Revenue Management with Forward-Looking Buyers
Posted Prices and Fire-sales

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The Problem

Seller owns $K$ units of a good

- Seller has $T$ periods to sell the goods.
- Buyers enter over time.
- Privately known values.
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- Buyers enter over time.
- Privately known values.

Big literature on revenue management

- Typically assume buyers are myopic.

Forward looking buyers

- Agents delay if expect prices to fall.
- Prefer to buy sooner rather than later.
Applications

RM is hugely successful branch of market design

- Historically: Airlines, Seasonal clothing, Hotels, Cars
- Online economy: Ad networks, Ticket distributors, e-Retailers

Buyers strategically time purchases

- Clothing (Soysal and Krishnamurthi, 2012)
- Airlines (Li, Granados and Netessine, 2012)
- Redzone contracts (e.g. YouTube)
- Price prediction sites (e.g. Bing Travel)

Questions

- What is the optimal mechanism?
- Is there a simple way to implement it?
Price and Cutoffs with One Units

Soysal and Krishnamurthi: Demand Dynamics in the Seasonal Goods Industry
Marketing Science 31(2), pp. 293–316, © 2012 INFORMS

6.1. Parameter Estimates


Table 2a reports the maximum likelihood estimates of the parameters of the markdown probability model (conditional on no previous markdown) specified in Equation (2).

Table 2b, on the other hand, reports the markdown probability model parameters (conditional on previous MD).

Table 2c reports the OLS estimates of the parameters of the markdown depth model specified in Equation (3). The estimates indicate that the probability of markdown conditional on a previous markdown is positively related to the product's retail price. The probability of markdown is also positively related to the time in the season (number of weeks since product j has been introduced) both conditional on no previous markdown and conditional on a previous markdown.

Table 2c reports the OLS estimates of the parameters of the markdown depth model specified in Equation (4). The estimates indicate that the natural logarithm of markdown depth is positively related to the natural logarithm of the product's retail price and negatively related to the markdown dummy. Products with higher retail prices face deeper markdowns compared to products with lower retail prices and first markdowns are deeper than later markdowns.


Table 3 reports the OLS estimates of the parameters of the availability expectations process specified in Equation (5). The estimates indicate that availability

Notes.

R² = 0.0535. Number of observations = 233; mean squared error = 0.0184.
Results

Allocations determined by deterministic cutoffs.

- Only depend on \((k, t)\),
- Not on \# of agents, their values, when sold units.

When demand gets weaker over time

- Cutoffs satisfy one-period-look-ahead property.

Implement in continuous time via posted prices

- With auction at time \(T\).
- Relies on cutoffs being deterministic.

Prices depend on when previous units were sold.

- Cutoffs are easy; prices are hard.
Outline

1. Allocations
   ▶ General demand - Cutoffs are deterministic
   ▶ Decreasing demand - One-period-look-ahead property

2. Implementation
   ▶ General demand - Use posted prices
   ▶ Decreasing demand - Prices given by differential equation

3. Applications
   ▶ Retailing - Storage costs
   ▶ Display ads - Third degree price discrimination
   ▶ Airlines - Changing distribution of arrivals
   ▶ House selling - Disappearing buyers
Literature

Gallien (2006)
- Infinite periods; Inter-arrival times have increasing failure rate.
- No delay in equilibrium.

Pai and Vohra (2013), Mierendorff (2009)
- Privately known value, entry time, exit time; No discounting.
- Show how to simplify problem, but do not fully characterize.

- Similar model to ours; only allow for two prices.

MacQueen and Miller (1960), McAfee and McMillan (1988)
- Optimal policy for single unit.
Model
Model

- Time discrete and finite \( t \in \{1, \ldots, T\} \)
- Seller has \( K \) goods.
- Seller can commit to mechanism.

Entrants

- At start of period \( t \), \( N_t \) buyers arrive
- \( N_t \) independently distributed, but distribution may vary
- \( N_t \) observed by seller but not other buyers

Preferences

- Buyer has value \( v_i \sim f(\cdot) \) for one unit.
- Utility is \( (v - p_t)\delta^t \)
Mechanisms

- Buyer makes report $\tilde{v}_i$ when enters market.
- Mechanism $\langle \tau_i, TR_i \rangle$ describes allocation and transfer.
- Feasible if award after entry, $K$ goods, adapted to seller’s info

Buyer’s problem

- Buyer chooses $\tilde{v}_i$ to maximise

$$u_i(\tilde{v}_i, v_i, t_i) = E_0\left[v_i \delta \tau_i(\tilde{v}_i, v_{-i}, t)|v_i, t_i\right]$$

where $E_t$ is expectation at the start of period $t$.

Mechanism is (IC) and (IR) if

**(INT)** $u_i(v_i, v_i, t_i) = E_0[\int_{v_i}^{v_i} \delta \tau_i(z, v_{-i}, t) dz]|v_i, t_i]$

**(MON)** $E_0[\delta \tau_i(v, t)|v_i, t_i]$ is increasing in $v_i$. 
Buyer’s expected rents

- Taking expectations over \((v_i, t_i)\) and integrating by parts,

\[
E_0[u_i(v_i, v_i, t_i)] = E_0 \left[ \delta_{i}(v, t) \frac{1 - F(v_i)}{f(v_i)} \right]
\]

Seller’s problem

- Define marginal revenue,

\[
m(v) := v - \frac{(1 - F(v))}{f(v)}
\]

- Seller chooses mechanism to solve

\[
\text{Profit} = E_0 \left[ \sum_i TR_i \right] = E_0 \left[ \sum_i \delta_{i}(v, t) m(v_i) \right]
\]

- Assume \(m(v)\) is increasing in \(v\), so (MON) satisfied.
Example: One Unit, IID Arrivals
Proposition 0.

Suppose $K = 1$ and $N_t$ is IID. The seller awards the good to the buyer with the highest valuation exceeding a cutoff $x_t$, where

$$m(x_t) = \delta E_{t+1}[\max\{m(v^1_{t+1}), m(x_t)\}] \quad \text{for } t < T$$

$$m(x_T) = 0$$

These cutoffs are constant in periods $t < T$, and drop at time $T$.

(i) Cutoffs deterministic: depend on $t$; not on # entrants, values.

(ii) Characterized by one-period-look-ahead rule.

(iii) Constant for $t < T$: Seller indifferent between selling/waiting. If delay, face same tradeoff tomorrow and indifferent again. Hence assume buy tomorrow.
Implementation in Continuous Time

- Buyers enter at Poisson rate $\lambda$.
- Optimal cutoffs are deterministic:
  \[ rm(x^*) = \lambda E \left[ \max \left\{ m(v) - m(x^*), 0 \right\} \right] \]

Implementation via Posted Prices

- At time $T$ hold SPA with reserve $m^{-1}(0)$.
- The final posted price
  \[ p_T = E_0 \left[ \max \left\{ v_{\leq T}^2, m^{-1}(0) \right\} | v_{\leq T}^1 = x^* \right] \]
- Posted price for $t < T$,
  \[ \dot{p}_t = -(x^* - p_t) \left( \lambda (1 - F(x^*)) + r \right) \]
Price and Cutoffs with One Units

Assumptions: Buyers enter with $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$. 
Implementation via Contingent Contract

Contingent Contract

- Netflix wishes to buy ad slot on front page of YouTube
- Buy-it-now price \( p_H \)
- Pay \( p_L \) to lock-in later if no other buyer

Implementation

- Fix price path \( p_t \) above, with final price \( p_T \)
- When buyer enters, bids \( b \)
- If \( b \geq p_T \), buyer locks-in contract at time \( \min\{t : p_t = b\} \)
- If \( b < p_T \), this is treated as bid in auction at \( T \)
Many Units: Allocations
Preliminaries

Seller has \(k\) units at start of period \(t\)

- Let \(y := \{y^1, y^2, \ldots, y^k\}\) be highest buyers at time \(t\).

Lemma 1.
The optimal mechanism uses cutoffs \(x_t^j(y^{-(k-j+1)})\), \(j \leq k\).

- Across buyers, seller allocates to high value buyers first
- For one buyer, allocations monotone in values
- Unit \(j\) awarded iff \(y^{k-\ell+1} \geq x_t^\ell(y^{k-\ell+1})\) for \(\ell \in \{j, \ldots, k\}\)

Highest values \((y^1, \ldots, y^k)\) act as state

- Buyer’s \(t_i\) doesn’t affect allocation, so seller need not know
- Optimal allocations independent of when past units sold
“Continuation profit” at time $t$ with $k$ units is

$$\Pi^k_t(y) := \max_{\tau_i \geq t} E_t \left[ \sum_i \delta_{\tau_i}(y) - t m(v_i) \right]$$

$$\tilde{\Pi}^k_t(y) := \max_{\tau_i \geq t} E_{t+1} \left[ \sum_i \delta_{\tau_i}(y) - t m(v_i) \right]$$

Lemma 2.

Suppose $x^j_t(\cdot)$ are decreasing in $j$. Then unit $j$ is allocated iff

$$y^{k-j+1} \geq x^j_t(y^{k-j+1})$$

Idea

If want to sell $j^{th}$ unit then want to sell units $\{j + 1, \ldots, k\}$
\[ \Delta \tilde{\Pi}_t^k (y) := \tilde{\Pi}_t^k (\text{sell 1 today}) - \tilde{\Pi}_t^k (\text{sell 0 today}) \]

- Cutoff \( x_t^j (\cdot) \) is \textit{deterministic} if it is independent of \( y^{-(k-j+1)} \)

\textbf{Lemma 3.}

Suppose \( \{x_s^j\}_{s \geq t+1} \) are deterministic and decreasing in \( j \). Then:

(a) \( \Delta \tilde{\Pi}_t^k (y) \) is independent of \( y^{-1} \)
(b) \( \Delta \tilde{\Pi}_t^k (y^1) \) is continuous and strictly increasing in \( y^1 \)
(c) \( \Delta \tilde{\Pi}_t^k (y^1) \) is increasing in \( k \).

\textbf{Idea}

(a) Allocation to \( y^j \) determined by rank relative to no. of goods. Decision today does not affect when \( y^j \) gets good. Hence value of \( y^j \) does not affect difference \( \Delta \tilde{\Pi}_t^k (y) \).
(b) A higher \( y^1 \) is more valuable if sell earlier.
(c) The option value of waiting declines with more goods.
Deterministic Allocations

**Theorem 1.**
The optimal cutoffs $x^k_t$ are deterministic, decreasing in $k$ and uniquely determined by $\Delta \tilde{\Pi}^k_t(x^k_t) = 0$

- At $T$, $m(x^k_T) = 0$. By induction, suppose $x^k_t(y^{-1}) > x^{k-1}_t$

  $$0 \geq \Delta \tilde{\Pi}^k_t(x^k_t(y^{-1})) > \Delta \tilde{\Pi}^k_t(x^{k-1}_t) \geq \Delta \tilde{\Pi}^{k-1}_t(x^{k-1}_t) = 0$$

  Using (i) $\tilde{\Pi}^k_t($sell $\geq 1$ today) $\geq \tilde{\Pi}^k_t($sell 1 today)

  (ii) monotonicity of $\Delta \tilde{\Pi}^k_t(y^1)$ in $y^1$

  (iii) monotonicity of $\Delta \tilde{\Pi}^k_t(y^1)$ in $k$

  (iv) induction.

- As $x^k_t(y^{-1}) \geq x^{k-1}_t$, $\Delta \tilde{\Pi}^k_t(x^k_t(y^{-1})) = 0$ and $x^k_t$ deterministic

- Hence seller need not elicit $y^{-1}$ to determine allocation.
Decreasing Demand

- $D\tilde{\Pi}^k_t(y^1) := \tilde{\Pi}^k_t(\text{sell 1 today}) - \tilde{\Pi}^k_t(\text{sell } \geq 1 \text{ tomorrow})$

- Note $D\tilde{\Pi}^k_t(y^1) \geq \Delta\tilde{\Pi}^k_t(y^1)$, with equality if $x^k_t \geq x^k_{t+1}$

**Theorem 2.**
Suppose $N_t$ is decreasing in FOSD. Then $x^k_t$ are decreasing in $t$, and determined by a one-period-look-ahead policy, $D\tilde{\Pi}^k_t(x^k_t) = 0$.

- If $\{x^k_s\}_{s \geq t+1}$ are decreasing in $s$, then $D\tilde{\Pi}^k_{t+1}(y^1) \geq D\tilde{\Pi}^k_t(y^1)$.
  Idea: Option value lower when fewer periods.

- By contradiction, if $x^k_t < x^k_{t+1}$ then
  \[0 \leq D\tilde{\Pi}^k_t(x^k_t) < D\tilde{\Pi}^k_t(x^k_{t+1}) \leq D\tilde{\Pi}^k_{t+1}(x^k_{t+1}) = 0.\]

- Using (i) $\tilde{\Pi}^k_t(\text{sell 0 today}) \geq \tilde{\Pi}^k_t(\text{sell } \geq 1 \text{ tomorrow})$
  (ii) monotonicity of $D\tilde{\Pi}^k_t(y^1)$ in $y^1$
  (iii) monotonicity of $D\tilde{\Pi}^k_t(y^1)$ in $t$
  (iv) induction.
Decreasing Demand: Indifference Equations

The optimal cutoffs $x^k_t$ are given by local indifference conditions

- At time $T$,
  \[ m(x^k_T) = 0 \]

- At time $T - 1$,
  \[ m(x^k_{T-1}) = \delta E_{T-1} \left[ \max\{m(x^k_{T-1}), m(v^k_T)\} \right] \]

- At time $t < T - 1$,
  \[
  m(x^k_t) + \delta E_{t+1} \left[ \tilde{\Pi}^{k-1}_{t+1}(v_{t+1}) \right] \\
  = \delta E_{t+1} \left[ \max\{m(x^k_t), m(v^1_{t+1})\} \right] + \delta E_{t+1} \left[ \tilde{\Pi}^{k-1}_{t+1} (\{x^k_t, v^1_{t+1}\})^2_k \right]
  \]
IMPLEMENTATION WITH POSTED PRICES
General Demand

- Assume Poisson arrivals $\lambda_t$, discount rate $r$, period length $h$
- Price mechanism: Single posted price in each period; if there is excess demand, good is rationed randomly.

**Theorem 3.**

Suppose $\lambda_t$ is Lipschitz continuous. Then lost profit from using posted prices and auction for final good in final period is $O(h)$.

(i) Cutoffs do not jump down by more than $O(h)$
   
   Idea: If $t < T - h$, follows from continuity of $\lambda_t$.
   
   For $t = T - h$, have $m(x^k_t) \approx 0$ for $k \geq 2$

(ii) Prices wrong because (1) don’t adjust cutoffs within a period; and (2) may ration incorrectly.
    
    But the prob. of 2 sales in one period is $O(h^2)$.

- Poisson arrivals important since imply common expectations
Decreasing Demand: Allocations

Poisson rate $\lambda_t$ decreasing in $t$.

- Optimal cutoffs given by infinitesimal-period-look-ahead rule:

$$rm(x^k_t) = \lambda_t E_v \left[ \max\{m(v) - m(x^k_t), 0\} + \Pi^{k-1}_t \left( \min\{v, x^k_t\} \right) - \Pi^{k-1}_t(v) \right]$$

$$m(x^k_T) = 0$$

where $v$ is drawn from $F(\cdot)$

End game, $t \to T$

- If $k \geq 2$, then $x^k_t \to m^{-1}(0)$.
- If $k = 1$, then $x^k_t$ jumps down to $m^{-1}(0)$
Decreasing Demand: Prices

Period $T$

- For $k = 1$, hold SPA with reserve $m^{-1}(0)$
- Final posted price

$$p_T = E_0 \left[ \max\{y^2, m^{-1}(0)\} \mid y^1 = \lim_{h \to 0} x_{T-h}^1, \{s_T(x)\}_{x \leq y^1} \right]$$

where $s_T(x)$ is last time the cutoff went below $x$.
- For $k \geq 2$, $p_t \to m^{-1}(0)$ as $t \to T$.

For $t < T$, prices determined by

$$\dot{p}_t^k = \dot{x}_t^k \left( \int_{s_t(x_t^k)}^t \lambda_s \, ds \right) f(x_t^k) - \lambda_t (1 - F(x_t^k)) \left[ x_t^k - p_t^k - U_{t-1}^k(x_t^k) \right] - r \left( x_t^k - p_t^k \right)$$

- If other units purchased earlier, $p_t^k$ is higher.
- Price falls over time but jumps with every sale.
Price and Cutoffs with Two Units

Penultimate Unit

Last Unit

Cutoffs

Prices
Probability of Sale

- Last Unit
- Penultimate Unit
Forward-Looking vs. Myopic Buyers

Myopic Buyers

- Buyers buy when enter, or leave forever
- Cutoffs $m(x^k_t) = \delta(V^k_{t+1} - V^{k-1}_{t+1})$, where $V^k_t$ is value in $(k, t)$.
- Implement with prices equal to cutoff.

Under forward-looking buyers

- Profits higher
- Total sales higher
- Sales later in season

Retailing data suggest forward-looking buyers

- Price reductions lead to large numbers of sales
- Burst of sales quickly dies down
- Prices fall rapidly near the end of season
Cutoffs, Prices and Sales with Myopic Buyers

Cutoffs/Prices

Probability of Sale

Last Unit

Penultimate Unit

Time, t
APPLICATIONS
Retail Markets - Inventory Costs

- Inventory cost $c_t$ if good held until time $t$.
- Assume marginal cost $\Delta c_t = c_{t+1} - c_t$ is increasing in $t$.

Cutoffs are deterministic and decreasing over time.

- For $t = T$, $m(x^k_T) = -\Delta c_T$. For $t < T$,

$$m(x^k_t) + E_{t+1} \left[ \tilde{\Pi}^{-1}_{t+1}(v_{t+1}) \right]$$

$$= E_{t+1} \left[ \max\{m(x^k_t), m(v^1_{t+1})\} \right] + E_{t+1} \left[ \tilde{\Pi}^{-1}_{t+1}(\{x^k_t, v_{t+1}\}_{k}^2) \right] - \Delta c_t$$

- In continuous time,

$$\dot{c}_t = \lambda_t E \left[ \max\{m(v) - m(x^k_t), 0\} + \Pi^{-1}_t(\min\{v, x^k_t\}) - \Pi^{-1}_t(v) \right]$$

$$\dot{p}^k_t = \left( \int_{st(x^k_t)}^{t} \lambda_s ds \right) f(x^k_t) - \lambda_t (1 - F(x^k_t)) \left[ x^k_t - p^k_t - U^{-1}_t(x^k_t) \right]$$
Display Ads - Price Discrimination

- Rich media ad buyers have values $v \sim f_R$
- Static ad buyers have values $v \sim f_S$

Solving the problem
- Letting $m_i \in \{m_R, m_S\}$, the seller maximizes

$$\text{Profit} = E_0 \left[ \sum_i \delta^{r_i} m_i(v_i) \right]$$

- State variable is now $k$ highest marginal revenues
- Cutoffs are deterministic in marginal revenue space

Implementation
- Use two price schedules for two types of buyer
- If rich media buyers have higher values, their marginal revenues are lower and prices are higher.
Airlines - Changing Distributions

- Demand $f_t$ gets stronger over time
- Seller maximizes
  \[ E_0 \left[ \sum_i \delta^{\tau_i} m_{t_i}(v_i) \right] \]

Optimal discriminations

- If $t_i$ observed, have cohort specific cutoffs/prices.
- Bias towards earlier cohorts.
- This is (IC) if $t_i$ not observed.
- e.g. If $f_t \sim \exp(\mu_t)$, then issue coupon of $\mu_t$ for cohort $t$. 
Selling a House - Disappearing Buyers

- Buyers exit the game with probability \( \in (0, 1) \).
- Now need to carry around all past entrants as state.

Cutoffs no longer deterministic

- If delay buyer \( y^1 \) may disappear, so value of \( y^2 \) matters.
- Prices no longer optimal.
- Explanation for indicative bidding in real estate.

Also have problem if

- Buyers have different discount rates.
- Mix of myopic and forward-looking buyers.
- General problem: ranking of buyer’s values changes.
Conclusion

Optimal cutoffs

- Deterministic (only depend on $k$ and $t$).
- Characterised by one-period-look-ahead rule.

Implemented by posted prices

- Sequence of prices with auction at time $T$.
- Prices depend on when sold previous units.

Extensions

- $N_t$ correlated (e.g. learning)
- Different quality of ad slots
- Cost of paying attention to prices.