Revenue Management with Forward-Looking Buyers
Posted Prices and Fire-sales

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The Problem

Seller owns K units of a good

- Seller has T periods to sell the goods.
- Buyers enter over time.
- Privately known values.

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Big literature on revenue management

- Typically assume buyers are myopic.

Forward looking buyers

- Agents delay if expect prices to fall.
- Prefer to buy sooner rather than later.

Applications

RM is hugely successful branch of market design

- Historically: Airlines, Seasonal clothing, Hotels, Cars
- Online economy: Ad networks, Ticket distributors, e-Retailers

Buyers strategically time purchases

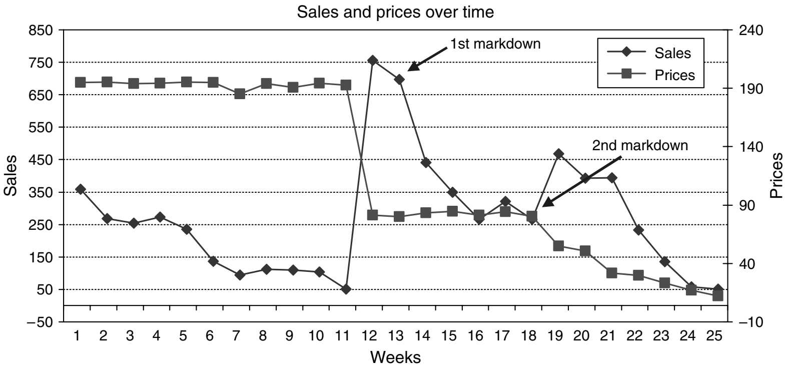
- Clothing (Soysal and Krishnamurthi, 2012)
- Airlines (Li, Granados and Netessine, 2012)
- Redzone contracts (e.g. YouTube)
- Price prediction sites (e.g. Bing Travel)

Questions

- What is the optimal mechanism?
- Is there a simple way to implement it?

Price and Cutoffs with One Units

Prices and Sales for a Sample Product



Results

Allocations determined by deterministic cutoffs.

- Only depend on (k, t),
- Not on $\#$ of agents, their values, when sold units.

When demand gets weaker over time

- Cutoffs satisfy one-period-look-ahead property.

Implement in continuous time via posted prices

- With auction at time T.
- Relies on cutoffs being deterministic.

Prices depend on when previous units were sold.

- Cutoffs are easy; prices are hard.

Outline

1. Allocations
   - General demand - Cutoffs are deterministic
   - Decreasing demand - One-period-look-ahead property
2. Implementation
   - General demand - Use posted prices
   - Decreasing demand - Prices given by differential equation
3. Applications
   - Retailing - Storage costs
   - Display ads - Third degree price discrimination
   - Airlines - Changing distribution of arrivals
   - House selling - Disappearing buyers

Literature

Gallien (2006)

- Infinite periods; Inter-arrival times have increasing failure rate.
- No delay in equilibrium.

Pai and Vohra (2013), Mierendorff (2009)

- Privately known value, entry time, exit time; No discounting.
- Show how to simplify problem, but do not fully characterize.

Aviv and Pazgal (2008), Elmaghraby et al (2008)

- Similar model to ours; only allow for two prices.

MacQueen and Miller (1960), McAfee and McMillan (1988)

- Optimal policy for single unit.

Model

Model

- Time discrete and finite $t \in \{1, \ldots, T\}$
- Seller has K goods.
- Seller can commit to mechanism.

Entrants

- At start of period t, N_t buyers arrive
- N_t independently distributed, but distribution may vary
- N_t observed by seller but not other buyers

Preferences

- Buyer has value $v_i \sim f(\cdot)$ for one unit.
- Utility is $(v - p_t)\delta^t$

Mechanisms

- Buyer makes report $\tilde{v}_i$ when enters market.
- Mechanism $\langle \tau_i, TR_i \rangle$ describes allocation and transfer.
- Feasible if award after entry, K goods, adapted to seller's info

Buyer's problem

- Buyer chooses $\tilde{v}_i$ to maximise

$$u_i(\tilde{v}_i, v_i, t_i) = E_0\left[v_i \delta^{\tau_i(\tilde{v}_i, \mathbf{v_{-i}}, \mathbf{t})} - TR_i(\tilde{v}_i, v_{-i}, \mathbf{t}) \middle| v_i, t_i\right]$$

where E_t is expectation at the start of period t.

Mechanism is (IC) and (IR) if

(INT) $u_i(v_i, v_i, t_i) = E_0[\int_{\underline{v}}^{v_i} \delta^{\tau_i(z, \mathbf{v_{-i}}, \mathbf{t})}\, dz | v_i, t_i]$

(MON) $E_0[\delta^{\tau_i(\mathbf{v}, \mathbf{t})} | v_i, t_i]$ is increasing in v_i.

Buyer's expected rents

- Taking expectations over (v_i, t_i) and integrating by parts,

$$E_0[u_i(v_i, v_i, t_i)] = E_0\left[\delta^{\tau_i(\mathbf{v},\mathbf{t})}\frac{1 - F(v_i)}{f(v_i)}\right]$$

Seller's problem

- Define marginal revenue, $m(v) := v - (1 - F(v))/f(v)$.
- Seller chooses mechanism to solve

$$\text{Profit} = E_0\Big[\sum_i TR_i\Big] = E_0\Big[\sum_i \delta^{\tau_i(\mathbf{v},\mathbf{t})} m(v_i)\Big]$$

- Assume $m(v)$ is increasing in v, so (MON) satisfied.

Example: One Unit, IID Arrivals

Single Unit

Proposition 0.

Suppose $K = 1$ and N_t is IID. The seller awards the good to the buyer with the highest valuation exceeding a cutoff x_t, where

$$m(x_t) = \delta E_{t+1}[\max\{m(v^1_{t+1}), m(x_t)\}] \qquad \text{for } t < T$$
$$m(x_T) = 0$$

These cutoffs are constant in periods $t < T$, and drop at time T.

(i) Cutoffs deterministic: depend on t; not on $\#$ entrants, values.

(ii) Characterized by one-period-look-ahead rule.

(iii) Constant for $t < T$: Seller indifferent between selling/waiting. If delay, face same tradeoff tomorrow and indifferent again. Hence assume buy tomorrow.

Implementation in Continuous Time

- Buyers enter at Poisson rate λ.
- Optimal cutoffs are deterministic:

$$rm(x^*) = \lambda E\big[\max\{m(v) - m(x^*), 0\}\big]$$

Implementation via Posted Prices

- At time T hold SPA with reserve $m^{-1}(0)$.
- The final posted price

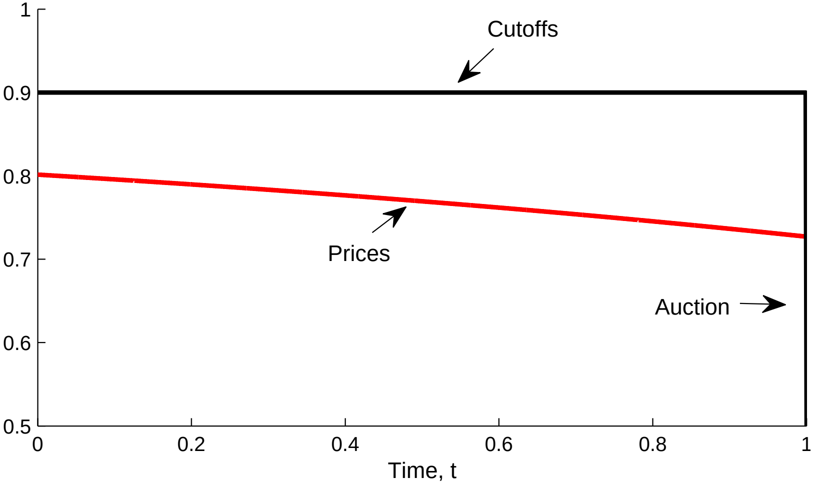
$$p_T = E_0\left[\max\{v^2_{\leq T}, m^{-1}(0)\}\middle| v^1_{\leq T} = x^*\right]$$

- Posted price for $t < T$,

$$\dot{p}_t = -(x^* - p_t)\big(\lambda(1 - F(x^*)) + r\big)$$

Price and Cutoffs with One Units

Assumptions: Buyers enter with $\lambda = 5$ and have values $v \sim U[0, 1]$. Total time is $T = 1$ and the interest rate is $r = 1/16$.



Implementation via Contingent Contract

Contingent Contract

- Netflix wishes to buy ad slot on front page of YouTube
- Buy-it-now price p_H
- Pay p_L to lock-in later if no other buyer

Implementation

- Fix price path p_t above, with final price p_T
- When buyer enters, bids b
- If $b \geq p_T$, buyer locks-in contract at time $\min\{t : p_t = b\}$
- If $b < p_T$, this is treated as bid in auction at T

Many Units: Allocations

Preliminaries

Seller has k units at start of period t

- Let $\mathbf{y} := \{y^1, y^2, \ldots, y^k\}$ be highest buyers at time t.

Lemma 1.
The optimal mechanism uses cutoffs $x_t^j(\mathbf{y}^{-(\mathbf{k}-\mathbf{j}+\mathbf{1})})$, $j \leq k$.

- Across buyers, seller allocates to high value buyers first
- For one buyer, allocations monotone in values
- Unit j awarded iff $y^{k-\ell+1} \geq x_t^\ell(\mathbf{y}^{\mathbf{k}-\ell+\mathbf{1}})$ for $\ell \in \{j, \ldots, k\}$

Highest values $(y^1, \ldots, y^k)$ act as state

- Buyer's t_i doesn't affect allocation, so seller need not know
- Optimal allocations independent of when past units sold

- "Continuation profit" at time t with k units is

$$\Pi_t^k(\mathbf{y}) := \max_{\tau_i \geq t} E_t\left[\sum_i \delta^{\tau_i(\mathbf{y})-t} m(v_i)\right]$$

$$\tilde{\Pi}_t^k(\mathbf{y}) := \max_{\tau_i \geq t} E_{t+1}\left[\sum_i \delta^{\tau_i(\mathbf{y})-t} m(v_i)\right]$$

Lemma 2.
Suppose $x_t^j(\cdot)$ are decreasing in j. Then unit j is allocated iff $y^{k-j+1} \geq x_t^j(\mathbf{y^{k-j+1}})$

Idea

- If want to sell j^{th} unit then want to sell units $\{j+1, \ldots, k\}$

- $\Delta\tilde{\Pi}_t^k(\mathbf{y}) := \tilde{\Pi}_t^k(\text{sell 1 today}) - \tilde{\Pi}_t^k(\text{sell 0 today})$
- Cutoff $x_t^j(\cdot)$ is *deterministic* if it is independent of $\mathbf{y}^{-(\mathbf{k}-\mathbf{j}+\mathbf{1})}$

Lemma 3.

Suppose $\{x_s^j\}_{s\geq t+1}$ are deterministic and decreasing in j. Then:
(a) $\Delta\tilde{\Pi}_t^k(\mathbf{y})$ is independent of $\mathbf{y}^{-\mathbf{1}}$
(b) $\Delta\tilde{\Pi}_t^k(y^1)$ is continuous and strictly increasing in y^1
(c) $\Delta\tilde{\Pi}_t^k(y^1)$ is increasing in k.

Idea

(a) Allocation to y^j determined by rank relative to no. of goods.
Decision today does not affect when y^j gets good.
Hence value of y^j does not affect difference $\Delta\tilde{\Pi}_t^k(\mathbf{y})$.

(b) A higher y^1 is more valuable if sell earlier.

(c) The option value of waiting declines with more goods.

Deterministic Allocations

Theorem 1.
The optimal cutoffs x_t^k *are deterministic, decreasing in* k *and uniquely determined by* $\Delta\tilde{\Pi}_t^k(x_t^k) = 0$

- At T, $m(x_T^k) = 0$. By induction, suppose $x_t^k(\mathbf{y^{-1}}) > x_t^{k-1}$

$$0 \geq \Delta\tilde{\Pi}_t^k(x_t^k(\mathbf{y}^{-1})) > \Delta\tilde{\Pi}_t^k(x_t^{k-1}) \geq \Delta\tilde{\Pi}_t^{k-1}(x_t^{k-1}) = 0$$

  Using (i) $\tilde{\Pi}_t^k$(sell ≥ 1 today) $\geq \tilde{\Pi}_t^k$(sell 1 today)
  (ii) monotonicity of $\Delta\tilde{\Pi}_t^k(y^1)$ in y^1
  (iii) monotonicity of $\Delta\tilde{\Pi}_t^k(y^1)$ in k
  (iv) induction.
- As $x_t^k(\mathbf{y^{-1}}) \geq x_t^{k-1}$, $\Delta\tilde{\Pi}_t^k(x_t^k(\mathbf{y}^{-1})) = 0$ and x_t^k deterministic
- Hence seller need not elicit $\mathbf{y^{-1}}$ to determine allocation.

Decreasing Demand

- $D\tilde{\Pi}_t^k(y^1) := \tilde{\Pi}_t^k(\text{sell } 1 \text{ today}) - \tilde{\Pi}_t^k(\text{sell} \geq 1 \text{ tomorrow})$
- Note $D\tilde{\Pi}_t^k(y^1) \geq \Delta\tilde{\Pi}_t^k(y^1)$, with equality if $x_t^k \geq x_{t+1}^k$

Theorem 2.
Suppose N_t is decreasing in FOSD. Then x_t^k are decreasing in t, and determined by a one-period-look-ahead policy, $D\tilde{\Pi}_t^k(x_t^k) = 0$.

- If $\{x_s^k\}_{s \geq t+1}$ are decreasing in s, then $D\tilde{\Pi}_{t+1}^k(y^1) \geq D\tilde{\Pi}_t^k(y^1)$.
  Idea: Option value lower when fewer periods.
- By contradiction, if $x_t^k < x_{t+1}^k$ then
  $$0 \leq D\tilde{\Pi}_t^k(x_t^k) < D\tilde{\Pi}_t^k(x_{t+1}^k) \leq D\tilde{\Pi}_{t+1}^k(x_{t+1}^k) = 0.$$
- Using (i) $\tilde{\Pi}_t^k(\text{sell } 0 \text{ today}) \geq \tilde{\Pi}_t^k(\text{sell} \geq 1 \text{ tomorrow})$
  (ii) monotonicity of $D\tilde{\Pi}_t^k(y^1)$ in y^1
  (iii) monotonicity of $D\tilde{\Pi}_t^k(y^1)$ in t
  (iv) induction.

Decreasing Demand: Indifference Equations

The optimal cutoffs x_t^k are given by local indifference conditions

- At time T,

$$m(x_T^k) = 0$$

- At time $T-1$,

$$m(x_{T-1}^k) = \delta E_{T-1}\left[\max\{m(x_{T-1}^k), m(v_T^k)\}\right]$$

- At time $t < T-1$,

$$m(x_t^k) + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\mathbf{v_{t+1}})\right]$$
$$= \delta E_{t+1}\left[\max\{m(x_t^k), m(v_{t+1}^1)\}\right] + \delta E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{x_t^k, \mathbf{v_{t+1}}\}_k^2)\right]$$

Implementation with Posted Prices

General Demand

- Assume Poisson arrivals λ_t, discount rate r, period length h
- Price mechanism: Single posted price in each period; if there is excess demand, good is rationed randomly.

Theorem 3.
Suppose λ_t is Lipschitz continuous. Then lost profit from using posted prices and auction for final good in final period is $O(h)$.

(i) Cutoffs do not jump down by more than $O(h)$
Idea: If $t < T - h$, follows from continuity of λ_t.
For $t = T - h$, have $m(x_t^k) \approx 0$ for $k \geq 2$

(ii) Prices wrong because (1) don't adjust cutoffs within a period; and (2) may ration incorrectly.
But the prob. of 2 sales in one period is $O(h^2)$.

- Poisson arrivals important since imply common expectations

Decreasing Demand: Allocations

Poisson rate λ_t decreasing in t.

- Optimal cutoffs given by infinitesimal-period-look-ahead rule:

$$rm(x_t^k) = \lambda_t E_v\Big[\max\{m(v) - m(x_t^k), 0\} + \Pi_t^{k-1}\big(\min\{v, x_t^k\}\big) - \Pi_t^{k-1}(v)\Big]$$
$$m(x_T^k) = 0$$

where v is drawn from $F(\cdot)$

End game, $t \to T$

- If $k \geq 2$, then $x_t^k \to m^{-1}(0)$.
- If $k = 1$, then x_t^k jumps down to $m^{-1}(0)$

Decreasing Demand: Prices

Period T

- For $k = 1$, hold SPA with reserve $m^{-1}(0)$
- Final posted price

$$p_T = E_0 \left[\max\{y^2, m^{-1}(0)\} \middle| y^1 = \lim_{h \to 0} x^1_{T-h}, \{s_T(x)\}_{x \leq y^1} \right]$$

where $s_T(x)$ is last time the cutoff went below x.

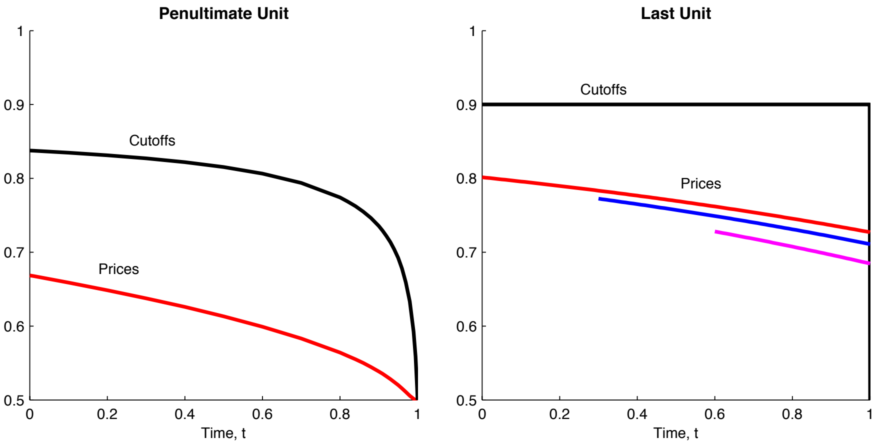
- For $k \geq 2$, $p_t \to m^{-1}(0)$ as $t \to T$.

For $t < T$, prices determined by

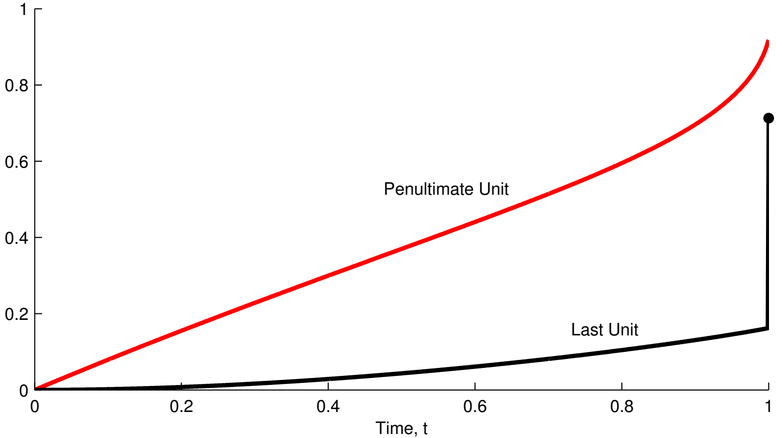
$$\dot{p}^k_t = \left[\dot{x}^k_t \left(\int_{s_t(x^k_t)}^t \lambda_s \, ds \right) f(x^k_t) - \lambda_t (1 - F(x^k_t)) \right] \left[x^k_t - p^k_t - U^{k-1}_t(x^k_t) \right] - r \left(x^k_t - p^k_t \right)$$

- If other units purchased earlier, p^k_t is higher.
- Price falls over time but jumps with every sale.

Price and Cutoffs with Two Units



Probability of Sale



Forward-Looking vs. Myopic Buyers

Myopic Buyers

- Buyers buy when enter, or leave forever
- Cutoffs $m(x_t^k) = \delta(V_{t+1}^k - V_{t+1}^{k-1})$, where V_t^k is value in (k, t).
- Implement with prices equal to cutoff.

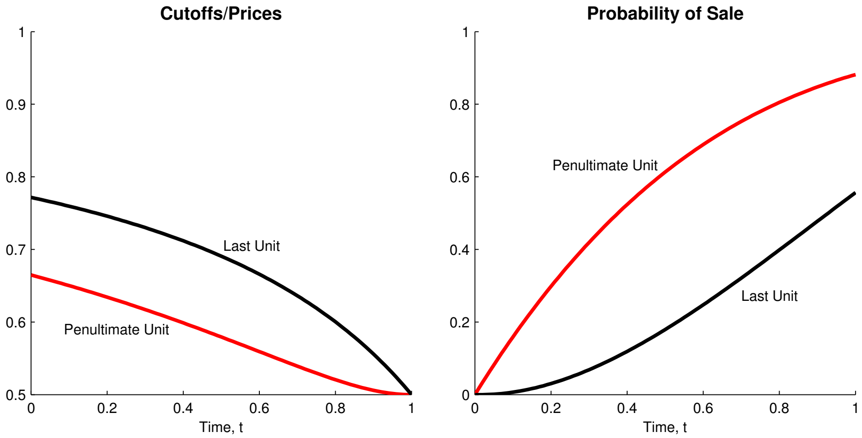
Under forward-looking buyers

- Profits higher
- Total sales higher
- Sales later in season

Retailing data suggest forward-looking buyers

- Price reductions lead to large numbers of sales
- Burst of sales quickly dies down
- Prices fall rapidly near the end of season

Cutoffs, Prices and Sales with Myopic Buyers



Applications

Retail Markets - Inventory Costs

- Inventory cost c_t if good held until time t.
- Assume marginal cost $\Delta c_t = c_{t+1} - c_t$ is increasing in t.

Cutoffs are deterministic and decreasing over time.

- For $t = T$, $m(x_T^k) = -\Delta c_T$. For $t < T$,

$$m(x_t^k) + E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\mathbf{v_{t+1}})\right] = E_{t+1}\left[\max\{m(x_t^k), m(v_{t+1}^1)\}\right] + E_{t+1}\left[\tilde{\Pi}_{t+1}^{k-1}(\{x_t^k, \mathbf{v_{t+1}}\}_k^2)\right] - \Delta c_t$$

- In continuous time,

$$\dot{c}_t = \lambda_t E\left[\max\{m(v) - m(x_t^k), 0\} + \Pi_t^{k-1}\big(\min\{v, x_t^k\}\big) - \Pi_t^{k-1}(v)\right]$$

$$\dot{p}_t^k = \left[\dot{x}_t^k\left(\int_{s_t(x_t^k)}^t \lambda_s ds\right) f(x_t^k) - \lambda_t(1 - F(x_t^k))\right]\left[x_t^k - p_t^k - U_t^{k-1}(x_t^k)\right]$$

Display Ads - Price Discrimination

- Rich media ad buyers have values $v \sim f_R$
- Static ad buyers have values $v \sim f_S$

Solving the problem

- Letting $m_i \in \{m_R, m_S\}$, the seller maximizes

$$\text{Profit} = E_0\Big[\sum_i \delta^{\tau_i} m_i(v_i)\Big]$$

- State variable is now k highest marginal revenues
- Cutoffs are deterministic in marginal revenue space

Implementation

- Use two price schedules for two types of buyer
- If rich media buyers have higher values, their marginal revenues are lower and prices are higher.

Airlines - Changing Distributions

- Demand f_t gets stronger over time
- Seller maximizes

$$E_0\Big[\sum_i \delta^{\tau_i} m_{t_i}(v_i)\Big]$$

Optimal discriminations

- If t_i observed, have cohort specific cutoffs/prices.
- Bias towards earlier cohorts.
- This is (IC) if t_i not observed.
- e.g. If $f_t \sim \exp(\mu_t)$, then issue coupon of μ_t for cohort t.

Selling a House - Disappearing Buyers

- Buyers exit the game with probability $\in (0, 1)$.
- Now need to carry around all past entrants as state

Cutoffs no longer deterministic

- If delay buyer y^1 may disappear, so value of y^2 matters
- Prices no longer optimal
- Explanation for indicative bidding in real estate

Also have problem if

- Buyers have different discount rates
- Mix of myopic and forward-looking buyers
- General problem: ranking of buyer's values changes

Conclusion

Optimal cutoffs

- Deterministic (only depend on k and t).
- Characterised by one-period-look-ahead rule.

Implemented by posted prices

- Sequence of prices with auction at time T.
- Prices depend on when sold previous units.

Extensions

- N_t correlated (e.g. learning)
- Different quality of ad slots
- Cost of paying attention to prices.