A Reputational Theory of Firm Dynamics*

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Abstract

We study the lifecycle of a firm who sells a product of uncertain quality, characterizing the optimal investment and exit decisions and the resulting firm dynamics. We investigate two model variations. If the firm shares the market's uncertainty, it learns about its product quality through past actions and public signals. This learning generates a level of self-esteem which coincides with its public reputation only on the equilibrium path. We show that the firm is incentivized to invest by the marginal value of self-esteem, and that the firm stops investing when its reputation approaches the exit threshold and its life-expectancy vanishes. In contrast, when the firm knows its product quality perfectly, both high- and low-quality firms invest at the threshold where low-quality firms exit the market. While the life-expectancy of a low-quality firm vanishes, investment remains profitable because investment success boosts the firm's quality and averts exit.

1 Introduction

Maintaining a good reputation is essential for survival in many industries. Professionals (e.g. consultants, lawyers, academics) invest in their skills to develop a reputation for solving problems, but may quit and change occupation if they do not succeed. Similarly, restaurants try to build a reputation for high quality food and service, but many fail with 25% of young restaurants exiting each year (Parsa et al. (2005)). This paper develops a simple model to study the lifecycle of such a firm. We characterize the optimal exit decision, and analyze how this impacts the firm's investment incentives.

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of

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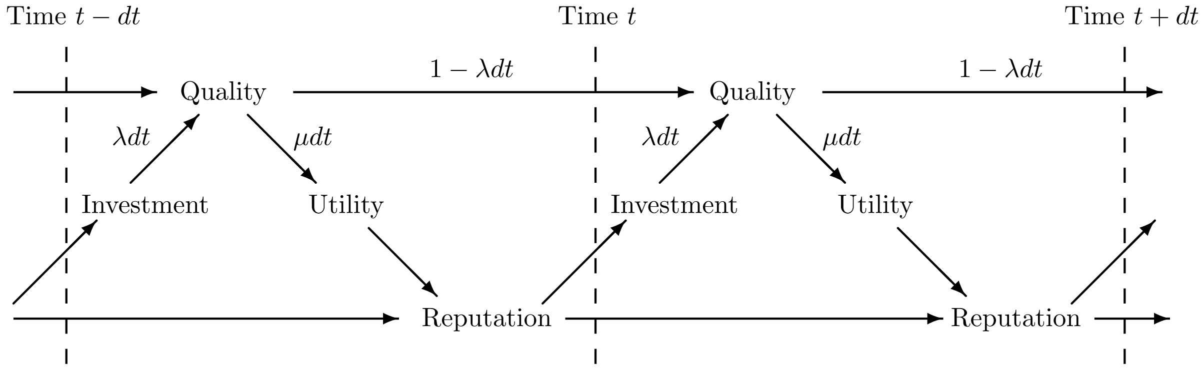

Figure 1: **Gameform.** Quality is a function of the firm's investment and its past quality. Consumer utility is an imperfect signal of quality that the market uses to update the firm's reputation.

the firm's past investments. Consumers observe neither quality nor investment directly and learn about the product quality through breakthroughs that can only be produced by a high-quality product. At each point in time, consumers' willingness to pay is determined by the market belief that the quality is high, x_t, which we call the reputation of the firm. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning via product breakthroughs. The firm can exit the market at any time, and does so when its reputation falls below some threshold.

Our analysis and the form of the results depend on whether or not the firm knows its own quality. Which case is more relevant will depend on the application at hand. For example, an academic who obtains breakthroughs by publishing papers, knows her past publishing success and how much she invests in her skills, but shares the profession's uncertainty of her current ability and future success. In contrast, a restaurateur, who obtains breakthroughs through newspaper reviews, can learn directly from customer feedback whether it has a potentially successful concept, a signal not available to the market as a whole.

We first suppose the firm does not know its own quality. In equilibrium, the state of the game is summarized by the firm's reputation. Off the equilibrium path the firm's belief about its quality, its self-esteem, diverges from its reputation because reputation is governed by believed investment while self-esteem is governed by actual investment. The firm's value is thus a function of both its reputation and its self-esteem, and investment incentives are determined by the marginal value of self-esteem. In our first major result, we show this marginal value of self-esteem can be written as a present value of future reputational dividends, which capture the immediate marginal benefit of self-esteem.

In equilibrium, the firm's self-esteem and reputation coincide. When this reputation is very low, the firm's losses exceed the option value of staying in the market, and it exits the market. For a firm above the exit threshold, investment incentives are hump-shaped in the firm's reputation. For low reputations near the exit threshold, the firm's life expectancy is very short and the firm stops investing. In equilibrium the market anticipates this effect and adverse market beliefs further accelerate the firm's demise. For intermediate reputation levels, the marginal value of self-esteem is bounded below and the firm invests when costs are sufficiently low. Finally, when the firm's reputation is close to 1, the firm cannot work in equilibrium. If the firm was believed to work at such a reputation, the lack of any breakthrough would be attributed to bad luck, undermining the incentive to actually invest.

Next, we suppose the firm knows its own quality. Here the firm's value is a function of its reputation and its quality, and investment is incentivized by the difference in value between a high and low quality firm. As above, we express this value of quality as the net present value of future reputational dividends, and use this expression to characterize incentives.

In equilibrium, the firm's quality affects its exit decision but not its investment. Specifically, there is a threshold where a low-quality firm exits, a high quality firm remains in the market, and the exit rate of the low-quality firm keeps the reputation of surviving firms at this threshold. In equilibrium, the firm's investment incentives are decreasing in its reputation and are maximized at the exit threshold, so the firm fights until the bitter end. While the low-quality firm's life expectancy vanishes as it approaches the exit threshold, investment success is observable and averts exit, resulting in investment incentives that are of first order. In contrast, with unknown quality, investment success still needs to be learned by the firm and investment incentives are of second order.

While we derive our main results for a perfect good news specification of market learning, many of these effects are robust to more general stationary learning structures with imperfect Poisson and Brownian signals. We also discuss how to model entry into the market in order complete the firm's lifecycle.

1.1 Literature

Our model is based on Board and Meyer-ter-Vehn (2010), which bridges classic models of reputation (e.g. Holmström (1999), Mailath and Samuelson (2001)) and models of repeated games (e.g. Fudenberg et al. (1990)). Our earlier paper characterizes firms' investment problems without considering entry or exit. As equilibrium investment does not depend on quality the distinction between the known and unknown quality cases is moot, and the issue of self-esteem does not arise.

Bar-Isaac (2003) analyses the optimal exit decision of a firm with fixed quality. He lays the foundation for our paper, introducing the distinction between known and unknown quality. He also shows that threshold exit rules are optimal and that a firm that knows it has high quality never exits because exit of low types bounds the reputational evolution from below. We build on Bar-Isaac's paper by analyzing how these exit decisions impact the firm's investment decisions at different stages of its lifecycle.

Kovrijnykh (2007) introduces exit into the career concerns model of Holmström (1999). This leads to the same issues of inner- and outer-reputation as in the present paper. Because of tractability problems with the normal-linear model, this paper only considers a three-period model, which limits the scope of the results.[1]

There are many other models of firm dynamics. In complete information models (e.g. Jovanovic (1982), Hopenhayn (1992), Ericson and Pakes (1995)), firms differ in their capabilities, and choose when to enter, exit and invest. In comparison, we allow quality to be imperfectly observed, introducing a role for reputation that affects the firm dynamics and investment incentives. In contrast to repeated games models (e.g. Gale and Rosenthal (1994), Rob and Fishman (2005)), our firm has a state variable, enabling us to impose more discipline on equilibria by focusing on Markovian equilbria.

2 Model

Overview: There is one firm and a continuum of consumers. Time $t \in [0, \infty)$ is continuous and infinite; the common interest rate is $r \in (0, \infty)$. At time t the firm produces one unit of a product that can have high or low quality, $\theta_t \in \{L = 0, H = 1\}$. The expected instantaneous value of the product to a consumer equals $\theta_t dt$. The market belief about product quality $x_t = \Pr(\theta_t = H)$ is called the firm's *reputation*. The firm chooses investment $\eta_t \in [0, 1]$ at cost $(c\eta_t + k)\, dt$, where c is the cost of investment and k is the operating cost; the firm can exit the market at any time.

Technology: Product quality θ_t is a function of past investments $(\eta_s)_{0 \le s \le t}$ via a Poisson process with arrival rate λ that models quality obsolescence. Absent a shock quality is constant, $\theta_{t+dt} = \theta_t$; when a shock occurs previous quality becomes obsolete and is determined by the level of investment, $Pr(\theta_{t+dt} = H) = \eta_t$.[2] Quality at time t is then a geometric sum of past investments,

$$\Pr(\theta_t = H) = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} \Pr(\theta_0 = H) . \tag{2.1}$$

Information: Consumers observe neither investment η nor product quality θ. We analyse both the case were the firm does not know its own quality (Section 3) and where it does (Section 4). Consumers (and the firm) learn about quality through product *breakthroughs* that arrive to high-quality firms at Poisson rate μ. The quality obsolescence process and the breakthrough process are statistically independent. The breakthroughs can be related to consumers' utility by assuming that $dU_t = 0$ almost always, with each breakthrough yielding utility $1/\mu$.

[1]This distinction between private and public beliefs is also present in Bonatti and Horner's (2010) model on strategic experimentation.

[2]This formulation provides a tractable way to model product quality as a function of past investments. For example, one can interpret investment as the choice of absorptive capacity, determining the ability of a firm to recognise new external information and apply it to commercial ends (Cohen and Levinthal (1990)).

Reputation Updating: The reputation increment $dx_t = x_{t+dt} - x_t$ is governed by product breakthroughs, their absence, and market beliefs about investment $\widetilde{\eta}$. A breakthrough reveals high quality, so the firm's reputation immediately jumps to one, $x_{t+dt} = 1$. Absent a breakthrough, dx is deterministic and by independence it can be decomposed additively:

$$dx = \lambda(\widetilde{\eta} - x)dt - \mu x(1-x)dt. \tag{2.2}$$

The first term is the differential version of equation (2.1); if expected quality after a technology shock $\widetilde{\eta}$ exceeds current expected quality x then reputation drifts up. The second term is the standard Bayesian increment in the absence of a breakthrough.

Profit and Consumer Surplus: The firm and consumers are risk-neutral. At time t the firm sets price equal to the expected value x_t, so consumers' expected utility is 0. The firm's flow profit is $(x_t - c\eta_t - k)dt$ and its discounted present value is thus given by:

$$V = \mathbb{E}\left[\int_{t=0}^{T} e^{-rt}(x_t - c\eta_t - k)dt\right] \tag{2.3}$$

Markov-Perfect-Equilibrium: We assume Markovian beliefs $\widetilde{\eta} = \widetilde{\eta}(x)$ and define Markov-Perfect-Equilibria in Sections 3 and 4 respectively.

3 Unknown Quality

If the firm does not know its own quality, its investment and exit decisions will depend on its public reputation $x_t = \Pr(\theta_t = H | U_s, s \in [0,t])$ and its *self-esteem* $z_t = \Pr(\theta_t = H | U_s, \eta_s, s \in [0,t])$. At time 0, we assume that $z_0 = x_0$. Subsequently, the dynamics of self-esteem are determined by

$$dz = \lambda\left(\eta\left(x,z\right) - z\right)dt - \mu z\left(1-z\right)dt \tag{3.1}$$

absent a breakthrough, and $z_{t+dt} = 1$ after a breakthrough. This differs from (2.2) in that self-esteem depends on actual investment η, whereas reputation depends on believed investment $\widetilde{\eta}$. In equilibrium these coincide, but investment incentives are determined by off-equilibrium considerations.

In a Markovian equilibrium we can write the firm's value as function of its reputation and self-esteem, $V(x,z)$. A *Markov-Perfect-Equilibrium* $\langle \eta, \widetilde{\eta} \rangle$ then consists of an investment function $\eta : [0,1]^2 \to [0,1]$, exit-region $R \subseteq [0,1]^2$, and market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that: (1) Investment maximizes firm value, $V(x,z)$; (2) The firm exits when value is negative, $V(x,z) \leq 0$;

and (3) Market beliefs are correct, $\widetilde{\eta}(x) = \eta(x, x)$.[3]

The value function has a number of basic properties. $V(x,z)$ is increasing in reputation x, since this leads directly to higher revenue. $V(x,z)$ is increasing in self-esteem z, since a higher quality ultimately leads to higher reputation. Finally, $V(x,z)$ is convex in self-esteem z, since information about quality is valuable to the firm.[4]

3.1 Optimal Investment and Exit Decisions

Lemma 1 shows that investment incentives are determined by the marginal value of self-esteem, $\partial_z V(x,z)$. This is because investment directly controls quality and thus the firm's belief about its quality, i.e. its self-esteem.

Lemma 1. *Equilibrium investment satisfies*

$$\eta(x,z) = \begin{cases} 1 & \text{if } \lambda \partial_z V(x,z) > c \\ 0 & \text{if } \lambda \partial_z V(x,z) < c \end{cases}$$

Proof. Using (3.1), investment over $[t, t+dt]$ increases self-esteem by λdt, and therefore yields the firm $\lambda \partial_z V(x,z)$. □

The firm exits the industry when its value is negative. Since reputation declines continuously (or jumps up) and the firm's value is increasing in its reputation and self-esteem, the firm exits when its reputation falls to zero, $V(x,z) = 0$. On the equilibrium path, the firm exits at x^e defined by

$$V(x^e, x^e) = 0.$$

3.2 Marginal Value of Self-Esteem

In order to understand the firm's investment incentives we decompose the value of incremental self-esteem, $V(x,z') - V(x,z)$ into (a) its immediate benefit, called the *reputational dividend (of self-esteem)*, and (b) its continuation value. First we develop the value of a firm with self-esteem z:

$$V(x,z) = \underbrace{rdt\,(x - \eta c - k)}_{\text{Today's Payoff}} + (1 - rdt)\underbrace{V(x+dx, z+dz)}_{\text{No Breakthrough}} + z\mu dt \underbrace{(V(1,1) - V(x+dx, z+dz))}_{\text{Breakthrough}}$$

[3] If the firm does not exit when its value become negative, the market interprets this as a mistake and updates based on market learning and $\widetilde{\eta} = \widetilde{\eta}(x^e)$, where x^e is the exit point.

[4] To prove monotonicity suppose the firm with the higher reputation (self-esteem) mimics the firm with the lower reputation (self-esteem). To prove convexity, suppose $\overline{z}$ is a convex combination of z' and z'', and let firms z' and z'' mimic $\overline{z}$.

where the increments dx and dz are conditional on no breakthroughs, as given by (2.2) and (3.1), and are thus deterministic. If we instead start with self-esteem z' we can write a similar expression. Adding and subtracting $V(x+dx, z+dz)$ then yields,

$$V\left(x, z'\right) = rdt\,(x - \eta c - k) + (1 - rdt)\,V\left(x + dx, z' + dz'\right) + z'\mu dt\,(V(1,1) - V(x+dx, z+dz)) + z'\mu dt\left(V(x+dx, z+dz) - V\left(x+dx, z'+dz'\right)\right).$$

Having higher self-esteem does not affect revenue today, but alters the evolution of future reputation and therefore future revenue. In particular, from these two equations, the value of the increment $z' - z$ is given by[5]

$$\begin{aligned} V\left(x, z'\right) - V(x, z) &= \left(z' - z\right) \underbrace{\mu\,(V(1,1) - V(x,z))}_{\text{Reputational Dividend}} dt \\ &+ (1 - (r + z\mu)\,dt) \underbrace{\left(V\left(x + dx, z' + dz'\right) - V(x + dx, z + dz)\right)}_{\text{Continuation Value}}. \end{aligned}$$

The first term is the reputational dividend: the immediate benefit of incremental self-esteem. The second term is the continuation value depreciated by interest rate r and rate of breakthroughs μz, that render incremental self-esteem obsolete. Integrating, and dividing by $(z_0' - z_0)$:

$$\begin{aligned} \frac{V(x_0, z_0') - V(x_0, z_0)}{z_0' - z_0} &= \int_0^T \exp\left(-\int_0^t r + \mu z_s ds\right) \frac{z_t' - z_t}{z_0' - z_0} \mu\,(V(1,1) - V(x_t, z_t))\,dt \\ &+ \exp\left(-\int_0^T r + \mu z_s ds\right) \frac{V(x_T, z_T') - V(x_T, z_T)}{z_0' - z_0} \end{aligned}$$

where $T = T(x_0, z_0)$ is the first time that the trajectory (x_t, z_t) hits the exit region. The second term vanishes in the limit because optimal exits implies $V(x_T, z) = 0$ for $z \leq z_T$, and smooth-pasting of the value function then implies $V(x_T, z_T') - V(x_T, z_T) \in o(z_T' - z_T)$.

By the updating equation of self-esteem (3.1), the increment decreases at rate $d\ln(z_t' - z_t)/dt = 1 - (\lambda + \mu(1 - 2z_t))$. It follows that

$$\partial_z V(x_0, z_0) = \int_0^T e^{-\int_0^t r + \lambda + \mu(1 - z_s) ds}\,\mu\,(V(1,1) - V(x_t, z_t))\,dt \tag{3.2}$$

Setting $z_t = x_t$, we conclude:

[5]When we cancel current cashflows we assume investment is identical on the two trajectories, $\eta(x,z) = \eta(x,z')$. This approximation is justified by the consideration that the amount of time at which $\eta(x_t, z_t) \neq \eta(x_t, z_t')$ is of order $z_0' - z_0$, and that the joint effect of the approximation at these times is of order $c - \partial_z V(x,z)$, which in equilibrium converges to 0 as $z_0' - z_0$ becomes small.

Proposition 1. *In equilibrium the marginal value of self-esteem is given by*

$$\Gamma(x_0) := \partial_z V(x_0, x_0) = \int_0^T e^{-\int_0^t r+\lambda+\mu(1-x_s)ds} D(x_t)dt \tag{3.3}$$

where $D(x) := \mu(V(1,1) - V(x,x))$ *is the equilibrium reputational dividend.*

Proposition 1 expresses the marginal value of self-esteem as the discounted sum of future reputational dividends. Since quality is persistent, investment does not pay off immediately but rather through a stream of dividends whose value depends on the future evolution of the firm's reputation. In equation (3.3), the dividends are discounted by the interest rate r, and the rate at which incremental self-esteem vanishes. The latter consists of two terms: if there is no breakthrough the gap closes at rate $-(\lambda + \mu(1-2z))\,dt$; if there is a breakthrough the gap completely closes, leading to a $-\mu z dt$ term.[6]

3.3 Shirk-Work-Shirk Equilibrium

In this Section we show that, when λ and c are sufficiently low, the firm shirks when its reputation is either low or high, and works for intermediate reputations. The intuition is as follows. For low reputations $x \approx x^e$, the firm is almost certain to go out of business soon, undermining its incentives to further invest into its product. In equilibrium the market anticipates that the firm gives up and the adverse beliefs accelerate the firm's demise. For intermediate reputations, incentives are bounded from below and the firm invests when the investment costs are sufficiently low. Finally, for high reputations $x \approx 1$, the firm cannot keep investing in equilibrium. If it did, its reputation would stay close to 1, because the market learns little from the absence of breakthroughs when it is sufficiently convinced of the firm's quality. This dynamic undermines the firm's incentive to actually invest.

While these effects are robust, their occurrence in the good news model relies on two restrictions on the model parameters. First, we assume that a firm's reputation declines in the absence of a breakthrough, even if the firm is believed to be investing. Using equation (2.2), this means that $x^e > \lambda/\mu$. Formally, we assume that

$$r(\lambda/\mu - k) + \lambda(1-k) < 0 \tag{A-λ}$$

which is satisfied if λ is sufficiently low, limiting the role of market beliefs. Equation (A-λ) says that the (negative) profits earned at $x = \lambda/\mu$ are lower than the option value of receiving a breakthrough. Second, we assume c is sufficiently low so that firms with intermediate reputations choose to invest.

[6] The astute reader will notice that we ignored the possibility that z' may have a breakthrough but not z, increasing $(dz' - dz)/(z' - z)$ at rate $\mu(1-z)dt$. However this term is captured by the reputational dividend.

Proposition 2. *Suppose (A-λ) holds and c is sufficiently low. In any equilibrium:*
(a) The optimal investment is characterized by cutoffs $\lambda/\mu < x^e < \underline{x} < \overline{x} < 1$ such that the firm

$$\begin{array}{ll} \textit{exits} & \textit{if } x \in [0, x^e] \\ \textit{shirks} & \textit{if } x \in [x^e, \underline{x}] \\ \textit{works} & \textit{if } x \in [\underline{x}, \overline{x}] \\ \textit{shirks} & \textit{if } x \in [\overline{x}, 1] \end{array}$$

(b) The optimal exit threshold x^e satisfies

$$(x^e - k) + x^e \mu V(1,1) = 0. \tag{3.4}$$

Proof. See Appendix A.1. □

The arguments leading up to the Proposition already show that every equilibrium must have (1) some shirking at the bottom, (2) some working in the middle, and (3) some shirking at the top. The proof, which is based on Proposition 1, strengthens these arguments to show that any equilibrium must be characterized by intervals.

Condition (3.4) follows from the indifference of a firm at reputation x^e to exit or stay. At this point, the firm's negative instantaneous profits $x^e - k$ are balanced by the option value $x^e \mu V(1,1)$ of staying in the market.

While our analysis focuses on learning through perfectly revealing good news signals, the spirit of these results extend to more general learning structures. Following Board and Meyer-ter-Vehn (2010), suppose the market learns through a signal Z_t, that is generated by a Brownian motion and finite number of Poisson processes. The Brownian component is given by $dU_{B,t} = \mu_B \theta_t dt + dW_t$, where W_t is a Wiener process. A Poisson process has a signal arrive at rate μ_θ. Such a signal is *good news* if the net arrival rate $\mu := \mu_H - \mu_L$ is positive, *perfect good news* if $\mu_L = 0$, *bad news* if $\mu < 0$, and *perfect bad news* if $\mu_H = 0$.

As shown in Appendix A.2, we can generalize equation (3.2) to express the marginal value of self-esteem as

$$\partial_z V(x_0, z_0) = \mathbb{E}\left[\int_0^T e^{-rt} \frac{\partial z(z_0, t)}{\partial z_0} D(x_t, z_t)\, dt\right] \tag{3.5}$$

where the reputational dividend is

$$D(x_t, z_t) = \left(\mathbb{E}_H\left[V(x_{t+dt}, z_{t+dt})\right] - \mathbb{E}_L\left[V(x_{t+dt}, z_{t+dt})\right]\right)/dt.$$

and $\mathbb{E}_\theta\left[V(x_{t+dt}, z_{t+dt})\right]$ conditions the evolution of x and z on θ. In the Appendix, we explicitly calculate the term $\partial z(z_0, t)/\partial z_0$ as a function of the learning process.

Using (3.5) one can extend our results to general learning structures. For low reputations

$x \approx x^e$, investment incentives disappear and the firm shirks if exit becomes imminent, $T \to 0$ a.s., as $x_0 \to x^e$. This condition holds under our good news specification or if there is any Brownian motion. For high reputations $x \approx 1$, investment incentives disappear and there will be some shirking at the top, as long as there is no perfect bad news signal. Under these circumstances, if the firm was thought to be working at $x \approx 1$, then x_t stays close to 1 with probability one for all t, yielding dividends $D(x_t, x_t) \approx 0$ forever.

3.4 Entry

We have characterized the firm's optimal investment and exit decisions. To complete the firm's lifecycle, suppose there is measure dt of potential entrants into the market over $[t, t+dt]$. Potential entrants have a public reputation x_0, and therefore only enter if $x_0 > x^e$. Once a firm enters the market he plays the game we have studied above, choosing his investment and exit decisions.[7]

As an application, consider the labor market for academics. When an agent enters the industry her type is unknown to her and the market, but her GPA is common knowledge and determines her initial reputation x_0. Agents with low GPAs choose not to enter the industry. Agents with high GPAs enter, invest in their skills over time and are free to exit at any point; the market then learns about their skills via their breakthroughs (e.g. publications). Proposition 2 predicts that an agent will stop investing in her skills shortly before she exits or after she has had a breakthrough.

4 Known Quality

We now turn to the case were the firm knows its quality, θ_t. In a Markovian equilibrium we can write the firm's value as function of its reputation and quality, $V_\theta(x)$. A *Markov-Perfect-Equilibrium* $\langle \eta, \widetilde{\eta} \rangle$ then consists of an investment function $\eta : [0,1] \times \{L, H\} \to [0,1]$, a exit region $R \subseteq [0,1] \times \{L, H\}$, and market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that: (1) Investment maximizes firm value $V_\theta(x)$; (2) The firm exits when its value is negative $V_\theta(x) \leq 0$; and (3) Market beliefs are correct, $\widetilde{\eta}(x) = (1-x)\eta_L(x) + x\eta_H(x)$.

It is straightforward to show that the value function $V_\theta(x)$ is increasing in reputation x, since this leads directly to higher revenue. In addition, $V_\theta(x)$ is increasing in quality θ, since this will ultimately lead to higher reputation.[8]

[7] This analysis implicitly assumes that the firm can only invest if they have paid the operating cost k to be in the market. For example, an academic can only invest in her skills if she no other job, where k is the opportunity cost.

[8] To prove, suppose the firm with the higher reputation (self-esteem) mimics the firm with the lower reputation (self-esteem).

4.1 Optimal Investment and Exit Decisions

The benefit of investment over $[t, t+dt]$ is the probability a technology shock hits, λdt, times the difference in value functions, $\Delta(x) := V_H(x) - V_L(x)$, which we call the *value of quality*. It follows that:

Lemma 2. *Optimal investment $\eta(x)$ is independent of quality θ and given by*

$$\eta(x) = \begin{cases} 1 & \text{if } \lambda\Delta(x) > c \\ 0 & \text{if } \lambda\Delta(x) < c \end{cases}$$

where $\Delta(x) := V_H(x) - V_L(x)$ is the value of quality.

The firm exits the industry when its value is negative. Since reputation declines continuously (or jumps up), the firm exits when its value falls to zero $V_\theta(x) = 0$. As quality is a valuable asset, the low-quality firm exits when the high-quality firm's value is strictly positive. This exit process of low-quality firms prevents a further decline in reputation, so a high quality firm never exits, as in Bar-Isaac (2001). As a result:

Lemma 3. *Define x^e by $V_L(x^e) = 0$.*
(a) The high-quality firm never exits, while
(b) The low-quality firm exits if $x_t \leq x^e$ and, if so, exits so that $x_{t+dt} = x^e$. At the cutoff x^e, the rate of exit is

$$q = \left[\mu - \frac{\lambda(\eta(x^e) - x^e)}{x^e(1-x^e)}\right] dt. \tag{4.1}$$

As a result, $x_t \in [x^e, 1]$ for $t > 0$.

Proof. Firm L quits to keep $x_{t+dt} = x^e$ when $x_t \leq x^e$. Hence $V_L(x) = 0$ and $V_H(x) > 0$ for $x \leq x^e$, and the high-quality firm never exits. When $x_t = x^e$, the low firm's quit probability can be calculated using Bayes rule:

$$q = 1 - \frac{1-x^e}{x^e} \times \frac{x^e + dx}{1-(x^e+dx)} \approx -\frac{dx}{x^e(1-x^e)}$$

so equation (2.2) yields (4.1). □

4.2 Value of Quality

In order to characterise investment incentives we need to evaluate the value of quality $\Delta(x) = V_H(x) - V_L(x)$. Following Board and Meyer-ter-Vehn (2010, Theorem 1), we develop the value functions into current profits and continuation values. Current profits cancel because both current revenue and costs depend on reputation but not on quality, yielding

$$\Delta(x) = (1 - rdt)(1 - \lambda dt)\mathbb{E}\left[V_H(x + d_H x) - V_L(x + d_L x)\right].$$

Adding and subtracting $V_H(x+d_L x)$, we can express the value of quality in terms of a reputational dividend and continuation value:

$$\Delta(x) = (1 - rdt - \lambda dt)\left(\hat{D}(x) + \mathbb{E}\left[\Delta(x + d_L x)\right]\right).$$

where

$$\hat{D}(x) := \left[V_H(x + d_H x) - V_H(x + d_L x)\right]/dt \tag{4.2}$$

is the *reputational dividend for known quality.* Evaluating (4.2) and integrating up, we express the value of quality as the discounted sum of future reputational dividends:

Proposition 3. *In equilibrium, the marginal value of quality is given by*

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\hat{D}(x_t)dt. \tag{4.3}$$

where $\hat{D}(x) = \mu(V_H(1) - V_H(x))$.

This representation of investment incentives with known quality differs from the unknown quality case in (3.3) in two respects. The substantial difference is that in (4.3) we integrate over $[0, \infty]$, whereas whereas in (3.3) the integral is over $[0, T]$. With known quality, a firm never exits with certainty because exit by low quality firms bounds reputational below at x^e and leaves even low quality firms indifferent about exiting. In contrast, with unknown quality the firm strictly prefers to exit after the exit time T. This difference can be reflected in the algebra by rewriting (4.3) as as $\Delta(x_0) = \int_0^{T^e} e^{-(r+\lambda)t}\hat{D}(x_t)dt + e^{-(r+\lambda)T^e}\Delta\left(x^e\right)$, where T^e is the time x_t hits x^e. In the known quality case the continuation value of the discrete quality increment $e^{-(r+\lambda)T^e}\Delta\left(x^e\right)$ is strictly positive, whereas in the unknown quality case, the continuation value of incremental self-esteem $V\left(x_T, z'_T\right) - V\left(x_T, z_T\right)$ is of second order because of smooth pasting.[9]

A second, more expositional, difference is that we choose simpler, if less canonical, functional forms in (4.3) than we did in (3.3). Specifically, we choose a constant discount rate $r + \lambda$ here. In Appendix A.3 we show that this is exactly compensated by replacing the reputational dividend $D\left(x\right) = V\left(1, 1\right) - V\left(x, x\right)$ with $\hat{D}(x) = \mu(V_H(1) - V_H(x))$.

[9]At the exit threshold, firm z_T receives zero profits when accounting for the option value of staying in the market. Hence firm z'_T receives small profits for a small period of time, which is of second order.

4.3 Work-Shirk Equilibrium

The reputational dividend equals the value of increasing the firm's reputation from x to 1, times the probability of a breakthrough. This dividend is decreasing in x, so equation (4.3) implies that the value of quality $\Delta(x_0)$ is decreasing in x_0. As a result, the firm's investment incentives are decreasing in its reputation:

Proposition 4. *Suppose (A-λ) holds and c is sufficiently low. In any equilibrium:*
(a) The optimal investment is characterized by cutoffs $\lambda/\mu < x^e < x^* < 1$ such that the firm[10]

$$\begin{array}{ll} \textit{exits} & \textit{if } x \in [0, x^e] \textit{ and quality is low} \\ \textit{works} & \textit{if } x \in [x^e, x^*] \\ \textit{shirks} & \textit{if } x \in [x^*, 1] \end{array}$$

(b) The optimal exit threshold x^e is characterized by

$$(x^e - c - k) + \lambda V_H(x^e) = 0 \tag{4.4}$$

at which point the low-quality firm exits at rate $q = \left[\mu - \frac{\lambda}{x^e}\right]$.

Proof. Assumption (A-λ) ensures that $x^e \geq \lambda/\mu$ and therefore x_t, as determined by (2.2), is decreasing in t. To see this suppose, by contradiction, that $x^e < \lambda/\mu$. Since the dynamics are stationary at $\hat{x} := \lambda/\mu$,

$$V_L(\hat{x}) = \frac{1}{r+\lambda}[r(\hat{x} - c\eta - k) + \lambda V_H(\hat{x})].$$

Observing that $V_H(\hat{x}) \leq 1 - k$, assumption (A-λ) implies that $V_L(\hat{x}) < 0$ as required.

The dividend $\hat{D}(x)$ is strictly decreasing in x, and x_t is decreasing in x_0, so $\Delta(x)$ is strictly decreasing in x. It remains to be shown that none of the intervals is trivial, i.e. that all the inequalities $\lambda/\mu < x^e < x^* < 1$ are strict. Assumption (A-λ) implies that the exit region $[0, x^e]$ exists. The cost is sufficiently small, so the work-region $[x^e, x^*]$ exists. Finally, the upper shirk-region $[x^*, 1]$ exists because $\Delta(x^*) = 0$ if $x^* = 1$.

Condition (4.4) is the indifference of a working firm with low quality to stay or exit when its reputation equals x^e. At this point, the negative instantaneous profits $x^e - c - k$ are balanced by the option value $\lambda V_H(x^e)$ of staying in the market. The exit rate follows from (4.1). □

Proposition 4 shows that, unlike the unknown quality case, the firm invests even as its reputation drops very low, and exit becomes imminent. This is because a technology shock increases the firm's product quality by a discrete amount and averts exit. Consider a firm who is just about

[10] Recall exit is probabilistic, as established in Lemma 3.

to exit: with known quality investment pays off if there is a technology shock, which occurs with probability λdt; with unknown quality investment pays off if the firm's quality increases and it achieves a breakthrough, which happens with probability $\lambda dt \times \mu dt$.

In our model, a firm's knowledge of its own quality is not directly relevant for its investment decision, but does enable it to make better exit decisions. This change in exit behavior, in turn, affects the firm's investment incentives. While our analysis does not lend itself to compare equilibria across the two cases, our results suggest that, at low reputations x an informed firm has higher incentives to invest than an ignorant firm because it can condition its exit decision on the outcome of the investment. On the other hand, at higher levels of reputation where this value of knowing one's quality is lower, one may suspect that the ignorant firm has higher incentives to invest in order to stay away from the low reputation region.

Finally, Proposition 4 can be extended to more general learning structures using the general expression for reputational dividends (4.2). For low reputations $x \approx x^e$, the dividend and value of quality are strictly positive, so the firm invests if the cost is sufficiently low. For high reputations $x \approx 1$, investment incentives disappear as long as there is no perfect bad news signal. This will lead to a shirk region at the top.

4.4 Entry

We can now complete the firm's lifecycle by introducing entry into the model.[11] Suppose potential entrants are endowed with quality $\theta_0 \in \{L, H\}$. Then all high types will enter the market, while low types enter until the reputation of an entrant falls to x^e. If there is a large enough pool of low-quality entrants, then firms enter with reputation $x_0 = x^e$. At this point, all entrants invest in their quality, and low-quality firms immediately start exiting the market.

As an application, consider the lifecycle of a restaurant. There are some potential entrants with a good concept, and many others who have no great idea, but are hopeful. Once a restaurant enters the market, it chooses to invest in food, decor and service. A high quality restaurant may then achieve a breakthrough in term of a good review, while a low quality restaurant may exit. The model predicts that many new restaurants will exit rapidly, but will invest even when exit is imminent. This is consistent with evidence that 25% of new restaurants close each year, and that these restaurants work very hard to stay afloat (Parsa et al. (2005)).

5 Conclusion

This paper has studied the lifecycle of a firm that sells a product of unknown quality. The firm chooses whether to enter the industry and, after entering, can invest in its quality and ultimately exit. We showed that the reputational dynamics depend on whether the firm knows its quality.

[11] We assume entry is observable. This may not be the case: see Tadelis (1999) and Mailath and Samuelson (2001).

When the firm is uninformed, it exits when its reputation falls too low, and shirks when exit is imminent. When the firm is informed, only low-quality firms exit and such firms work no matter how low their reputation.

We have studied the lifecycle of a single firm, ignoring firm interaction by assuming that industry demand is perfectly elastic. As an extension, one could embed the model into a competitive industry, assuming consumers have heterogenous demand for quality. Since our model allows for entry and exit, the steady state would exhibit turnover related to the speed of learning μ and the rate of technological change λ.[12]

[12] There are a couple of papers along these lines. Vial (2008) introduces perfect competition into Mailath and Samuelson (2001) but has no exit. Atkeson, Hellwig, and Ordonez (2010) studies entry and exit in a monopolistically competitive industry, but has no investment.

A Omitted Material

A.1 Proof of Proposition 2

To prove Proposition 2 we first establish a lower bound on the marginal value of reputation.

Lemma 4. *Fix model parameters λ, μ, k, r. There exists $\delta > 0$ such that for all c, all equilibria η,*

$$\partial_x V(x, x) \geq \delta \quad \text{for all } x \in [k, 1].$$

A fortiori, the marginal reputational dividend has the upper bound:

$$D'(x) = -\partial_x V(x, x) - \partial_z V(x, x) \leq -\delta \quad \text{for all } x \in [k, 1].$$

Proof. In equilibrium we have $x^e < k$ because the firm exits only when cash flows are negative. Hence any firm with a reputation exceeding k stays in the market for a period of time that is bounded below.

Consider two firms with different reputations but the same self-esteem: (x_0, x_0) and (x_0', x_0), where $x_0' > x_0$. If the high firm mimics the investment strategy of the low firm, its reputation x_t', and thus its profits $x_t' - c\eta_t - k$, exceed those of the low firm at all times t. While the reputational increment $x_t' - x_t$ may decrease over time, it does so at a finite rate by (2.2). Thus the high firm can guarantee itself an incremental value $(V_{\text{mimic}} - V(x_0, x_0))$ that is bounded below by a linear function of $(x_0' - x_0)$. In equilibrium the high firm achieves a weakly higher value $V(x_0', x_0) \geq V_{\text{mimic}}$, finishing the proof. □

Proof of Proposition 2. Fix model parameters λ, μ, k, r. Assumption (A-λ) ensures that $x^e \geq \lambda/\mu$ and therefore x_t, as determined by (2.2), is decreasing in t. To see this suppose, by contradiction, that $x^e < \lambda/\mu$. Since the dynamics are stationary at $\hat{x} := \lambda/\mu$,

$$V(\hat{x}, \hat{x}) = \frac{1}{r + \hat{x}\mu} \left[r(\hat{x} - c\eta - k) + \hat{x}\mu V(1, 1) \right]$$

Using $\hat{x} = \lambda/\mu$ and $V(1, 1) \leq 1 - k$, assumption (A-λ) implies that $V(\hat{x}, \hat{x}) < 0$ as required.

We now show that there exists $\varepsilon > 0$, and $c > 0$, such that for any equilibrium η

1. $\lambda\Gamma$ is strictly increasing on $[x^e, x^e + \varepsilon]$ with $\lambda\Gamma(x^e) < c$.

2. $\lambda\Gamma$ is greater than c on $[x^e + \varepsilon, 1 - \varepsilon]$.

3. $\lambda\Gamma(x)$ crosses c once and from above on $[1 - \varepsilon, 1]$ with $\lambda\Gamma(1) < c$.

Part (1). Differentiating (3.3), the marginal value of self-esteem obeys the following differential equation:

$$\frac{d}{dt}\Gamma(x_t) = (r + \lambda + \mu(1-x))\,\Gamma(x_t) - D(x_t) \tag{A.1}$$

Since $\Gamma(x^e) = 0$, $d\Gamma(x_t)/dt < 0$ for $x_t \in [x^e, x^e + \varepsilon]$. Since $dx_t/dt < 0$, $\Gamma(x)$ is increasing in x.

Part (2). The dividend $D(x)$ is bounded below for $x \in [x^e, 1-\varepsilon]$. The time to exit T is bounded below for $x_0 \in [x^e + \varepsilon, 1-\varepsilon]$. Therefore, we can choose c low enough such that $c < \lambda\Gamma(x)$ for all $x \in [x^e + \varepsilon, 1-\varepsilon]$

Part (3). By part (2) we know that $\lambda\Gamma(1-\varepsilon) > c$. We also know that $\lambda\Gamma(1) \leq c$; otherwise the firm would invest at $x = 1$ which would imply that $dx = 0$ and $\Gamma(1) = 0$, yielding a contradiction.

Thus, $\lambda\Gamma(x)$ crosses c at least once from above. Suppose, by contradiction, that $\lambda\Gamma(x)$ crosses c at more than one point. Then there exist $x_1, x_2 \in [1-\varepsilon, 1]$ with $x_1 < x_2$, such that Γ has a local minimum at x_1 and a local maximum at x_2 with $\Gamma'(x_1) = \Gamma'(x_2) = 0$ and $\Gamma(x_1) \leq \Gamma(x_2)$.[13] Equation (A.1) implies that

$$\Gamma(x) = \frac{\mu D(x)}{r + \lambda + \mu(1-x)}$$

for $x = x_1, x_2$. We will now show that the RHS is strictly decreasing on $[1-\varepsilon, 1]$; this contradicts $\Gamma(x_1) \leq \Gamma(x_2)$ and finishes the proof. Differentiating the logarithm of the RHS yields

$$\frac{D'(x)}{D(x)} - \frac{-\mu}{r + \lambda + \mu(1-x)}.$$

The second term is bounded, while the first term is unboundedly negative for $x \approx 1$ because $D(x) \approx 0$ and $D'(x) \leq -\delta$ by Lemma 4. Hence the derivative of the RHS is negative, as required.

Finally, part (b) of the Proposition is explained in the text. □

A.2 General Market Learning: Derivation of (3.5)

For any general payoff function $f(x)$, the value function is given by

$$V(x, z) = rdtf(x) + (1 - rdt)\,\mathbb{E}_z[V(x_{dt}, z_{dt})] + O\left(dt^2\right)$$

where (x_{dt}, z_{dt}) are the stochastic values of reputation and self-esteem after dt, and

$$\mathbb{E}_z[V(x_{dt}, z_{dt})] := z\mathbb{E}_H[V(x_{dt}, z_{dt})] + (1-z)\mathbb{E}_L[V(x_{dt}, z_{dt})].$$

[13] For this to be true, it is sufficient that Γ is differentiable in the interior of work-regions and shirk-regions but it does not matter that Γ has kinks on the cutoffs.

The marginal value of self-esteem is then

$$\begin{aligned} V\left(x, z'\right) - V\left(x, z\right) &= \left(1 - rdt\right)\left(\mathbb{E}_{z'}\left[V\left(x_{dt}, z'_{dt}\right)\right] - \mathbb{E}_z\left[V\left(x_{dt}, z_{dt}\right)\right]\right) \\ &= \left(z' - z\right)\underbrace{\left(\mathbb{E}_H\left[V\left(x_{dt}, z_{dt}\right)\right] - \mathbb{E}_L\left[V\left(x_{dt}, z_{dt}\right)\right]\right)}_{\text{Dividend } D(x,z)dt} + \left(1 - rdt\right)\mathbb{E}_z\left[V\left(x_{dt}, z'_{dt}\right) - V\left(x_{dt}, z_{dt}\right)\right] \end{aligned}$$

Define the reputational dividend of self-esteem by

$$D\left(x, z\right) = \left(\mathbb{E}_H\left[V\left(x_{dt}, z_{dt}\right)\right] - \mathbb{E}_L\left[V\left(x_{dt}, z_{dt}\right)\right]\right)/dt$$

The rental value of marginal self-esteem then equals the dividend plus the appreciation:

$$rdt\left(V\left(x, z'\right) - V\left(x, z\right)\right) = \left(z' - z\right)D\left(x, z\right)dt + \mathbb{E}_z\left[d\left(V\left(x, z'\right) - V\left(x, z\right)\right)\right].$$

If we integrate and divide by $(z'_0 - z_0)$

$$\frac{V\left(x_0, z'_0\right) - V\left(x_0, z_0\right)}{z'_0 - z_0} = \mathbb{E}\left[\int_0^T e^{-rt}\frac{z'_t - z_t}{z'_0 - z_0}D\left(x_t, z_t\right)dt\right].$$

In the limit, this yields

$$\partial_z V\left(x_0, z_0\right) = \mathbb{E}\left[\int_0^T e^{-rt}\frac{\partial z\left(z_0, t\right)}{\partial z_0}D\left(x_t, z_t\right)dt\right]$$

as in equation (3.5).

We can further develop the $\partial z\left(z_0, t\right)/\partial z_0$ term for Brownian motion and Poisson shocks $y \in Y$.

$$\frac{z'_{dt} - z_{dt}}{z' - z} - 1 = \begin{cases} -\lambda dt & \text{equilibrium beliefs} \\ -\left(1 - 2z\right)\sum_y \mu_y dt & \text{absence of Poisson shocks} \\ \left(1 - 2z\right)\mu_B dW & \text{Brownian motion} \\ \mu_y \frac{(1-2z)\left(z\mu_{y,H} + (1-z)\mu_{y,L}\right) - \mu_y z(1-z)}{\left(z\mu_{y,H} + (1-z)\mu_{y,L}\right)^2} & \text{at Poisson shock } y \end{cases}$$

Taking the limit

$$\begin{aligned} \partial z\left(z_0, t\right)/\partial z_0 = &\exp\left(\int_0^t -\lambda + \sum_y \mu_y\left(1 - 2z_s\right)ds\right) && \text{(Drift)} \\ &\times \exp\left(-\int_0^t\left(\mu_B^2\left(1 - 2z_s\right)^2/2\right)ds + \int_0^t\left(1 - 2z_s\right)\mu_B dW_s\right) && \text{(Brownian)} \\ &\times \prod_{y \in Y, t_y \in P_y}\left(1 + \mu_y\frac{\left(1 - 2z\right)\left(z\mu_{y,H} + \left(1 - z\right)\mu_{y,L}\right) - \mu_y z\left(1 - z\right)}{\left(z\mu_{y,H} + \left(1 - z\right)\mu_{y,L}\right)^2}\right) && \text{(Poisson)} \end{aligned}$$

The first term in the second line is the Ito-term accounting for the fact that $\exp\left(f\left(W\right)dW\right) = 1 + f\left(W\right)dW + f\left(W\right)^2 dt/2 + o\left(dt\right)$. In the third line, $P_y \subseteq [0,t]$ is the finite number of times that Poisson shock y hits.

A.3 The Value of Quality: An Alternative Expression

We can now apply Proposition 1 to provide an alternative derivation of the firm's investment incentives with known quality. Define

$$V\left(x,z\right) := zV_H\left(x\right) + \left(1-z\right)V_L\left(x\right).$$

Note that $V(x,z)$ is linear in z, whereas it is convex in the unknown quality case. Repeating the analysis in Section 3 yields:

$$\Delta(x_0) = \int_0^\infty e^{-\int_0^t (r+\lambda+\mu(1-x_s))ds}\, D(x_t)\, dt \tag{A.2}$$

where $D(x) = \mu(V\left(1,1\right) - V\left(x,x\right))$ is the reputational dividend.

To see that (A.2) and (4.3) are the same, we differentiate them and obtain:

$$(r+\lambda+\mu(1-x_t))\Delta(x_t) = \mu[V(1,1) - V(x,x)] + \frac{d}{dt}[\Delta(x_t)]$$
$$(r+\lambda)\Delta(x_t) = \mu[V_H(1) - V_H(x)] + \frac{d}{dt}[\Delta(x_t)]$$

which coincide since $\mu(1-x_t)\Delta(x_t) = \mu[V_H(x) - V(x,x)]$.

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