Contract Theory: Final

8:00am–11:00am, 6th December, 2015

1. Investing in a Project

A firm hires an agent to work on a project. Time is continuous $t \in [0, \infty)$. While the project is operational, the firm gives the agent c to invest in the project at each moment in time. The agent can choose to invest the money, $a_t = 1$, leading to instantaneous probability of breakthrough λa_t. Or, he can steal the money, $a_t = 0$, gaining $\phi(1 - a_t)$. To ensure investment is efficient, assume $\lambda B > c > \phi$.

The firm can commit to a contract. The agent has limited liability, is risk-neutral and there is no discounting. We restrict ourselves to contracts where (i) the project is operational for some interval $[0, T]$, and the agent exerts effort the entire time, $a_t = 1$ for $t \leq T$, and (ii) the agent is paid a lump-sum $R_\tau \geq 0$ if the breakthrough occurs at time τ.

Suppose the agent has yet to obtain a breakthrough. Denote his continuation utility by

$$V_t = E_\tau \left[R_\tau \mathbf{1}_{\tau \leq T} + \int_t^{\min\{\tau, T\}} \phi(1 - a_s) ds \right]$$

and the firm's profit by

$$\Pi_t = E_\tau[(B - R_\tau)\mathbf{1}_{\tau \leq T} - c \min\{\tau, T\}]$$

where B is the benefit of the breakthrough.

(a) Write down the differential equation for the agent's value V_t in terms of R_{t+dt} and V_{t+dt}. Use it to show that effort, $a = 1$, is incentive compatible if

$$\lambda(R_t - V_t) \geq \phi$$

(b) Assume $t < T$. Argue that that (IC) binds in the profit-maximizing contract, and show that $\dot{V}_t = -\phi$. Derive expressions for V_t and R_t as a function of T.

(c) Write down profits as a function of T. Derive the profit-maximizing T.

(d) Throughout this question we assumed the firm pays the agent a lump-sum R_t when a breakthrough occurs. Is this without loss? (This argument can be heuristic)

2. Implementation without Single-Crossing

Two agents $i \in \{1,2\}$ compete for one good. Each agent has type $\theta \in \{\theta_h, \theta_l\}$, where $\theta_h > \theta_l > 0$. Suppose $\Pr(\theta = \theta_h) = 1/2$ for both agents. As in a standard auction, a mechanism $\langle p_i(\hat{\theta}_i, \hat{\theta}_j), t_i(\hat{\theta}_i, \hat{\theta}_j)\rangle$ maps the reports of the agents into (i) a probability of allocating the good to agent i and (ii) a transfer from agent i to the principal.

Suppose agent i has payoffs

$$u_i(\theta_i, \theta_j | \hat{\theta}_i, \hat{\theta}_j) = v_i(\theta_i, \theta_j) p_i(\hat{\theta}_i, \hat{\theta}_j) - t_i(\hat{\theta}_i, \hat{\theta}_j) = (\theta_i + 2\theta_j) p_i(\hat{\theta}_i, \hat{\theta}_j) - t_i(\hat{\theta}_i, \hat{\theta}_j)$$

so the agent cares more about his opponent's type than his own.

(a) What is the (symmetric) efficient allocation? [Hereafter, when I write "efficient allocation", I mean the symmetric one].

(b) Suppose the types are independently distributed. Show there exist no mechanism that implements the efficient allocation as a Bayesian equilibrium. [Hint: show that (IC) cannot be satisfied].

For the rest of the question, suppose types are positively correlated. In particular, the correlation matrix is

	θ_l	θ_h
θ_l	3/4	1/4
θ_h	1/4	3/4

(c) Show that there exists a mechanism that implements the efficient allocation as a Bayesian equilibrium and also fully extracts the rents of the agents.

(d) Show there exist no mechanism that implements the efficient allocation as an ex-post equilibrium.

3. Ratchet Effect and Learning

A firm employs an agent over two periods, $t \in \{1,2\}$. Each period runs as follows. First, the principal offers the agent a wage function, $w_t(q_t)$. Second, the agent chooses private effort a_t at cost $c(a_t)$, where the cost function is convex and satisfies $c'(0) = 0$ and $\lim_{a\to\infty} c'(a) = \infty$.

Finally, the publicly observed output is realized and the wage is paid, where output is

$$q_t = a_t + \theta + \epsilon_t.$$

In the output equation, $\theta \sim N(\mu_0, 1/h_0)$ is an unknown state of the world, and $\epsilon_t \sim N(0, 1/h_\epsilon)$ is an IID shock.

The principal chooses a spot contract in each period to maximize its profits

$$\Pi_t = \sum_{s \geq t} [q_s - w(q_s)]$$

subject to the agent accepting the contract. The agent has outside option 0 in each period, and is risk neutral. Hence his utility is

$$V_t = \sum_{s \geq t} [w(q_s) - c(a_s)]$$

We solve the problem by backwards induction. It will be helpful to recall that if $z = \eta + \epsilon$, where $\eta \sim N(\mu_\eta, 1/h_\eta)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$, then

$$\eta | z \sim N\left(\frac{h_\epsilon z + h_\eta \mu_\eta}{h_\eta + h_\epsilon}, \frac{1}{h_\eta + h_\epsilon}\right)$$

(a) In period 2, suppose both the firm and agent believe that $\theta \sim N(\mu_1^*, 1/h_1)$. What is the optimal contract?

(b) Suppose the agent believes that $\theta \sim N(\mu_1, 1/h_1)$, while the firm still believes that $\theta \sim N(\mu_1^*, 1/h_1)$ (and thinks the agent has the same beliefs). What is the agent's expected utility at the start of period 2?

(c) Now, consider the first period. Suppose the firm wishes to implement a pure strategy effort $a_1^* > 0$. Write down the agent's period 1 lifetime utility. What is the agent's benefit from raising/lowering his effort a little? Argue that the principal cannot implement any pure strategy effort $a_1^* > 0$.