Economics 2102: Final

10 December, 2006

Time limit: 24 hours

1. Auctions with Correlated Values

A seller wants to sell a good to one of two symmetric buyers. Buyer i gains utility $v_i x_i - t_i$, where v_i is his valuation, x_i is the probability he gets the good and t_i is his payment to the seller. The seller wishes to maximise expected payments.

A seller designs a mechanism $(x_i(v_1, v_2), t_i(v_1, v_2))$, $i \in \{1, 2\}$, where the allocation probability and payments are a function of the agents' reports. The mechanism must allocate the good to the highest valuation buyer if valuations are different, and to each buyer with probability 1/2 if the valuations are the same. We consider only symmetric mechanisms, where payments depend on the agents' reports and not their identities. Denote $t_{ab} := t_1(v_a, v_b) = t_2(v_b, v_a)$.

Each buyer has one of two valuations, v_l or v_h, where $v_h > v_l$. The probability that the agents have valuations a, b is given by p_{ab}, where $a, b \in \{l, h\}$. We assume $p_{hh}p_{ll} > p_{hl}^2$, so valuations are positively correlated.

(a) The seller wants to design an ex–post individually rational (EPIR) and ex–post incentive compatible (EPIC) mechanism to maximise their expected revenue.[1] Determine the optimal transfers and the expected utility of a high and low type.

(b) The seller now drops the EPIR and EPIC requirement. The mechanism only has to be interim individually rational (IR) and interim incentive compatible (IC). Show that the seller can fully extract from the buyers. [Hint: Choose $t_{hh} = v_h/2$ and $t_{hl} = v_h$.] Intuitively, why can the seller fully extract the buyers' rent?

(c) The seller is concerned the buyers may collude. Suppose that if the buyers collude, they choose a pair of reports that minimises the sum of the transfers they pay. Show that if the buyers collude in the mechanism from part (a), they pay a total of v_l. Show that if the buyers collude in the mechanism from part (b), they pay less than v_l.

(d) Show that any (IR) and (IC) mechanism where buyers pay at least v_l by colluding, gives them at least as much rent as the mechanism from part (a).

[1]That is, every type should be happy to participate and reveal their type truthfully after knowing their opponent's type.

2. Supermodularity

An agent facing strictly positive prices (p_1, p_2) consumes two goods $(x_1, x_2) \in \mathbb{R}^2_+$. Her utility quasi–linear,

$$u(x_1, x_2) - p_1 x_1 - p_2 x_2$$

Assume $u(\cdot, \cdot)$ is continuous, so that an optimal choice exists. We wish to examine how a change in the price of good 1 affects demands for the two goods. Let

$$x_1^*(p_1, x_2) = \sup\left\{\text{argmax}_{x_1} u(x_1, x_2) - p_1 x_1 - p_2 x_2\right\} \tag{1}$$

be the largest solution to the consumers x_1–problem, taking x_2 as fixed. Similarly, let

$$x_2^*(p_1) = \sup\left\{\text{argmax}_{x_2} u(x_1^*(p_1, x_2), x_2) - p_1 x_1^*(p_1, x_2) - p_2 x_2\right\} \tag{2}$$

be the largest solution to the consumer's x_2–problem.

(a) Suppose $u(x_1, x_2)$ is supermodular and let $p_1' \geq p_1$. Show that

$$x_1^*(p_1, x_2(p_1)) \geq x_1^*(p_1', x_2(p_1)) \geq x_1^*(p_1', x_2(p_1'))$$

(b) Suppose that $u(x_1, x_2)$ is submodular and let $p_1' \geq p_1$. Show that, once again,

$$x_1^*(p_1, x_2(p_1)) \geq x_1^*(p_1', x_2(p_1)) \geq x_1^*(p_1', x_2(p_1'))$$

(c) Consider N goods $(y_1, \ldots, y_N)$, which we divide into arbitrary sets x_1 and x_2. Can we generalise the results in (a) and (b)? [Note: proofs are not required for this part of the question].

3. Theory of A Market Maker

Suppose a risk–neutral agent wishes to trade one unit of a share with a risk–neutral intermediary. That is, the agent can buy one share, sell one share, or choose not to trade. All parties start with a common prior on the value of the share, $\theta \sim g(\theta)$. The game is as follows.

1. The intermediary sets bid price B and ask prices A. Assume the market for intermediaries is competitive, so they make zero profits on each trade.
2. With probability $1 - \alpha \in (0, 1)$ the agent is irrational, buying one share at price A and selling one share at price B.[2] With probability α the agent is rational. In this case, the agent receives a signal $s \in [\underline{s}, \overline{s}]$ with nondegenerate distribution $f(s|\theta)$, and chooses to buy at A or sell at B. Assume $f(s|\theta)$ obeys the MLRP.
3. The value of the share, θ, is revealed. The agent and intermediary receive their payoffs. The rational agent's payoffs are as follows: if he buys, he receives $\theta - A$; if he sells he receives $B - \theta$; and if he does not trade he receives 0. The intermediaries payoffs are the opposite.

(a) Fix prices (A, B). For which signals will the rational agent trade?

(b) Given the zero profit condition for the intermediary, how are equilibrium prices (A, B) determined?

(c) Show that, in equilibrium, $A \geq E[\theta] \geq B$. Show that some rational agents will not trade.

(d) Suppose α increases. Show how this affects (a) the equilibrium prices, and (b) the proportion of rational agents trading.

(e) What happens as $\alpha \to 1$?

[2] This is rather unrealistic, but it makes the maths easier.

4. Team problem

Two agents, $i \in \{1, 2\}$, simultaneously choose effort $e_i \in \{0, 1\}$ on a project. Exerting effort costs costs c_i, where $c_1 + c_2 < 1$ and $1 - x > c_i$. The output produced is given by

		Agent 2: 1	Agent 2: 0
Agent 1	1	1	x
	0	x	0

Fix (c_1, c_2). Suppose agent i gets share β_i of the output, where $\beta_1 + \beta_2 \leq 1$. We say the efficient outcome can be *implemented* if there exists an equilibrium where both agents exert high effort.

(a) For which values of x do there exist (β_1, β_2) such that the efficient outcome can be implemented?

(b) Show there exist sharing rules (β_1, β_2) which only depend on (c_1, c_2) and implement the efficient outcome whenever it is implementable.

We say the efficient outcome can be *implemented in dominant strategies* if the high effort choice is a dominant strategy.

(c) For which values of x do there exist (β_1, β_2) such that the efficient outcome can be implemented in dominant strategies?

(d) Show there exist sharing rules (β_1, β_2) which only depend on (c_1, c_2) and implement the efficient outcome in dominant strategies whenever it is implementable.

5. Debt Contracts

An entrepreneur has access to a project requiring one unit of capital. If taken, the project succeeds with probability p and produces output $R(p)$, or fails with probability $1-p$ and produces 0. The entrepreneur can costlessly choose $p \in [0,1]$. This choice is unobservable to investors.

The entrepreneur is risk neutral and has initial wealth $w \in [0,1]$. The entrepreneur must raise the additional capital by issuing debt to perfectly competitive risk neutral investors.[3] This debt is secured only by the assets of the project. Both the investors and the entrepreneur have available a safe investment paying an interest rate 0 if they do not invest.

(a) For $w \in [0,1]$, determine the equation that defines the equilibrium relationship between w and p. (Assume an interior solution for p).

(b) Let $R(p) = 5 - 4p$. If $w = 1$, what value of p would the entrepreneur choose? If instead, $w \in (\frac{7}{32}, 1)$, show there are 2 possible equilibrium choices for p. Which of these solutions is more reasonable? What happens if $w < \frac{7}{32}$?

(c) Let $R(p) = 5 - 4p$. Plot the entrepreneur's expected final wealth as a function of initial wealth $w \in [0,1]$. Discuss the effect of agency costs on the return to wealth.

[3] A debt contract states that the first D dollars from the project goes to the investors.