Practice Problems 3: Multiagent Asymmetric Information

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1. Bilateral Trade

Suppose two agents wish to trade a single good. The seller has privately known cost $c \sim g(\cdot)$ on $[0,1]$. The buyer has privately known value $v \sim f(\cdot)$ on $[0,1]$. These random variables are independent of each other. The agents' payoffs are

$$U_S = t - cp$$
$$U_B = vp - t$$

where $t \in \Re$ is a transfer and $p \in [0,1]$ is the probability of trade. If an agent abstains from trade, they receive 0.

In class, we showed that it is impossible to implement the ex–post efficient allocation. We now wish to find the welfare maximising mechanism.

(a) Consider the problem of a middleman who runs mechanism $\langle p(\tilde{v},\tilde{c}), t_B(\tilde{v},\tilde{c}), t_S(\tilde{v},\tilde{c})\rangle$ where t_B and t_S are the transfers from the buyer and to the seller respectively. Show that a middleman can make profit

$$\Pi = E\Big[[MR(v) - MC(c)]p(v,c)\Big] - U_B(\underline{v}) - U_S(\bar{c})$$

where

$$MR(v) = v - \frac{1-F(v)}{f(v)} \quad \text{and} \quad MC(c) = c + \frac{G(c)}{g(c)}$$

(b) Maximise expected welfare subject to $\Pi = 0$. [Note: We have not shown that $\Pi = 0$ implies one can find a common transfer function $t(v,c)$. We leave this for another day.]

2. Auction with Correlated Values

A seller wants to sell a good to one of two symmetric buyers. Buyer i gains utility $v_i x_i - t_i$, where v_i is his valuation, x_i is the probability he gets the good and t_i is his payment to the seller. The seller wishes to maximise expected payments.

A seller designs a mechanism $(x_i(v_1, v_2), t_i(v_1, v_2))$, $i \in \{1, 2\}$, where the allocation probability and payments are a function of the agents' reports. The mechanism must allocate the good to the highest valuation buyer if valuations are different, and to each buyer with probability 1/2 if the valuations are the same. We consider only symmetric mechanisms, where payments depend on the agents' reports and not their identities. Denote $t_{ab} := t_1(v_a, v_b) = t_2(v_b, v_a)$.

Each buyer has one of two valuations, v_l or v_h, where $v_h > v_l$. The probability that the agents have valuations a, b is given by p_{ab}, where $a, b \in \{l, h\}$. We assume $p_{hh}p_{ll} > p_{hl}^2$, so valuations are positively correlated.

(a) The seller wants to design an ex–post individually rational (EPIR) and ex–post incentive compatible (EPIC) mechanism to maximise their expected revenue.[1] Determine the optimal transfers and the expected utility of a high and low type.

(b) The seller now drops the EPIR and EPIC requirement. The mechanism only has to be interim individually rational (IR) and interim incentive compatible (IC). Show that the seller can fully extract from the buyers. [Hint: Choose $t_{hh} = v_h/2$ and $t_{hl} = v_h$.] Intuitively, why can the seller fully extract the buyers' rent?

(c) The seller is concerned the buyers may collude. Suppose that if the buyers collude, they choose a pair of reports that minimises the sum of the transfers they pay. Show that if the buyers collude in the mechanism from part (a), they pay a total of v_l. Show that if the buyers collude in the mechanism from part (b), they pay less than v_l.

(d) Show that any (IR) and (IC) mechanism where buyers pay at least v_l by colluding, gives them at least as much rent as the mechanism from part (a).

3. All Pay Auction

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[0, 1]$. Suppose the good is sold via an all–pay auction.

(a) Derive the symmetric equilibrium bidding strategy directly.

(b) Derive the symmetric equilibrium bidding strategy via revenue equivalence.

[1] That is, every type should be happy to participate and reveal their type truthfully after knowing their opponent's type.

4. Negotiations and Auctions

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[\underline{V}, \overline{V}]$. Define marginal revenue as

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

(a) Show that $E[MR(v)] = \underline{V}$.

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be v_0. In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume $\underline{V} \geq v_0$, i.e. all bidders are "serious". How is revenue affected if one bidder is swapped for a reservation price?

5. Asymmetric Auctions

(a) There is one bidder with value $v_1 \sim U[a, a+1]$, where $a \geq 0$. What is the optimal auction? Intuitively, why is the optimal reservation price increasing in a?

(b) Now there is a second bidder with value $v_2 \sim U[0, 1]$, where agents' types are independent. What is the optimal auction?

6. Grants

Each of N agents have a project which needs funding. The value they place on funding is $\theta \sim F$ on $[0, 1]$. The SSHRC wants to fund the most worthwhile project, but cannot observe θ. Agents write proposals which are time consuming: an agent who spends time t on a proposal gains utility $u_i(\theta_i) = P_i \cdot \theta - t_i$, where the project is funded with probability P_i. The SSHRC can only observe the time t_i each agent spends writing their proposal. Their aim is to maximise welfare which, since writing proposals is wasteful, is the same as maximising $\sum_i u_i$.

(a) Specify the problem as a DRM and write down the agents' utility.

(b) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(c) Suppose $(1-F(x))/f(x)$ is strictly decreasing in x. Show the SSHRC's optimal policy is to allocate the grant randomly.

7. Auctions with Hidden Quality

The economics department is trying to procure teaching services from one of N potential assistant professors. Candidate i has an outside option of wage $\theta_i \in [0,1]$ with distribution function F. This wage is private information and can be thought of as the candidate's type. The department gets value $v(\theta_i)$ from type θ_i.

Consider a direct revelation mechanism consisting of an allocation function $P(\tilde{\theta}_1, \dots, \tilde{\theta}_N)$ and a transfer function $t(\tilde{\theta}_1, \dots, \tilde{\theta}_N)$. Suppose candidate i's utility is $u(\theta_i, \tilde{\theta}_i) = E_{-i}[t(\tilde{\theta}) - P(\tilde{\theta})\theta_i]$ and the department's profit is $\pi = E[P(\tilde{\theta})v(\theta_i) - t(\tilde{\theta})]$.

(a) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(b) Using (a), what is the department's profit?

For the rest of the question assume that

$$1 \geq \frac{d}{d\theta_i}\frac{F(\theta_i)}{f(\theta_i)} \geq 0$$

(c) If $v'(\theta_i) \leq 1$ what is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

(d) Suppose $v'(\theta_i) \geq 2$ and $E[v(\theta_i)] \geq 1$. What is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

8. Double Auction

A seller and buyer participate in a double auction. The seller's cost, $c \in [0,1]$, is distributed according to F_S. The buyer's value, $v \in [0,1]$, is distributed according to F_B. The seller names a price s and the buyer a price b. If $b \geq s$ the agents trade at price $p = (s+b)/2$, the seller gains $p - c$ and the buyer gains $v - p$. If $s < b$ there is no trade and both gain 0.

(a)Write down the utilities of buyer and seller. Derive the FOCs for the optimal bidding strategies.

For the rest of the question assume $c \sim U[0,1]$ and $v \sim U[0,1]$.

(b) Show that $S(c) = \frac{2}{3}c + \frac{1}{4}$ and $B(v) = \frac{2}{3}v + \frac{1}{12}$ satisfy the FOCs.

(c) Under which conditions on (v, c) does trade occur?

9. Auctions with Endogenous Entry

This question studies optimal auction design with endogenous entry. There are a large number of potential bidders who must pay k in order to enter an auction. After the entry decision, each entering bidder learns their private value θ_i which are distributed independently and identically with positive density $f(\theta)$, distribution function $F(\theta)$ and support $[\underline{\theta}, \overline{\theta}]$. The auctioneer has known valuation θ_0.

Denote the direct mechanism by $\langle N, P_i, t_i \rangle$, which is common knowledge. The auctioneer first allows bidders in the set N to enter. Each entering bidder learns their type θ_i and reports $\tilde{\theta}_i$. If the other bidders report truthfully, bidder i wins the good with probability $P_i(\tilde{\theta}_i, \theta_{-i})$ and pays $t_i(\tilde{\theta}_i, \theta_{-i})$ yielding utility,

$$u_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}}\left[\theta_i P_i(\tilde{\theta}_i, \theta_{-i}) - t_i(\tilde{\theta}_i, \theta_{-i})\right]$$

where the lowest type gets utility $u_i(\underline{\theta})$.

(a) Show that incentive compatibility (IC) implies that utility obeys an integral equation and a monotonicity constraint.

(b) Write down the ex–ante individual rationality (IR) constraint which ensures that each bidder is happy to pay the entry cost and participate.

(c) Write down the auctioneer's program or maximising revenue, equal to the sum of payments, subject to (IC) and (IR).

(d) Show that the (IR) constraint will bind at the optimum.

(e) Optimal allocation function. Show that the revenue maximising mechanism awards the object to the agent with the highest valuation if that value exceeds θ_0.

(f) Optimal entry policy. Define welfare with n bidders by

$$W(n) := E_\theta \max\{\theta_0, \theta_1, \ldots, \theta_n\}$$

Show that $W(n+1) - W(n)$ decreases in n. Use this to show that the optimal number of bidders, n^*, obeys $W(n^*) - W(n^* - 1) \geq k \geq W(n^* + 1) - W(n^*)$.

(g) Argue that the optimal mechanism can be implemented by a standard auction with reserve price, entry fee and having bidders make their entry decisions sequentially. What are the optimal entry fee and reserve price?