

Homework 2: Moral Hazard

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Question 1 (Hidden Savings I)

There are two periods. In period 1 the agent (privately) chooses to consume c . In period 2 they choose effort $a \in \{L, H\}$ at cost $g(a)$, where $g(H) > g(L)$. Output is binomial, $q \in \{0, 1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H > p_L$. The principal commits to the wage schedule at the start of the game. Wages are paid in period 2: denote the wage paid in state $q \in \{0, 1\}$ by (w_1, w_0) .

Suppose the agent's utility is given by

$$u(c_a) + p_a u(w_1 - c_a) + (1 - p_a)u(w_0 - c_a) - g(a)$$

where $u(\cdot)$ is increasing and strictly concave, and c_a is the consumption of the agent in period 1 if they plan to take action a in period 2.

Suppose the principal wishes to implement high effort. The two-period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H) + (1 - p_H)u(w_0 - c_H) - g(H) \\ \geq u(c_L) + p_L u(w_1 - c_L) + (1 - p_L)u(w_0 - c_L) - g(L) \end{aligned} \tag{1}$$

- (a) Show that $w_1 > w_0$ and $c_H > c_L$. [Note: there is an elegant proof and an ugly proof].
- (b) Use (1) to show that the second-period (IC) constraint (after c_H has been chosen) is slack.
- (c) Why does this matter?

Question 2 (Hidden savings II)

There are two periods. In period 1 the agent (privately) chooses to consume c . In period 2 they choose effort $a \in \{L, H\}$ at monetary cost $\{0, g\}$ respectively. Output is binomial, $q \in \{0, 1\}$,

where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H \geq p_L$. The principal chooses wages (w_1, w_0) .

The two-period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H - g(H)) + (1 - p_H) u(w_0 - c_H - g(H)) \\ \geq u(c_L) + p_L u(w_1 - c_L - g(L)) + (1 - p_L) u(w_0 - c_L - g(L)) \end{aligned} \quad (2)$$

where c_a is the optimal consumption when the agent plans to choose a .

Show that under CARA utility, $u(c) = -\exp(-rc)$, we have $c_H = c_L$ when the (IC) constraint binds. Why is this important?

Question 3 (Short-term and long-term contracts)

Suppose there are three periods, $t \in \{1, 2, 3\}$. Each period a principal and an agent must share a good; let $x_t \in \mathbb{R}$ be the share obtained by the agent. The principal gets $\sum_t \pi_t(x_t)$ and the agent gets $\sum_t u_t(x_t)$, where $\pi_t(x_t)$ is decreasing in x_t and $u_t(x_t)$ is increasing in x_t . The agents' outside option is a share of the assets $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

(a) Suppose the principal can write a long term contract. Write down the program of maximising profit subject to individual rationality.

(b) Now suppose the principal offered a spot contract each period. Using backwards induction derive the optimal sequence of spot contracts. Explain why this may differ from the long-term contract.

(c) Suppose the principal offers two-period contracts. In the first period they offer $({}_1x_1, {}_1x_2)$. If it is rejected the agent gets \underline{x}_1 . At the start of the second period a new contract $({}_2x_2, {}_2x_3)$ may be proposed by the principal. If this is rejected the agent gets ${}_1x_2$ if they accepted the first contract or \underline{x}_2 otherwise. In the third period a spot contract is offered to the agent. If this is rejected, the agent gets ${}_2x_3$ if they accepted the second contract, or \underline{x}_3 otherwise. Show that if $\lim_{x \rightarrow -\infty} u_t(x) = -\infty$ and $\lim_{x \rightarrow \infty} u_t(x) = \infty$ then this can implement the optimal long term contract.

(d) Provide an example (outside options, utility functions, profit function) where the two-period contracts cannot implement the long-term contract.

Question 4 (Normal learning model)

Suppose that $z_t = \theta + \epsilon_t$, where $\theta \sim N(m_0, 1/h_0)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$ are IID. Show that

$$\theta|z_1 \sim N\left(\frac{h_\epsilon z_1}{h_0 + h_\epsilon} + \frac{h_0 m_0}{h_0 + h_\epsilon}, \frac{1}{h_0 + h_\epsilon}\right)$$

Question 5 (Credible Wage Paths)

There are two periods, with no discounting. The firm proposes a contract (w_0, w_s) which the agent accepts if the sum of period 1 and period 2 utilities exceeds \bar{u} in expectation. Their utility function is given by the increasing, strictly concave function $u(\cdot)$.

In the first period the worker gets paid w_0 (if they accept the contract). They then produce q for the firm.

In the second period, the state of the world $s \in S$ is realised with probability f_s . The firm offers w_s , while there is an outside offer, \bar{w}_s . The worker accepts the larger. If they work for the firm, the worker produces $q > \max_s \bar{w}_s$.

- The firm's problem is to maximise two-period profits subject to the first-period and second-period (IR) constraints. Write down this problem.
- Characterise the optimal wage path. If s is the state of the economy, how are wages affected by slumps and booms?
- Suppose the agent can commit to his period 2 behaviour in period 1. Describe the optimal contract.

Question 6 (Relational Contracting)

Suppose a firm employs two workers. It signs a stationary relational contract (w^i, b^i, e^i) with each worker i . The firm gets profit $y(e^i) - W^i$ from each worker, while the agents get $W^i - c^i(e^i)$, where $W^i = w^i + b^i$. Outside utility/profits equal 0.

First, consider a bilateral contract, where deviation by the firm or agent in relationship i leads to Nash reversion in this relationship only.

(a) Characterise the self-enforcing contracts by no deviation constraints on both agents and the principal.

(b) Sum across the constraints to derive conditions on surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self-enforcing.]

Second, consider a joint contract where deviation by the firm or any worker leads all workers to revert to noncooperation.

(c) Characterise the self-enforcing contracts by no deviation constraints on both agents and the principal.

(d) Sum across the constraints to derive a condition of surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self-enforcing.]

(e) Show that the total surplus is higher under the joint contract than under bilateral contracts. Intuitively, when is the joint contract strictly better? In this case, why is it better?