Homework 0: Monotone Comparative Statics and Stochastic Orders

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Question 1

Profits equal $\pi = pk^\alpha l^\beta - wk - rl$, where $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$.

(a) Verify that π is supermodular in (k, l).

(b) Let $x = (k, l)$ and $t = (p, -r, -w)$. Verify that π satisfies increasing differences in (x, t).

(c) How does an increase in p, r or w affect the optimal choices of k and l?

Question 2

A consumer faces prices p_t over time $t \in \{1, 2, \ldots\}$. The agent chooses her optimal purchase time τ to maximise utility $u(\theta, \tau) = \theta\delta^\tau - p_\tau$, where $\delta < 1$ and $\theta > 0$. How does the optimal purchase time vary with θ?

Question 3

Consider a sealed bid auction, where all agents simultaneously submit bids, and the highest bidder wins and pays her bid. Consider the problem of agent 1. Suppose the highest bid of her opponents has cumulative distribution function $F(\cdot)$ on $[0, \infty)$, and the agent has value v. If she bids b, her expected utility is thus

$$u(v, b) = (v - b)F(b)$$

How does the agent's optimal bid change with v?

Question 4

(a) Suppose $f(x)$ and $g(x)$ are supermodular. Show that $f(x) + g(x)$ is supermodular.

(b) Suppose $f(x)$ and $g(x)$ are quasi–supermodular. Show, by example, that $f(x) + g(x)$ may not be quasi–supermodular.

(c) Suppose $f(x)$ is supermodular and $g : \mathbb{R} \to \mathbb{R}$ is increasing. Show, by example, that $g(f(x))$ may not be supermodular.

(d) Suppose $f(x)$ is quasi–supermodular and $g : \mathbb{R} \to \mathbb{R}$ is increasing. Show that $g(f(x))$ is quasi–supermodular.

Question 5

An agent facing strictly positive prices (p_1, p_2) consumes two goods $(x_1, x_2) \in \mathbb{R}^2_+$. Her utility quasi–linear,

$$u(x_1, x_2) - p_1 x_1 - p_2 x_2$$

Assume $u(\cdot, \cdot)$ is continuous, so that an optimal choice exists. We wish to examine how a change in the price of good 1 affects demands for the two goods. Let

$$x_1^*(p_1, x_2) = \sup \left\{ \text{argmax}_{x_1} u(x_1, x_2) - p_1 x_1 - p_2 x_2 \right\} \tag{1}$$

be the largest solution to the consumers x_1–problem, taking x_2 as fixed. Similarly, let

$$x_2^*(p_1) = \sup \left\{ \text{argmax}_{x_2} u(x_1^*(p_1, x_2), x_2) - p_1 x_1^*(p_1, x_2) - p_2 x_2 \right\} \tag{2}$$

be the largest solution to the consumer's x_2–problem.

(a) Suppose $u(x_1, x_2)$ is supermodular and let $p_1' \geq p_1$. Show that

$$x_1^*(p_1, x_2(p_1)) \geq x_1^*(p_1', x_2(p_1)) \geq x_1^*(p_1', x_2(p_1'))$$

(b) Suppose that $u(x_1, x_2)$ is submodular and let $p_1' \geq p_1$. Show that, once again,

$$x_1^*(p_1, x_2(p_1)) \geq x_1^*(p_1', x_2(p_1)) \geq x_1^*(p_1', x_2(p_1'))$$

(c) Consider N goods $(y_1, \ldots, y_N)$, which we divide into arbitrary sets x_1 and x_2. Intuitively, can we generalise the results in (a) and (b)?

Question 6

Suppose that $X_1 \geq_{st} Y_1$ and $X_2 \geq_{st} Y_2$.

(a) Suppose $\{X_i, Y_i\}$ are independent. Show that $X_1 + X_2 \geq_{st} Y_1 + Y_2$.

(b) Suppose the variables are not independent. Show, by counterexample, that the result in part (a) may fail.

Question 7

Suppose that X and Y have continuous distribution functions F and G and equal means. Suppose $g(z) - f(z)$ changes sign twice, and the order is $+, -, +$.

(a) Show that $G(z) - F(z)$ has one change of sign, and the order is $+, -$.

(b) Show that $Y \geq_{cx} X$.