Contract Theory: Final

10:00am–1:00pm, 15th December, 2014

1. Teamwork and Tournaments

Two agents work in a team. They simultaneously choose effort e_i at cost $c(e_i)$, yielding output $y(e_1+e_2)$. Utility is linear, so an agent's utility equals the share of the output they receive minus the cost of effort. Assume c is differentiable, strictly convex with $c'(0) = 0$ and $\lim_{e\to\infty} c'(e) = \infty$. Assume y is differentiable and concave.

(a) Write down the FOC for the first-best effort, e^*

Now, suppose there is a measure of which agent produces the most output. Given (e_1, e_2), agent 1 wins with probability $p(e_1, e_2)$ and agent 2 wins with $1-p(e_1, e_2)$. Suppose we give the winner share s of the output, while the loser gains share $1 - s$. Assume p is increasing in e_1, differentiable and symmetric in (e_1, e_2), so that $p(e_1, e_2) = 1 - p(e_2, e_1)$

(b) Write down the FOC for the agent's problem. Derive an expression for s in order to implement the first-best.

(c) Suppose $y = \alpha(e_1 + e_2)$, $c(e) = e^2/2$ and $p = \frac{e_1^r}{e_1^r+e_2^r}$, where $r \geq 1$ reflects the responsiveness of the signal to effort. Which shares $(s, 1 - s)$ implement first-best?

2. Allocation via Queue

A pro-bono law firm has two lawyers, A and B. There are an infinite number of clients waiting. Each client is a good match with one partner and a bad match with the other; if the match is good (bad) the client receives value v_H (v_L). The client prefers A with probability 1/2; a client's preference is private information. Clients are also impatient, so a client who receives value v at time t obtains utility ve^{-rt}, where r is the interest rate.

It takes time for a lawyer to solve the problem of a client. The amount of time is Poisson distributed with arrival rate λ. Ideally we would like to allocate clients to the lawyer that matches their needs, but clients preferences are private information.

(a) Suppose lawyers are allocated via a queue. That is, we line up the clients according to, say, their name. Each time a lawyer becomes available, we approach clients in order. If a client accepts, he sees the lawyer; if he rejects, then he retains his place in the queue. Show that a client in n^{th} position will tell the truth about their type iff

$$v_H \left(\frac{\lambda}{r+\lambda} \right)^n \geq v_L.$$

[Hint: You may find it useful to note that the arrival time $t \geq 0$ of the nth arrival from a Poisson arrival process obeys the Erlang distribution, $f_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$.]

(b) Suppose we instead use a "preferred group" allocation system. Suppose there are k clients in the "preferred group". Each time a lawyer becomes available, we ask clients in the preferred group in random order. If one of the clients accepts, we bring a new client into the group. If none of the clients accept, we allocate the lawyer to someone outside the group. Show that truth-telling is an ex-post equilibrium for clients in the preferred group if k satisfies

$$v_H \frac{1}{k} \left(\frac{\lambda}{r+\lambda} \right) \geq v_L$$

(c) Assume $r \geq \lambda$. Which system is better?

3. Auctions with Endogenous Quantity

There are N agents bidding for a procurement contract. The agents have constant marginal costs that are distributed iid, $c_i \sim f[0,1]$, where the hazard rate $f(c_i)/F(c_i)$ is decreasing in c_i. The principal has value function $V(q)$ for quantity q. Assume V is differentiable and concave, with $\lim_{q\to\infty} V'(q) = 0$.

Consider the mechanism design problem. An agent reports cost $\tilde{c}_i$, is paid $t_i(\tilde{c}_i, \tilde{c}_{-i})$ and is allocated quantity $q_i(\tilde{c}_i, \tilde{c}_{-i})$. It is assumed that only one agent can win the contract, so $q_i = 0$ for all but one agent. The winning agent obtains utility $u_i = t_i - q_i c_i$, while losing agents receive $u_j = t_j$. The principal obtains profit $\pi = V(q_i) - \sum_j t_j$. Aside: we are assuming the mechanism is deterministic; this is without loss here.

(a) What is the optimal mechanism? Suppose $c_i \sim U[0,1]$ and $V(q) = q - q^2/2$ for $q \in [0,1]$. What is the optimal allocation function?

For the rest of this question we suppose the agents compete via an auction and the principal then chooses the quantity q afterwards so that $V'(q) = p$, where p is the price from the auction.

(b) First, lets consider a first-price auction. As a benchmark, suppose agents bid as if quantity q is fixed (say $q = 1$). Characterize agents' symmetric bidding function, $\beta(c_i)$. What is the bidding function when $c_i \sim U[0, 1]$ and $N = 2$? Fixing the bids, suppose $V(q) = q - q^2/2$ for $q \in [0, 1]$, and the quantity is determined so that $V'(q) = \beta_{(1)}$, where $\beta_{(1)}$ is the lowest bid. How does the allocation function differ from the optimal allocation in part (a)?

For the rest of the question, suppose when the agents bid they take into account the fact the demand function $q(p) = (V')^{-1}(p)$ is downward sloping.

(c) Considering a FPA, write down the FOC for the agent's bidding function, $b(c)$. Argue that the agent bids more aggressively than in part (b). That is, $b(c) \leq \beta(c)$. [Hint: one can do this by comparing the FOCs, or by integrating the bidding function up.]

(d) How would agents bid in a SPA? Using part (c), argue that the FPA yields lower expected prices than the SPA.