0, \gamma]$, so we express the integrand as an integral as

$$\frac{\partial \ell_t^\gamma}{\partial \ell_0}X'(\ell_t^\gamma) - \frac{\partial \ell_t}{\partial \ell_0}X'(\ell_t) = \int_{\beta=0}^{\gamma}\bigg(\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2}X'(\ell_t^\beta) + \bigg(\frac{\partial \ell_t^\beta}{\partial \ell_0}\bigg)^2 X''(\ell_t^\beta)\bigg)d\beta.$$

Decomposing the integrand of this expression into a sum of three terms, we substitute back into (E.17):

$$\begin{aligned}& \Pi'_H(\ell+\gamma) - \Pi'_H(\ell) \\ \text{(E.18)}\quad &\le \mathbb{E}^H\bigg[\int_{t=0}^{T^\geq} e^{-rt}\int_{\beta=0}^{\gamma}\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2}X'(\ell_t^\beta)\,d\beta\,dt\bigg] \\ \text{(E.19)}\quad &\quad + \mathbb{E}^H\bigg[\int_{t=0}^{T^\geq} e^{-rt}\int_{\beta=0}^{\gamma}\bigg(\frac{\partial \ell_t^\beta}{\partial \ell_0}\bigg)^2 \max\{X''(\ell_t^\beta), 0\}\,d\beta\,dt\bigg]\end{aligned}$$

[38]As initial quality is high and the firm invests until the cutoff time, $\theta_0 = H$ and $A(\ell_t) = 1$ for $t < T$, firm quality stays high until the cutoff time, allowing us to replace the probability measure $\mathbb{E}^{A,H}$ by $\mathbb{E}^H$.

$$-\mathbb{E}^H\left[\int_{t=0}^{T^{\geq}} e^{-rt}\int_{\beta=0}^{\gamma}\left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{-X''(\ell_t^\beta), 0\}\,d\beta\,dt\right] + k_3 e^{-2\ell^*}. \tag{E.20}$$

We then proceed by showing that the negative term (E.20) is bounded below by some $\kappa e^{-\ell_0}$ when $\ell_0 \in [\ell^\Gamma, \ell^*)$, while terms (E.18) and (E.19) are of order $O(e^{-2\ell_0})$.

Negative term (E.20): Fix $\tau = 1$, say, and note that $\lim_{\ell\to\infty} X''(\ell)e^\ell = \frac{e^{2\ell}-e^{3\ell}}{(1+e^\ell)^3} = -1$ so that $-X''(\ell) \geq e^{-\ell}/2$ for all sufficiently high ℓ. Since reputational drift on (ℓ^g, ℓ^*) takes values in $[-\mu, 0]$, we have $\ell_t^{\beta,\varnothing} \in [\ell_0 - \mu\tau, \ell_0 + \gamma]$ for all $t \in [0, \tau]$, so for sufficiently large ℓ^Γ, we have $-X''(\ell_t^{\beta,\varnothing}) \geq e^{-\ell_t^{\beta,\varnothing}}/2 \geq e^{-(\ell_0+\gamma)}/2$ for all $\ell_0 > \ell^\Gamma$, $t \in [0, \tau]$, and $\beta \in [0, \gamma]$. This allows us to show

$$\begin{aligned}
&\mathbb{E}^H\left[\int_{t=0}^{T^{\geq}} e^{-rt}\int_{\beta=0}^{\gamma}\left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{-X''(\ell_t^\beta), 0\}\,d\beta\,dt\right]\\
&\quad\geq \int_{t=0}^{\tau} e^{-(r+\mu_H)t}\int_{\beta=0}^{\gamma}\left(\frac{\partial \ell_t^{\beta,\varnothing}}{\partial \ell_0}\right)^2\left(-X''(\ell_t^{\beta,\varnothing})\right)d\beta\,dt\\
&\quad\geq \tau e^{-(r+\mu_H)\tau}\gamma e^{-2\lambda\tau}\frac{e^{-(\ell_0+\gamma)}}{2} =: \kappa e^{-\ell_0}.
\end{aligned}$$

The first line truncates the integral at τ and keeps only histories without signal arrivals, which have probability at least $e^{-\mu_H t}$ at time t. The second line uses $\partial\ell_t^\varnothing/\partial\ell_0 \geq \exp(-\lambda\tau)$ from (E.12) and $-X''(\ell_t^{\beta,\varnothing}) \geq e^{-(\ell_0+\gamma)}/2$ for $t < \tau$.

First smaller term (E.18): First observe that for all $\beta \in [0, \gamma]$,

$$\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2} = \frac{\partial \ell_t^\beta}{\partial \ell_0}\int_{s=0}^{t}\frac{\partial \ell_s^\beta}{\partial \ell_0} g''(\ell_s^\beta)\,ds \leq \int_{s=0}^{t}\lambda e^{-\ell_s^\beta}\,ds \leq \lambda t e^{-\min_{s\leq t}\{\ell_s\}}, \tag{E.21}$$

where the equality uses (E.4), the first inequality uses $\partial\ell_s^\beta/\partial\ell_0 < 1$ and that $g''(\ell) = \lambda e^{-\ell}$ in the work region, and the second inequality uses $\ell_s^\beta \geq \ell_s$. Then we show

$$\begin{aligned}
&\mathbb{E}^H\left[\int_{t=0}^{T^{\geq}} e^{-rt}\int_{\beta=0}^{\gamma}\frac{\partial^2 \ell_t^\beta}{\partial \ell_0^2} X'(\ell_t^\beta)\,d\beta\,dt\right]\\
&\quad\leq \mathbb{E}^H\left[\int_{t=0}^{\infty} e^{-rt}\gamma\lambda t e^{-\min_{s\leq t}\{\ell_s\}} e^{-\ell_t}\,dt\right]\\
&\quad\leq \gamma\lambda\int_{t=0}^{\infty} t e^{-rt} e^{-2(\ell_0-\mu t)}\,dt\\
&\quad= \frac{\gamma\lambda}{(r-2\mu)^2} e^{-2\ell_0}.
\end{aligned}$$

The first inequality uses (E.21), $X'(\ell) \le e^{-\ell}$, and $\ell_t^\beta \ge \ell_t$. The second inequality uses that both ℓ_t and $\min_{s\le t}\{\ell_s\}$ are bounded below by $\ell_0 - \mu t$. The last equality uses assumption (E.1) and $\int_0^\infty te^{-(r-2\mu)t}\,dt = 1/(r-2\mu)^2$.

Second smaller term (E.19): We now show

$$\begin{aligned}
&\mathbb{E}^H\left[\int_{t=0}^{T^{\ge}} e^{-rt}\int_{\beta=0}^{\gamma}\left(\frac{\partial \ell_t^\beta}{\partial \ell_0}\right)^2 \max\{X''(\ell_t^\beta), 0\}\,d\beta\,dt\right] \\
&\quad\le \int_{t=\ell_0/\mu}^{\infty} \gamma e^{-rt}\,dt = \frac{\gamma}{r}e^{-r\ell_0/\mu} \le \frac{\gamma}{r}e^{-2\ell_0}.
\end{aligned}$$

The first inequality uses $\ell_t^\beta \ge \ell_0 - \mu t \ge 0$ for $t \le \ell_0/\mu$, so $X''(\ell_t^\beta) < 0$ and the integrand is 0; for $t \ge \ell_0/\mu$, the integrand is bounded above by γe^{-rt} because $\partial \ell_t^\beta/\partial \ell_0 \le 1$ and $X''(\ell) \le 1$. The second inequality uses assumption (E.1). *Q.E.D.*

LEMMA 14G: *Assume* (E.1). *There exists ℓ^Δ such that for any $c > 0$ and $\ell^* > \ell^\Delta$ with indifference at the cutoff*, (E.8), *we have $\lambda\Delta'(\ell) < 0$ for $\ell \in [\ell^\Delta, \ell^*)$.*

PROOF: As in the proof of Lemma 14B, we decompose the integral expansion of $\Delta'(\ell_0)$ into its positive and negative contributions,

$$\Delta'(\ell_0) = \mathbb{E}^L\left[\int_{t=0}^{\infty} e^{-(r+\lambda)t}\frac{\partial \ell_t}{\partial \ell_0}\max\{\Gamma_H'(\ell_t), 0\}\,dt\right] \tag{E.22}$$

$$- \mathbb{E}^L\left[\int_{t=0}^{\infty} e^{-(r+\lambda)t}\frac{\partial \ell_t}{\partial \ell_0}\max\{-\Gamma_H'(\ell_t), 0\}\,dt\right]. \tag{E.23}$$

Recall from Lemma 13G that ℓ^Γ is such that $\Gamma_H'(\ell) \le -k_4e^{-\ell}$ for all $\ell \in [\ell^\Gamma, \ell^*) \setminus \{\ell^* - \gamma\}$. Now assume that ℓ^Δ is sufficiently large; specifically assume that $\ell^\Delta \ge \max\{3\ell^\Gamma, \ell^\Gamma + \mu\}$. We prove $\Delta'(\ell_0) < 0$ for $\ell_0 \in [\ell^\Delta, \ell^*)$ by establishing that there exists $\kappa > 0$ such that the negative term (E.23) is bounded below by some $\kappa e^{-\ell_0}$, while the term (E.22) is of order $O(e^{-(4/3)\ell_0})$.

Negative term (E.23): We show that for every $\ell_0 \in [\ell^\Gamma, \ell^*)$,

$$\begin{aligned}
&\mathbb{E}^L\left[\int_{t=0}^{\infty} e^{-(r+\lambda)t}\frac{\partial \ell_t}{\partial \ell_0}\max\{-\Gamma_H'(\ell_t), 0\}\,dt\right] \\
&\quad\ge \int_{t=0}^{\tau} e^{-(r+\lambda+\mu_L)t}\frac{\partial \ell_t^\varnothing}{\partial \ell_0}\left(k_4e^{-\ell_t^\varnothing}\right)dt \\
&\quad\ge \tau e^{-(r+\lambda+\mu_L)\tau}e^{-\lambda\tau}k_4e^{-\ell_0} =: \kappa e^{-\ell_0}.
\end{aligned}$$

The first line truncates the integral at τ, say $\tau = 1$, discards any histories with a signal arrival, and applies the bound $-\Gamma_H'(\ell_t^\varnothing) \ge k_4e^{-\ell_t^\varnothing}$ from Lemma 13G to

$\ell_t^{\varnothing} \geq \ell_0 - \mu \geq \ell^\Gamma$ for all $t \in [0, \tau]$. The second line applies $\partial \ell_t^{\varnothing}/\partial \ell_0 \geq \exp(-\lambda\tau)$ (from (E.12)) and $e^{-\ell_t^{\varnothing}} \geq e^{-\ell_0}$ for $t \leq \tau$.

Smaller positive term (E.22): We first establish the upper bound,

$$\max\{\Gamma_H'(\ell), 0\} \leq \begin{cases} \mu/r & \text{for } \ell < \ell^\Gamma, \\ 0 & \text{for } \ell \in [\ell^\Gamma, \ell^*) \setminus \{\ell^* - \gamma\}, \\ \mu k_3 e^{-2\ell^*} & \text{for } \ell > \ell^*. \end{cases} \tag{E.24}$$

The first case follows by $\Gamma_H'(\ell) = \mu(\Pi_H'(\ell+\gamma) - \Pi_H'(\ell)) \leq \mu\Pi_H'(\ell+\gamma)$ and $\Pi_H'(\ell) = \mathbb{E}^{A,\theta}[\int_0^\infty e^{-rt} \frac{\partial \ell_t}{\partial \ell_0} X'(\ell_t)\, dt] \leq 1/r$, where the last inequality uses $\partial\ell_t/\partial\ell_0 < 1$ and $X'(\ell) < 1$. The second case follows by Lemma 13G. The third case follows by $\Gamma_H'(\ell) \leq \mu\Pi_H'(\ell+\gamma)$ and (E.11).

Now fix $\ell_0 \geq \ell^\Delta \geq 3\ell^\Gamma$ so that $\frac{1}{3}\ell_0 > \ell^\Gamma$ and $\frac{2}{3}\ell_0 \leq (\ell_0 - \ell^\Gamma)$. For $t \leq \frac{4}{3}\ell_0/r$, assumption (E.1) ensures $t \leq \frac{2}{3}\ell_0/\mu \leq (\ell_0 - \ell^\Gamma)/\mu$ and so $\ell_t \geq \ell_0 - \mu t \geq \ell^\Gamma$. Then $\partial\ell_t/\partial\ell_0 < 1$ and the bound (E.24) imply the upper bound for (E.22):

$$\begin{aligned} &\mathbb{E}^L\left[\int_{t=0}^\infty e^{-(r+\lambda)t}\frac{\partial \ell_t}{\partial \ell_0}\max\{\Gamma_H'(\ell_t), 0\}\, dt\right] \\ &\quad \leq \int_{t=0}^\infty e^{-(r+\lambda)t}\mu k_3 e^{-2\ell^*}\, dt + \int_{t=(4/3)\ell_0/r}^\infty e^{-(r+\lambda)t}\frac{\mu}{r}\, dt \\ &\quad \leq \frac{\mu k_3 e^{-2\ell^*}}{r+\lambda} + \frac{\mu}{r(r+\lambda)} e^{-((r+\lambda)/r)(4/3)\ell_0} \\ &\quad = O\big(e^{-(4/3)\ell_0}\big). \end{aligned}$$

Q.E.D.

LEMMA 15G: *Assume* (E.1) *and fix* ℓ^Δ *from Lemma* 14G. *There exists* $\bar{\bar{c}} > 0$ *such that for all* $\ell^* > \ell^\Delta$, $c < \bar{\bar{c}}$ *with indifference at the cutoff*, (E.8), *we have* $\lambda\Delta(\ell) > c$ *for all* $\ell \in [-\infty, \ell^\Delta]$.

The proof is the same as the proof of Lemma 15B.

E.4. *Good News—Shirk Above Cutoff*

Lemma 16G shows that firms with high reputations shirk. The idea of the proof is to write the payoff of quality above ℓ^* as the sum of dividends until ℓ_t hits ℓ^* plus a continuation payoff and then to show that the dividends are less than the annuity value of $\Delta(\ell^*)$.

LEMMA 16G: *Assume* (E.1) *and fix* ℓ^Δ *from Lemma* 14G. *For any* $c > 0$ *and* $\ell^* > \ell^\Delta$ *with indifference at the cutoff*, (E.8), *we have* $\lambda\Delta(\ell) < c$ *for all* $\ell \in [\ell^*, \infty]$.

PROOF: Fix $\ell_0 = \ell^*$ and $\ell'_0 > \ell^*$, and let $T' = T(\ell'_0, \ell^*) = \min\{t : \ell'_t = \ell^*\}$. Then

$$\begin{aligned}\text{(E.25)}\quad \Delta(\ell') - \Delta(\ell^*) &= \mathbb{E}^L\left[\int_0^{T'} e^{-(r+\lambda)t}\Gamma_H(\ell'_t)\,dt + e^{-(r+\lambda)T'}\Delta(\ell^*)\right] - \Delta(\ell^*) \\ &= \mathbb{E}^L\left[\int_0^{T'} e^{-(r+\lambda)t}\big(\Gamma_H(\ell'_t) - (r+\lambda)\Delta(\ell^*)\big)\,dt\right].\end{aligned}$$

By (E.16), we know that $\Gamma_H(\ell'_t)$ is of order $O(e^{-2\ell^*})$, because $\ell'_t > \ell^*$.

To conclude the proof, we show that there exists $\kappa > 0$ with $\Delta(\ell^*) \geq \kappa e^{-\ell^*}$. Subject to having chosen ℓ^Δ high enough in Lemma 14G, the downward drift $|\lambda(1+e^{-\ell}) - \mu|$ exceeds $(\mu - \lambda)/2$ for $\ell \in [\ell^* - \gamma, \ell^*]$; hence for any $\ell^* > \ell^\Delta$ and $\tau = 2\gamma/(\mu - \lambda)$, we have $\ell^\varnothing_\tau + \gamma < \ell^*$ and $\ell^\varnothing_{\tau+1} > \ell^\Delta - \mu(\tau+1) > \ell^\Pi$. We can then show

$$\begin{aligned}\Delta(\ell^*) &= \mathbb{E}^L\left[\int_0^\infty e^{-(r+\lambda)t}\Gamma_H(\ell_t)\,dt\right] \\ &\geq \int_{t=0}^\infty e^{-(r+\lambda+\mu_L)t}\Gamma_H(\ell^\varnothing_t)\,dt \\ &\geq \int_{t=\tau}^{\tau+1} e^{-(r+\lambda+\mu_L)t}\left(\int_{\ell^\varnothing_t}^{\ell^\varnothing_t+\gamma}\mu\Pi'_H(\ell')\,d\ell'\right)dt \\ &\geq e^{-(r+\lambda+\mu_L)(\tau+1)}\mu\gamma k_1 e^{-(\ell^\varnothing_\tau+\gamma)} \\ &\geq \kappa e^{-\ell^*}.\end{aligned}$$

The second line discards histories with signal arrivals, the third line discards terms with $t \notin [\tau, \tau+1]$ and writes $\Gamma_H(\ell^\varnothing_t)$ as an integral over marginal payoffs, and the fourth line applies the lower bound (E.9) to $\Pi'_H(\ell')$ and uses $\ell^\varnothing_\tau + \gamma \leq \ell^*$.

Thus, $\Gamma_H(\ell'_t)$ is of order $O(e^{-2\ell^*})$ while $\Delta(\ell^*) \geq \kappa e^{-\ell^*}$, implying that (E.25) is negative, and so $\lambda\Delta(\ell) < \lambda\Delta(\ell^*) = c$ for all $\ell \in [\ell^*, \infty]$. *Q.E.D.*

APPENDIX F: PROOF OF THEOREM 6

Part (i). We first establish that the firm prefers to work at all intermediate levels of reputation. Formally, we show that for any $\varepsilon > 0$, there exists $c'_\varepsilon > 0$ such that for any beliefs $\tilde{A}$, any $c < c'_\varepsilon$, and any $x \in (\varepsilon, 1-\varepsilon)$, we have $\lambda D(x) > c$.

Lemma 2 shows that value functions are strictly monotone by arguing that for fixed history h, time-t reputation $x_t(x_0, h, \tilde{A})$ is nondecreasing in x_0. Its proof actually implies a lower bound on $V_\theta(\hat{x}_0) - V_\theta(x_0)$ when $\hat{x}_0 > x_0$. To see this, let $\{\hat{x}^\varnothing_t\}_{t\geq 0}$ and $\{x^\varnothing_t\}_{t\geq 0}$ be the reputational trajectories of a "high" and a "low" firm in the absence of signals. When the distance between the trajectories, $\hat{x}_t^{\varnothing} - x_t^{\varnothing} \geq 0$, is decreasing, the rate of decrease is bounded above by $2(\lambda + |\mu|)$ because the drift of either trajectory is bounded above in absolute terms by $\lambda + |\mu|$. Thus, $\hat{x}_t^{\varnothing} - x_t^{\varnothing} \geq \hat{x}_0 - x_0 - 2(\lambda + |\mu|)t$. This allows us to show

$$\begin{aligned} & V_\theta(\hat{x}_0) - V_\theta(x_0) \\ & \quad \geq \mathbb{E}^{a,\theta}\left[\int_0^\infty e^{-rt}(\hat{x}_t - x_t)\,dt\right] \\ & \quad \geq \int_0^{(\hat{x}_0 - x_0)/(2(\lambda+|\mu|))} e^{-rt} e^{-\max\{\mu_L,\mu_H\}t}\big(\hat{x}_0 - x_0 - 2\big(\lambda + |\mu|\big)t\big)\,dt \\ & \quad \geq e^{-((r+\max\{\mu_L,\mu_H\})(\hat{x}_0 - x_0))/(2(\lambda+|\mu|))} \frac{(\hat{x}_0 - x_0)^2}{4(\lambda + |\mu|)} =: \kappa_1(\hat{x}_0 - x_0)^2. \end{aligned}$$

The first line uses the revealed preference argument from the proof of Lemma 2; the second line discards histories with signal arrivals (the probability of histories with no signal arrival before t is bounded below by $e^{-\max\{\mu_L,\mu_H\}t}$), truncates the integral at $t = \frac{\hat{x}_0 - x_0}{2(\lambda+|\mu|)}$, discards the continuation value, and uses the lower bound for $\hat{x}_t^{\varnothing} - x_t^{\varnothing}$. The third line evaluates the integral and uses $\hat{x}_0 - x_0 < 1$ to drop the $\hat{x}_0 - x_0$ term from the exponent.

This inequality implies a uniform lower bound on reputational dividends,

$$\begin{aligned} \mu\big(V_\theta\big(j(x)\big) - V_\theta(x)\big) & \geq |\mu|\kappa_1\big(j(x) - x\big)^2 \\ & = |\mu|\kappa_1\left(\frac{\mu x(1-x)}{\mu_H x + \mu_L(1-x)}\right)^2 \\ & \geq \frac{|\mu|^3\kappa_1}{(\max\{\mu_L, \mu_H\})^2}\big(\varepsilon(1-\varepsilon)\big)^2 \end{aligned}$$

for $x \in (\varepsilon, 1-\varepsilon)$, which in turn implies a uniform lower bound on $D(x)$ for $x \in (\varepsilon, 1-\varepsilon)$ by Theorem 1. This establishes part (i) of the proof.

Part (ii). Next, we show that if the firm is believed to work at intermediate levels of reputation $x \in (x_1^*, 1-\varepsilon]$, then it also prefers to work just below x_1^*. Intuitively, at such a point, investment incentives are bounded below by the continuation value at some $z > x_1^*$ times the discounted probability that z is reached. This latter term is positive because (HOPE) guarantees that reputation x_t rises above x_1^* with positive probability in finite time; once above x_1^*, the favorable beliefs in (x_1^*, z) quickly push reputation to z.

Fix

$$z = \begin{cases} 1 - \lambda/|\mu| > 0 & \text{if learning is via bad news,} \\ \min\{\lambda/(2\mu), 1/2\} > 0 & \text{if learning is via good news,} \end{cases}$$

and let $T^{\geq}(x_0, z) := \min\{t : x_t \geq z\}$ be the first time that a reputational trajectory starting at $x_0 \in (0,1)$ reaches or exceeds z. If the firm is believed to be working on (x_1^*, z) and x_0 is just below x_1^*, then $T^{\geq}(x_0, z)$ is "bounded above with positive probability." More precisely, we make the following claim.

CLAIM: *There exists* $\kappa_2 \in (0,1)$ *such that for any* $x_* \in (0, z)$ *and any* $\tilde{A}$ *with* $\tilde{A} = 1$ *on* (x_*, z), *there exists* $\delta > 0$ *such that for all* $x_0 \in (x_* - \delta, x_*)$, *we have*

$$\mathbb{E}^L\left[e^{-(r+\lambda)T^{\geq}(x_0,z)}\right] \geq \kappa_2.$$

PROOF: *Bad news with* $|\mu| > \lambda$. By $x_* < z = 1 - \lambda/|\mu|$, the drift $g(x) = \lambda(\tilde{A}(x) - x) + |\mu| x(1-x)$ is strictly positive on $(0, x_*)$ even when $\tilde{A} = 0$ and is bounded below by $\lambda(1-z)$ on (x_*, z). Thus, we can choose $\delta > 0$ so that $x_t^{\varnothing} > z$ for all $x_0 \in (x_* - \delta, x_*)$ and $t > z/(\lambda(1-z))$. To summarize, we have $T^{\geq}(x_0, z) \leq z/(\lambda(1-z))$ with probability at least $e^{-\mu_L T^{\geq}(x_0,z)}$. This implies

$$\mathbb{E}^L\left[e^{-(r+\lambda)T^{\geq}(x_0,\varepsilon)}\right] \geq e^{-\mu_L z/(\lambda(1-z))} e^{-(r+\lambda)z/(\lambda(1-z))} =: \kappa_{2,B}.$$

Good news. For $x \in (x_*, z)$, reputational drift is bounded below by $\lambda(1-x) - \mu x(1-x) = (\lambda - \mu x)(1-x) \geq \lambda/4$; then $x_t > x_*$ implies $x_{t'} > z$ for all $t' > t + 4/\lambda$. Let $\delta > 0$ be such that $L(x_*) - L(x_* - \delta) > \gamma/2$, where $L(x) = \log(x/(1-x))$ is the log-likelihood-ratio transformation introduced in Appendix C.2, and $\gamma = \log(\mu_H/\mu_L)$ is the (constant) jump size $j(\ell) - \ell$ in ℓ-space. Below $L(x_*)$, reputational drift in ℓ-space is bounded above in absolute terms by $|\lambda(1+e^\ell) + \mu| \leq 2\lambda + \mu$. Thus, we have $L(x_t) > L(x_*) - \gamma$ for all $x_0 \in (x_* - \delta, x_*)$ and $t \leq \gamma/(4\lambda + 2\mu)$. Thus, if a signal arrives in $(0, \gamma/(4\lambda + 2\mu))$, we get $x_t > x_*$ for $t = \gamma/(4\lambda + 2\mu)$ and then $x_{t'} > z$ for $t' = t + 4/\lambda$.

To summarize, there exists $\delta > 0$ such that for all $x_0 \in (x_* - \delta, x_*)$, the probability that $T^{\geq}(x_0, z) \leq \gamma/(4\lambda + 2\mu) + 4/\lambda$ is bounded below by $1 - e^{-\mu_L \gamma/(4\lambda+2\mu)}$, so

$$\mathbb{E}^L\left[e^{-(r+\lambda)T^{\geq}(x_0,z)}\right] \geq \left(1 - e^{-\mu_L(1-\gamma/(4\lambda+2\mu))}\right) e^{-(r+\lambda)(\gamma/(4\lambda+2\mu)+4/\lambda)} =: \kappa_{2,G}.$$

This establishes the claim for $\kappa_2 = \min\{\kappa_{2,B}, \kappa_{2,G}\}$. *Q.E.D.*

To finish the proof of Theorem 6, fix any $\varepsilon > 0$ and any $c < c_\varepsilon := \min\{c'_\varepsilon, \kappa_2 c'_z\}$. By part (i) of the proof, the firm prefers to invest on $(z, 1-\varepsilon)$. Assume that, contrary to the statement of the theorem, there exists an equilibrium with a shirk–work cutoff $x_1^* \in (0, z)$ and a work region (x_1^*, z). Then our claim implies that there exists $\delta > 0$ such that for all $x_0 \in (x_1^* - \delta, x_1^*)$,

$$\lambda D(x_0) \geq \lambda \mathbb{E}^L\left[e^{-(r+\lambda)T^{\geq}(x_0,z)} D(x_{T^{\geq}(x_0,z)})\right] \geq \kappa_2 c'_z > c.$$

Thus, the firm prefers to work on $(x_1^* - \delta, x_1^*)$. This contradiction finishes the proof.

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