Practice Problems 2: Asymmetric Information

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Question 1 (Sequential Screening with Different Priors)

At time I, a principal signs a contract $\langle q_1, t_1, q_2, t_2 \rangle$ with an agent for trade conducted at time II. At the time of contracting, the principal and agent are both uninformed of the agent's period II utility.

At time II, the state $s \in \{1, 2\}$ is revealed. The agent's utility in state s is $u_s(q) - t$. The cost to the principal of providing quantity q is $c(q)$ in both states. A contract $\langle q_1, t_1, q_2, t_2 \rangle$ then specifies the quantity $q \in \Re_+$ and transfer $t \in \Re$ in both states of the world. Assume that $u_1'(q) > u_2'(q)$ $(\forall q)$. For technical simplicity, also assume that utility functions are increasing and concave, while the cost function is increasing and convex.

The agent and principal have different priors over the state. The principal is experienced and knows that state 1 will occur with probability p. The agent is mistaken, and believes that state 1 will occur with probability θ. Assume that $\theta > p$, so the agent is more confident than the principal.

(a) Suppose that the state s is publicly observable. The principal thus maximises her profit

$$\Pi = p[t_1 - c(q_1)] + (1 - p)[t_2 - c(q_2)]$$

subject to the individual rationality constraint of the agent,

$$\theta[u_1(q_1) - t_1] + (1 - \theta)[u_2(q_2) - t_2] \geq 0$$

Describe the principal's profit–maximising contract.

For the rest of this question, suppose the state s is only observed by the agent.

(b) Show that your optimal contract from (a) is not incentive compatible after the state has been revealed.

(c) Suppose the principal maximises her profit subject to individual rationality and incentive compatibility. Derive the optimal contract. [Hint: you can ignore one of the (IC) constraints and later show that it does not bind at the optimal solution].

Question 2 (Theory of A Market Maker)

Suppose a risk–neutral agent wishes to trade one unit of a share with a risk–neutral intermediary. That is, the agent can buy one share, sell one share, or choose not to trade. All parties start with a common prior on the value of the share, $\theta \sim g(\theta)$. The game is as follows.

1. The intermediary sets bid price B and ask prices A. Assume the market for intermediaries is competitive, so they make zero profits on each trade.
2. With probability $1 - \alpha \in (0, 1)$ the agent is irrational, buying one share at price A and selling one share at price B.[1] With probability α the agent is rational. In this case, the agent receives a signal $s \in [\underline{s}, \overline{s}]$ with nondegenerate distribution $f(s|\theta)$, and chooses to buy at A or sell at B. Assume $f(s|\theta)$ obeys the MLRP.
3. The value of the share, θ, is revealed. The agent and intermediary receive their payoffs. The rational agent's payoffs are as follows: if he buys, he receives $\theta - A$; if he sells he receives $B - \theta$; and if he does not trade he receives 0. The intermediaries payoffs are the opposite.

(a) Fix prices (A, B). For which signals will the rational agent trade?

(b) Given the zero profit condition for the intermediary, how are equilibrium prices (A, B) determined?

(c) Show that, in equilibrium, $A \geq E[\theta] \geq B$. Show that some rational agents will not trade.

(d) Suppose α increases. Show how this affects (a) the equilibrium prices, and (b) the proportion of rational agents trading.

(e) What happens as $\alpha \to 1$?

[1] This is rather unrealistic, but it makes the maths easier.

Question 3 (Screening without Transfers)

[25 points] A principal employs an agent who privately observes the state of the world $\theta \in [\underline{\theta}, \overline{\theta}]$ which is distributed with density $f(\theta)$. The principal first makes a report to the principal who chooses an action $q \in \{1, 2\}$. Consider the following direct–revelation mechanism:

1. The principal commits to a mechanism $q(\hat{\theta}) \in \{1, 2\}$.
2. The agent observes the state θ.
3. The agent then sends a message to the principal $\hat{\theta}$.
4. The principal receives payoff $v(\theta, q)$ and the agent receive payoff $u(\theta, q)$.

(a) Suppose $u(\theta, q)$ is supermodular in that

$$u(\theta_H, q_H) + u(\theta_L, q_L) > u(\theta_H, q_L) + u(\theta_L, q_H)$$

for $\theta_H > \theta_L$ and $q_H > q_L$. Show incentive compatibility implies that $q(\theta)$ is increasing.

(b) Characterise the mechanism, $q(\cdot)$, that maximises the principal's expected payoff.

(c) Intuitively, what happens to the optimal mechanism as the principal's preferences converge to those of the agent's? That is, $v(\theta, q) \to u(\theta, q)$ in L^1.

Question 4 (Holdup and Private Information)

[25 points] Suppose a buyer invests b at cost $c(b)$, where $c(\cdot)$ is increasing and convex. Investment b induces a stochastic valuation v for one unit of a good. The valuation is observed by the buyer and is distributed according to $f(v|b)$.

The seller then makes a TIOLI offer to the buyer of a price p. The buyer accepts or rejects.

(a) First suppose the seller observes v. How much will the buyer invest?

For the rest of the question, suppose that the seller observes neither b nor v. Assume that buyer's and seller's optimisation problems are concave.

(b) Assume $f(v|b)$ satisfies the hazard rate order in that

$$\frac{f(v|b)}{1-F(v|b)} \quad \text{decreases in } b \tag{HR}$$

Derive the seller's optimal price. How does the optimal price vary with b?

(c) Derive the buyer's optimal investment choice. Notice that (HR) implies that $F(v|b)$ decreases in b. How does the optimal investment vary with the expected price, p?

(d) Argue that there will be a unique Nash equilibrium in (b,p) space.

(e) How does the level of investment differ from part (a)? Why?

Question 5 (Moral Hazard and Asymmetric Information)

[25 points] A firm employs an agent who is risk–neutral, but has limited liability (i.e. they cannot be paid a negative wage). There is no individual rationality constraint. The agent can choose action $a \in \{L,H\}$ at cost $\{0,c\}$. There are two possible outputs $\{q_L,q_H\}$. The high output occurs with probability p_L or p_H if the agent takes action L or H, respectively. The agent's payoff is

$$w - c(a)$$

where w is the wage and $c(a)$ the cost of the action. The principal's payoff is

$$q - w$$

where q is the output and w is the wage.

(a) Characterise the optimal wages and action.

Suppose there are two types of agents, $i \in \{1,2\}$. The principal cannot observe an agent's type but believes the probability of either type is 1/2. The agents are identical except for their cost of taking the action: for agent $i \in \{1,2\}$ the cost of $a \in \{L,H\}$ is $\{0,c^i\}$, where $c^2 > c^1$.

(b) What are the optimal wages if the principal wishes to implement $\{a^1,a^2\} = \{L,L\}$?

(c) What are the optimal wages if the principal wishes to implement $\{a^1,a^2\} = \{H,H\}$?

(d) What are the optimal wages if the principal wishes to implement $\{a^1,a^2\} = \{L,H\}$?

(e) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{H, L\}$?

Question 6 (Pricing)

Consider the pricing problem of a monopolist who has 300 units to sell and is only allowed to choose a price p per unit (i.e. no first degree price discrimination). There are 100 agents who are identical and have the following demand:

$$\begin{aligned} D(p) &= 0 \quad \text{if} \quad p > 2 \\ &= 1 \quad \text{if} \quad p \in (1,2] \\ &= 5 \quad \text{if} \quad p \in [0,1] \end{aligned}$$

(a) Suppose the firm can charge a single price, p, per unit. What is the best they can do?

(b) Suppose the firm can separate the agents into two groups. The first group of N are charged price p_1 per unit. The second are charged p_2 per unit. What is the best they can do?

(c) Agents are identical so, intuitively, how can splitting them into two groups help? Does this relate to anything we covered in class?

Question 7 (Pricing)

Consider a second degree price discriminating firm facing customers with two possible types $\theta \in \{3, 4\}$ with equal probability. An agent with type θ gains utility $u(\theta) = \theta q - p$ from quality q supplied at price p. If the agent does not purchase they gain utility 0. The cost of quality q is $c(q) = q^2/2$.

(a) Suppose the firm could observe each agents type θ. What quantity would she choose for each type?

For the next two parts assume the firm cannot observe agents' types. She can choose two quantity–price bundles $\{q(\theta), p(\theta)\}$ for $\theta \in \{3, 4\}$.

(b) Suppose there is a single outside good of quality $q^* = 1$ and price $p^* = 1$. What quantity would the firm choose for each type?

(c) Now suppose the outside good has quality $q^* = 6$ and price $p^* = 18$. What quantity would the firm choose for each type?