Homework 3

November 23, 2009

1. Nonlinear Pricing with Three Types

Consider the nonlinear pricing model with three types, $\theta_3 > \theta_2 > \theta_1$. The utility of agent θ_i is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent θ_i by (q_i, t_i). We now have six (IC) constraint and three (IR) constraints. For example, (IC_1^2) says that θ_1 must not want to copy θ_2, i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \qquad (\text{IC}_1^2)$$

The firm's profit is

$$\sum_{i=1}^{3} \pi_i [t_i - c(q_i)]$$

where π_i is the proportion of type θ_i agents and $c(q)$ is increasing and convex.

(a) Show that (IR_2) and (IR_3) can be ignored.
(b) Show that $q_3 \geq q_2 \geq q_1$.
(c) Using (IC_2^1) and (IC_3^2) show that we can ignore (IC_3^1). Using (IC_2^3) and (IC_1^2) show that we can ignore (IC_1^3).
(d) Show that (IR_1) will bind.
(e) Show that (IC_2^1) will bind.
(f) Show that (IC_3^2) will bind.
(g) Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC_1^2) and (IC_2^3) can be ignored.

2. Downward Sloping Demand I

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2} q^2 - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi) t_2 \tag{1}$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit–maximising qualities.

3. Downward Sloping Demand II

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i (q - \frac{1}{2} q^2) - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi) t_2 \tag{2}$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities and prices.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit–maximising qualities.

4. Bilateral Trade

Suppose two agents wish to trade a single good. The seller has privately known cost $c \sim g(\cdot)$ on $[0, 1]$. The buyer has privately known value $v \sim f(\cdot)$ on $[0, 1]$. These random variables are independent of each other. The agents' payoffs are

$$U_S = t - cp$$
$$U_B = vp - t$$

where $t \in \Re$ is a transfer and $p \in [0,1]$ is the probability of trade. If an agent abstains from trade, they receive 0.

In class, we showed that it is impossible to implement the ex–post efficient allocation. We now wish to find the revenue and welfare maximising mechanisms.

(a) Consider the problem of a middleman who runs mechanism $\langle p(\tilde{v},\tilde{c}), t_B(\tilde{v},\tilde{c}), t_S(\tilde{v},\tilde{c})\rangle$ where t_B and t_S are the transfers from the buyer and to the seller respectively. Show that a middleman can make profit

$$\Pi = E\Big[[MR(v) - MC(c)]p(v,c)\Big] - U_B(\underline{v}) - U_S(\overline{c})$$

where

$$MR(v) = v - \frac{1-F(v)}{f(v)} \quad \text{and} \quad MC(c) = c + \frac{G(c)}{g(c)}$$

(b) Maximise the middleman's expected profits.

(c) Maximise expected welfare subject to $\Pi = 0$. [Note: We have not shown that $\Pi = 0$ implies one can find a common transfer function $t(v,c)$. We leave this for another day.]

5. Costly State Verification

There is a risk–neutral entrepreneur E who has a project with privately observed return y with density $f(y)$ on $[0,Y]$. The project requires investment $I < E[y]$ from an outside creditor C.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports y they pay $s(y) \leq y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies E they pay cost $c(y)$ and get to observe E's type.

The game is as follows:

- E chooses $(s(y), B(y))$ to raise I from a competitive financial market.
- Output y is realised.
- E claims the project yields $\hat{y}$. If $B(\hat{y}) = 0$ then E pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then C pays $c(y)$ and observes E's true type. If they are telling the truth they pay $s(y)$; if not, then C can take everything.

- Payoffs. E gets $y - s(y)$, while C gets $s(y) - c(y)B(y) - I$.

(a) Show that a contract is incentive compatible if and only if there exists a D such that $s(y) = D$ when $B(y) = 0$ and $s(y) \leq D$ when $B(y) = 1$.

Consider E's problem:

$$\begin{aligned}
\max_{s(y),B(y)} \quad & E[y - s(y)] \\
\text{s.t.} \quad & s(y) \leq y \qquad (MAX) \\
& E[s(y) - c(y)B(y) - I] \geq 0 \qquad (IR) \\
& s(y) \leq D \quad \forall y \in B^V \qquad (IC1) \\
& s(y) = D \quad \forall y \notin B^V \qquad (IC2)
\end{aligned}$$

where B^V is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now E's problem becomes

$$\begin{aligned}
\min_{s(y),B(y)} \quad & E[c(y)B(y)] \\
\text{s.t.} \quad & (MAX),\ (IC1),\ (IC2) \\
& E[s(y) - c(y)B(y) - I] = 0 \qquad (IR)
\end{aligned}$$

(c) Show that any optimal contract $(s(y), B(y))$ has a verification range of the form $B^V = [0, D]$ for some D. [Hint: Proof by contradiction.]

(d) Show that any optimal contract $(s(y), B(y))$ sets $s(y) = y$ when $B(y) = 1$. [Hint: Proof by contradiction.]

(e) A contract is thus characterised by D. Which D maximises E's utility? Can you give a financial interpretation to this contract?

6. Ironing

Consider the continuous–type price discrimination problem from class, where the principal chooses $q(\theta)$ to maximise

$$E[q(\theta)MR(\theta) - c(q(\theta))]$$

subject to $q(\theta)$ increasing in θ.

For $v \in [0,1]$, let

$$H(v) = \int_0^v MR(F^{-1}(x))dx$$

be the expected marginal revenue up to $\theta = F^{-1}(v)$. Let $\overline{H}(v)$ be the highest convex function under $H(v)$. Then define $\overline{MR}(\theta)$ by

$$\overline{H}(v) = \int_0^v \overline{MR}(F^{-1}(x))dx$$

Finally, let $\Delta(\theta) = H(F(\theta)) - \overline{H}(F(\theta))$.[1]

(a) Argue that $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)$ is flat. Also argue that $\Delta(\underline{\theta}) = \Delta(\overline{\theta}) = 0$.

(b) Since $q(\theta)$ is an increasing function, show that

$$E[q(\theta)MR(\theta) - c(q(\theta))] = E[q(\theta)\overline{MR}(\theta) - c(q(\theta))] - \int_{\underline{\theta}}^{\overline{\theta}} \Delta(\theta)dq(\theta)$$

(c) Derive the profit–maximising allocation $q(\theta)$.

[1]Note, it is important that we take the convex hull in quantile space. If we use θ–space, then $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)f(\theta)$ is flat, which is not particularly useful.