Homework 1: Basic Moral Hazard

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Question 1 (Normal–Linear Model)

The following normal–linear model is regularly used in applied models. Given action \( a \in \mathbb{R} \), output is \( q = a + x \), where \( x \sim N(0, \sigma^2) \). The cost of effort is \( g(a) \) is increasing and convex. The agent’s utility equals \( u(w(q) - g(a)) \), while the principals is \( q - w(q) \). Suppose the agent’s outside option is \( u(0) \).

We make two large assumptions. First, the principal uses a linear contract:

\[
w(q) = \alpha + \beta q
\]

Second, the agent’s utility is CARA, i.e., \( u(w) = -e^{-w} \).

(a) Suppose \( w \sim N(\mu, \sigma^2) \). Denote the certainty equivalent of \( w \) by \( \bar{w} \), where

\[
u(\bar{w}) = E[u(w)]
\]

Show that \( \bar{w} = \mu - \sigma^2/2 \).

(b) Suppose effort is unobservable. The principal’s problem is

\[
\begin{align*}
\max_{w(q), a} & \quad E[q - w(q)] \\
\text{s.t.} & \quad E[u(w(q) - g(a))|a] \geq u(0) \\
& \quad a \in \arg\max_{a' \in \mathbb{R}} E[u(w(q) - g(a'))|a']
\end{align*}
\]

Using the first order approach, characterise the optimal contract \( (\alpha, \beta, a) \). [Hint: write utilities in terms of their certainty equivalent.]

(c) How would the solution change if the agent knows \( x \) before choosing his action (but after signing the contract)?
**Question 2 (Signal of Effort)**

Consider the same normal–linear model as in Question 1. After $a$ is chosen, the principal observes output $q$ and a signal $y$ that is correlated with $x$. For example, if the agent is selling cars, then $y$ could be the sales of the dealer next door. Let

$$
\begin{pmatrix}
x \\
y
\end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}\right)
$$

Suppose the principal uses the linear wage function:

$$w = \alpha + \beta(q + \gamma y)$$

Using the same approach as above, solve for the optimal value of $\gamma$. How does $\gamma$ vary with $\sigma_{xy}$? Provide an intuition.

**Question 3 (Insurance)**

An agent has increasing, concave utility $u(\cdot)$. They start with wealth $W_0$ and may have an accident costing $x$ of their wealth. Assume $x$ is publicly observable. The agent has access to a perfectly competitive market of risk–neutral insurers who offer payments $R(x)$ net of any insurance premium. The distribution of $x$ is as follows

$$
\begin{align*}
f(0, a) &= 1 - p(a) \quad (1) \\
f(x, a) &= p(a)g(x) \quad \text{for} \quad x > 0 \quad (2)
\end{align*}
$$

where $\int g(x)dx = 1$. The agent can affect the probability of an accident through their choice of $a$. The cost is given by increasing convex function, $\psi(a)$. The function $p(a)$ is decreasing and convex. Utility is then given by $u(W_0 - x + R(x)) - \psi(a)$.

(a) Suppose there is no insurance market. What action $\hat{a}$ does the agent take?

(b) Suppose $a$ is contractible. Describe the first–best payment schedule $R(x)$ and the effort choice, $a^*$.

(c) Suppose $a$ is not contractible. Describe the second–best payment schedule $R(x)$.

(d) Interpret the second–best payment schedule. Would the agent ever have an incentive to
hide an accident? (i.e. report \( x = 0 \) when \( x > 0 \)).

## Question 4 (Private Evaluations with Limited Liability)

A principal employs an agent. The game is as follows.

1. The agent privately chooses an action \( a \in \{L, H\} \). The cost of this action is \( g(a) \).

2. The principal *privately* observes output \( q \sim f(q|a) \) on \([q, \bar{q}]\). Assume this distribution function satisfies strict MLRP. That is,

\[
\frac{f(q|H)}{f(q|L)}
\]

is strictly increasing in \( q \).

3. Suppose the principal reports that output is \( \tilde{q} \). The principal then pays out \( t(\tilde{q}) \), while the agent receives \( w(\tilde{q}) \), where \( w(\tilde{q}) \leq t(\tilde{q}) \). The difference is burned. The payments \( \langle t, q \rangle \) are contractible.

Payoffs are as follows. The principal obtains

\[ q - t \]

The agent obtains

\[ u(w) - g(a) \]

where \( u(\cdot) \) is strictly increasing and concave, and \( g(\cdot) \) is increasing and convex. The agent has no (IR) constraint, but does have limited liability. That is, \( w(q) \geq 0 \) for all \( q \).

First, assume the principal wishes to implement \( a = L \).

(a) Characterise the optimal contract.

Second, assume the principal wishes to implement \( a = H \).

(b) Write down the principal’s problem as maximising expected profits subject to the agent’s (IC) constraint, the principal’s (IC) constraint, the limited liability constraint and the constraint that \( w(q) \leq t(q) \).
(c) Argue that \( t(q) \) is independent of \( q \).

(d) Characterise the optimal contract. How does the wage vary with \( q \)?

**Question 5 (Debt Contracts)**

A risk neutral agent seeks funding from a risk neutral principal. The game is as follows:

1. The project requires investment \( I \) from the principal.
2. The agent chooses effort \( a \in \{L, H\} \) at cost \( c(a) \). Assume \( c(H) > c(L) \).
3. Output \( q \) is realised. Assume \( q \) takes values \( \{q_1, \ldots, q_N\} \), where \( q_{i+1} > q_i \). Output is distributed according to \( f(q_i|a) \).
4. If \( q_i \) is realised, the principal obtains payment \( B_i \) and the agent obtains \( q_i - B_i \). The agent’s utility is \( u = q_i - B_i - c(a) \); the principal’s profit is \( \pi = B_i - I \).

A contract specifies the payment to the principal as a function of the output \( \langle B_i \rangle \). Assume the principal has outside option 0 and the agent makes a TIOLI offer to the principal. We also assume the contract satisfies feasibility (FE):

\[
0 \leq B_i \leq q_i
\]

and monotonicity (MON):

\[
B_i \text{ is increasing in } i
\]

Finally assume that \( f(q_i|a) \) satisfies the monotone hazard rate principle (MHRP):

\[
\frac{f(q_i|L)}{1 - F(q_i|L)} \geq \frac{f(q_i|H)}{1 - F(q_i|H)} \quad \text{for each } q_i.
\]

Assume the agent wishes to implement the high action. Show that a debt contract is optimal.

[Aside: In class we showed that MLRP implies debt contracts are optimal. The key insight is that, if we use the (MON) condition, we can use the weaker MHRP assumption.]
Question 6 (Bargaining Power)

Suppose a risk neutral principal employs a risk averse agent. The two parties both sign a contract stating wage profile \( w(q) \). The agent then chooses action \( a \in A \) at cost \( g(a) \).

Payoffs are as follows. The agent gets

\[
u(w - g(a))\]

where \( g(a) \) is increasing and convex. Utility is strictly increasing and strictly concave. The principal gets

\[
q - w
\]

The principal has reservation profit 0; the agent has reservation utility \( u(0) \).

First, suppose the principal makes a TIOLI offer to the agent.

(a) Assume the effort \( a \) is observable. Set up and solve the principal’s optimal contract.

(b) Assume effort \( a \) is not observable. Set up the principal’s problem.

Next, suppose the agent makes a TIOLI offer to the principal.

(c) Assume the effort \( a \) is observable. Show that the optimal contract induces the same effort as when the principal proposes the contract.

(d) Assume effort \( a \) is not observable. Set up the agent’s problem. Next, suppose that utility is CARA, i.e. \( u(w) = -\exp(-w) \), which implies that \( u(w + x) = u(w)e^{-x} \). Show that the optimal contract induces the same effort as when the principal proposes the contract (part (b)).