Contract Theory: Final

9:00am–12:00pm, 9th December, 2013

1. Financing of Investments

An entrepreneur (E) has assets $\theta \sim G[\underline{\theta}, \overline{\theta}]$ that it privately knows. It also has an opportunity to make a further investment that yields random returns v that are independent of θ. In order to make this investment, E needs an investor (I) to inject funds. The two parties can only contract on the total output of the firm, $y = \theta + v$. Suppose $y \sim f(\cdot|\theta)$ obeys MLRP.

We suppose I offers E a menu of contracts. Using the revelation principle, we can denote the contracts $\{t(y, \tilde{\theta})\}$, where $\tilde{\theta}$ is the agent's report. We consider contracts $t \in [0, y]$ that are weakly increasing in y. If E does not invest, he makes $U^{NI} = \theta$; if E does invest he makes $U^I = E[y - t(y, \tilde{\theta})|\theta]$ and I makes $\pi = E[t(y, \tilde{\theta})]$.

(a) Show that any mechanism $\{t(y, \tilde{\theta})\}$ induces investment from some decreasing set, $[\underline{\theta}, \hat{\theta}]$.

Suppose I wishes to implement a cutoff $\hat{\theta}$.

(b) Argue that I can limit herself to contracts $\{t(y)\}$ that are independent of E's report

(c) A debt contract is defined by $t^D(y) = \min\{y, D\}$. Show that a debt contract is the cheapest way to implement $\hat{\theta}$. That is, pick any arbitrary contract $\{t(y)\}$ such that the same types invest, and show that moving to a debt contract raises expected payments to I.

[Useful fact: A function $\phi(y)$ is single-crossing in y if $\phi(y) \leq 0$ for $y < y^*$ and $\phi(y) \geq 0$ for $y > y^*$. If $y \sim f(\cdot|\theta)$ obeys MLRP and $\phi(y)$ is single-crossing in y, then $E[\phi(y)|\theta]$ is single-crossing in θ.]

2. Disaster Prevention

A firm employs a single agent. Time is continuous $t \in [0, \infty)$. Both players are risk neutral. The agent has discount rate ρ, while the firm has discount rate r. Assume $\rho > r$.

The agent exerts private effort to avert a disaster (e.g. an oil spill, or a product explosion). The agent chooses effort $a_t \in \{0, 1\}$ corresponding to low/high effort. Under high effort, a disaster arrives according to Poisson rate λ; under low effort a disaster arrives at Poisson rate $\lambda + \delta$. If the agent chooses low effort, he also gains flow benefit b. Let N_t be the number of disasters that have occurred by time t. The agent's flow wage is $w_t \geq 0$. The firm earns μ flow profits but loses K every time there is a disaster. If the firm fires the agent and liquidates the project then both parties get zero thereafter.

The agent's wealth at time t is thus

$$V_t = E_t\left[\int_t^\tau e^{-\rho(s-t)}[w_s + (1-a_s)b]ds\right]$$

where τ is the (random) liquidation time. The firm's time t profits are

$$\Pi_t = E_t\left[\int_t^\tau e^{-r(s-t)}[\mu ds - w_s ds - K dN_s]\right]$$

The firm chooses to a contract $\{w_t, \tau\}$ to maximize her time-0 profits. Assume that δ is sufficiently large that she wishes to implement high effort, $a_t = 1$, at all times.

(a) Observe that a contract can be described recursively using V as a state variable. Write the agent's HJB equation and use it to derive a condition for high effort to be (IC). When evaluating how the agent's wealth changes over time, you will find it useful to separate between the increase if there is no disaster, denoted $\dot{V}$, and the decrease if there is a disaster, denoted H.

(b) Let the firm's profit function be denoted $\Pi(V)$. Write the firm's HJB assuming $a_t = 1$ (and ignoring (IC)).

(c) Assume $\Pi(V)$ is concave. Using the agent's HJB to substitute for $\dot{V}$ in the firm's HJB, argue that there is a V^* such that the the firm only pays the agent when his wealth exceeds this threshold.

(d) Using (IC), what is the optimal punishment H?

(e) When is the project shut down?

3. Naïve Price Discrimination.

Agents have values $v \sim F$, where the lowest value is zero. Assume the marginal revenue curve is strictly increasing and linear, i.e.

$$MR(v) := v - \frac{1 - F(v)}{f(v)} = av - b$$

Agents' utility is given by

$$u = vq - p$$

where q is quality and p is price.

(a) Give two examples of distributions of values that satisfy the linearity assumption.

(b) Suppose the firm offers a single good of quality q at cost $c(q)$ (e.g. VW is selling a Golf). What is the optimal price?

(c) Suppose the firm can choose to sell any good of quality $q \in [0, \infty]$ and cost $c(q)$, where $c(\cdot)$ is increasing and convex with $c(0) = 0$ and $\lim_{q \to \infty} c'(q) = \infty$ (e.g. VW is selling a Golf, Jetta, Passat etc.). Show the optimal prices are the same as in part (b).