Economics 211A: Homework 3

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Question 1 (Nonlinear Pricing with Three Types)

Consider the nonlinear pricing model with three types, $\theta_3 > \theta_2 > \theta_1$. The utility of agent θ_i is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent θ_i by (q_i, t_i). We now have six (IC) constraint and three (IR) constraints. For example, (IC_1^2) says that θ_1 must not want to copy θ_2, i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \qquad (\mathrm{IC}_1^2)$$

The firm's profit is

$$\sum_{i=1}^{3} \pi_i [t_i - c(q_i)]$$

where π_i is the proportion of type θ_i agents and $c(q)$ is increasing and convex.

(a) Show that (IR_2) and (IR_3) can be ignored.
(b) Show that $q_3 \geq q_2 \geq q_1$.
(c) Using (IC_2^1) and (IC_3^2) show that we can ignore (IC_3^1). Using (IC_2^3) and (IC_1^2) show that we can ignore (IC_1^3).
(d) Show that (IR_1) will bind.
(e) Show that (IC_2^1) will bind.
(f) Show that (IC_3^2) will bind.
(g) Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC_1^2) and (IC_2^3) can be ignored.

Question 2 (Downward Sloping Demand I)

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2} q^2 - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi) t_2 \tag{1}$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit–maximising qualities.

Question 3 (Downward Sloping Demand II)

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi) t_2 \tag{2}$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities and prices.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit–maximising qualities.

Question 4 (Optimal Taxation)

There are two types of agents, $\theta_H > \theta_L$. Proportion β have productivity θ_L. An agent of type θ who exerts effort e produces output $q = \theta e$. The utility of an agent who produces quantity q with effort e is then

$$u(q - t - g(e))$$

where t is the net tax. Assume $g(e)$ is increasing and strictly convex, and $u(\cdot)$ is strictly concave.

Suppose that output is contractible so that a mechanism consists of a pair $(q(\theta), t(\theta))$. The government's problem is to maximise

$$\beta u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) + (1-\beta)u\left(q_H - t_H - g\left(\frac{q_H}{\theta_H}\right)\right)$$

subject to budget balance (BB), $\beta t_L + (1-\beta)t_H \geq 0$. Notice that there are no (IR) constraints here.

(a) First, suppose the government can observe agents' types. Solve for the first–best contract. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type L, for example, is

$$u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) \geq u\left(q_H - t_H - g\left(\frac{q_H}{\theta_L}\right)\right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum $u'_L \geq u'_H$, where u'_i is the marginal utility of type i.

(d) Show that at the optimum (IC_H) binds.

(e) Consider the government's relaxed problem of maximising welfare subject to (BB) and (IC_H), ignoring (IC_L). Show the optimal contract satisfies:

$$1 - \frac{1}{\theta_H} g'\left(\frac{q_H}{\theta_H}\right) = 0 \tag{3}$$

$$1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) = \frac{u'_L - u'_H}{u'_L}(1-\beta)\left(1 - \frac{1}{\theta_H} g'\left(\frac{q_L}{\theta_H}\right)\right) \tag{4}$$

(f) Show that (4) implies

$$1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) \geq 0 \tag{5}$$

(g) Using equations (3) and (5) show that $q_H \geq q_L$. Use this and the fact that (IC_H) binds, to show that (IC_L) holds.

(h) What does (5) imply about the level of work performed by the low type. Provide an intuition for this distortion.

Question 5 (Costly State Verification)

There is a risk–neutral entrepreneur E who has a project with privately observed return y with density $f(y)$ on $[0, Y]$. The project requires investment $I < E[y]$ from an outside creditor C.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports y they pay $s(y) \leq y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies E they pay cost $c(y)$ and get to observe E's type.

The game is as follows:

- E chooses $(s(y), B(y))$ to raise I from a competitive financial market.
- Output y is realised.
- E claims the project yields $\hat{y}$. If $B(\hat{y}) = 0$ then E pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then C pays $c(y)$ and observes E's true type. If they are telling the truth they pay $s(y)$; if not, then C can take everything.
- Payoffs. E gets $y - s(y)$, while C gets $s(y) - c(y)B(y) - I$.

(a) Show that a contract is incentive compatible if and only if there exists a D such that $s(y) = D$ when $B(y) = 0$ and $s(y) \leq D$ when $B(y) = 1$.

Consider E's problem:

$$\begin{aligned}
\max_{s(y),B(y)} \quad & E[y - s(y)] \\
\text{s.t.} \quad & s(y) \leq y \qquad (MAX) \\
& E[s(y) - c(y)B(y) - I] \geq 0 \qquad (IR) \\
& s(y) \leq D \quad \forall y \in B^V \qquad (IC1) \\
& s(y) = D \quad \forall y \notin B^V \qquad (IC2)
\end{aligned}$$

where B^V is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now E's problem becomes

$$\begin{aligned}\min_{s(y),B(y)} \quad & E[c(y)B(y)] \\ \text{s.t.} \quad & (MAX),\ (IC1),\ (IC2) \\ & E[s(y) - c(y)B(y) - I] = 0 \qquad (IR)\end{aligned}$$

(c) Show that any optimal contract $(s(y), B(y))$ has a verification range of the form $B^V = [0, D]$ for some D. [Hint: Proof by contradiction.]

(d) Show that any optimal contract $(s(y), B(y))$ sets $s(y) = y$ when $B(y) = 1$. [Hint: Proof by contradiction.]

(e) A contract is thus characterised by D. Which D maximises E's utility? Can you give a financial interpretation to this contract?

Question 6 (Ironing)

Consider the continuous–type price discrimination problem from class, where the principal chooses $q(\theta)$ to maximise

$$E[q(\theta)MR(\theta) - c(q(\theta))]$$

subject to $q(\theta)$ increasing in θ.

For $v \in [0, 1]$, let

$$H(v) = \int_0^v MR(F^{-1}(x))dx$$

be the expected marginal revenue up to $\theta = F^{-1}(v)$. Let $\overline{H}(v)$ be the highest convex function under $H(v)$. Then define $\overline{MR}(\theta)$ by

$$\overline{H}(v) = \int_0^v \overline{MR}(F^{-1}(x))dx$$

Finally, let $\Delta(\theta) = H(F(\theta)) - \overline{H}(F(\theta))$.[1]

(a) Argue that $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)$ is flat. Also argue that $\Delta(\underline{\theta}) = \Delta(\overline{\theta}) = 0$.

[1] Note, it is important that we take the convex hull in quantile space. If we use θ–space, then $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)f(\theta)$ is flat, which is not particularly useful.

(b) Since $q(\theta)$ is an increasing function, show that

$$E[q(\theta)MR(\theta) - c(q(\theta))] = E[q(\theta)\overline{MR}(\theta) - c(q(\theta))] - \int_{\underline{\theta}}^{\overline{\theta}} \Delta(\theta)dq(\theta)$$

(c) Derive the profit–maximising allocation $q(\theta)$.