Homework 2: Dynamic Moral Hazard

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Question 1 (Hidden Savings I)

There are two periods. In period 1 the agent (privately) chooses to consume c. In period 2 they choose effort $a \in \{L, H\}$ at cost $g(a)$, where $g(H) > g(L)$. Output is binomial, $q \in \{0, 1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H > p_L$. The principal commits to the wage schedule at the start of the game. Wages are paid in period 2: denote the wage paid in state $q \in \{0, 1\}$ by (w_1, w_0).

Suppose the agent's utility is given by

$$u(c_a) + p_a u(w_1 - c_a) + (1 - p_a)u(w_0 - c_a) - g(a)$$

where $u(\cdot)$ is increasing and strictly concave, and c_a is the consumption of the agent in period 1 if they plan to take action a in period 2.

Suppose the principal wishes to implement high effort. The two–period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H) + (1 - p_H)u(w_0 - c_H) - g(H) \\ \geq u(c_L) + p_L u(w_1 - c_L) + (1 - p_L)u(w_0 - c_L) - g(L) \end{aligned} \tag{1}$$

(a) Show that $w_1 > w_0$ and $c_H > c_L$. [Note: there is an elegant proof and an ugly proof].

(b) Use (1) to show that the second–period (IC) constraint (after c_H has been chosen) is slack.

(c) Why does this matter?

Question 2 (Hidden savings II)

There are two periods. In period 1 the agent (privately) chooses to consume c. In period 2 he chooses effort $a \in \{L, H\}$ at monetary cost $g(a)$, where $g(H) > g(L)$. Output is binomial,

$q \in \{0,1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H \geq p_L$. The principal chooses wages (w_1, w_0).

The two–period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H - g(H)) + (1 - p_H) u(w_0 - c_H - g(H)) \\ \geq u(c_L) + p_L u(w_1 - c_L - g(L)) + (1 - p_L) u(w_0 - c_L - g(L)) \end{aligned} \tag{2}$$

where c_a is the optimal consumption when the agent plans to choose a.

Show that under CARA utility, $u(c) = -\exp(-rc)$, we have $c_H = c_L$ when the (IC) constraint binds. Why is this important?

Question 3 (Short–term and long–term contracts)

Suppose there are three periods, $t \in \{1, 2, 3\}$. Each period a principal and an agent must share a good; let $x_t \in \mathbb{R}$ be the share obtained by the agent. The principal gets $\sum_t \pi_t(x_t)$ and the agent gets $\sum_t u_t(x_t)$, where $\pi_t(x_t)$ is decreasing in x_t and $u_t(x_t)$ is increasing in x_t. The agent's outside option is a share of the assets $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

(a) Suppose the principal can write a long term contract. Write down the program of maximising profit subject to individual rationality.

(b) Now suppose the principal offered a spot contract each period. Using backwards induction derive the optimal sequence of spot contracts. Explain why this may differ from the long–term contract.

(c) Suppose the principal offers two–period contracts. In the first period they offer $({}_1x_1,\ {}_1x_2)$. If it is rejected the agent gets $\underline{x}_1$. At the start of the second period a new contract $({}_2x_2,\ {}_2x_3)$ may be proposed by the principal. If this is rejected the agent gets ${}_1x_2$ if they accepted the first contract or $\underline{x}_2$ otherwise. In the third period a spot contract is offered to the agent. If this is rejected, the agent gets ${}_2x_3$ if they accepted the second contract, or $\underline{x}_3$ otherwise. Show that if $\lim_{x \to -\infty} u_t(x) = -\infty$ and $\lim_{x \to \infty} u_t(x) = \infty$ then this can implement the optimal long term contract.

(d) Provide an example (outside options, utility functions, profit function) where the two–period contracts cannot implement the long–term contract.

Question 4 (Normal learning model)

Suppose that $z_t = \theta + \epsilon_t$, where $\theta \sim N(m_0, 1/h_0)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$ are IID. Show that

$$\theta | z_1 \sim N\left(\frac{h_\epsilon z_1}{h_0 + h_\epsilon} + \frac{h_0 m_0}{h_0 + h_\epsilon}, \frac{1}{h_0 + h_\epsilon}\right)$$

Question 5 (Credible Wage Paths)

There are two periods, with no discounting. The firm proposes a contract (w_0, w_s) which the agent accepts if the sum of period 1 and period 2 utilities exceeds $\overline{u}$ in expectation. Their utility function is given by the increasing, strictly concave function $u(\cdot)$.

In the first period the worker gets paid w_0 (if they accept the contract). They then produce q for the firm.

In the second period, the state of the world $s \in S$ is the realised with probability f_s. The firm offers w_s, while there is an outside offer, $\overline{w}_s$. The worker accepts the larger. If they work for the firm, the worker produces $q > \max_s \overline{w}_s$.

(a) The firms problem is to maximise two–period profits subject to the first–period and second–period (IR) constraints. Write down this problem.

(b) Characterise the optimal wage path. If s is the state of the economy, how are wage affected by slumps and booms?

(c) Suppose the agent can commit to his period 2 behaviour in period 1. Describe the optimal contract.

Question 6 (Relational Contracting)

Suppose a firm employs two workers. It signs a stationary relational contract (w^i, b^i, e^i) with each worker i. The firm gets profit $y(e^i) - W^i$ from each worker, while the agents get $W^i - c^i(e^i)$, where $W^i = w^i + b^i$. Outside utility/profits equal 0.

First, consider a bilateral contract, where deviation by the firm or agent in relationship i leads to Nash reversion in this relationship only.

(a) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(b) Sum across the constraints to derive conditions on surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

Second, consider a joint contract where deviation by the firm or any worker leads all workers to revert to noncooperation.

(c) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(d) Sum across the constraints to derive a condition of surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

(e) Show that the total surplus is higher under the joint contract than under bilateral contracts. Intuitively, when is the joint contract strictly better? In this case, why is it better?