1 Inverse Marginal Utility is Martingale (Rogerson '85)

1.1 Setup

- Two periods, no discounting
- Actions $a_t \in A$
- Output q_t
- Time-separable and stationary
  - Production $q_t \sim f(q_t|a_t)$ - no technological link
  - Agent utilty $\sum_t (u(w_t) - g(a_t))$ no preference link
  - Principal payoff $R = \sum_t (q_t - w_t)$

1.2 Principal's Problem

- Let $a = (a_1, a_2(q_1))$ be agent's action plan
- Principal chooses $a^*, w_1^*(q_1), w_2^*(q_1, q_2)$ to maximize

$$
\begin{aligned}
\mathbb{E}\left[(q_1 - w_1(q_1) + q_2 - w_2(q_1,q_2))\,|a\right] \text{ subject to } &: \\
\mathbb{E}\left[u(w_1^*(q_1)) - g(a_1^*) + u(w_2^*(q_1,q_2)) - g(a_2^*)\,|a^*\right] &\geq \mathbb{E}\left[...|\widetilde{a}\right] \quad \text{(IC)} \\
\mathbb{E}\left[u(w_1^*(q_1)) - g(a_1^*) + u(w_2^*(q_1,q_2)) - g(a_2^*)\,|a^*\right] &\geq 2\overline{u} \quad \text{(IR)}
\end{aligned}
$$

- Note: Can't save or borrow

1.3 Result

Proposition 1 *The optimal long-term contract satisfies*

$$
\frac{1}{u'(w_1(q_1))} = \mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))}|q_1, a\right] \tag{*}
$$

for all q_1.

Idea:

- LHS is marginal cost of providing utility today
- RHS is expected marginal cost of providing utility tomorrow
- Agent is indifferent between receiving utility today or tomorrow

- If LHS<RHS principal could profit by front-loading utility

Proof.

- Let $w_1(q_1), w_2(q_1, q_2)$ be optimal contract
- Fix q_1
- Shift $\varepsilon \lesseqgtr 0$ utility to period 1

$$\begin{aligned} u(\widehat{w}_1(q_1)) &= u(w_1(q_1)) + \varepsilon \\ u(\widehat{w}_2(q_1, q_2)) &= u(w_2(q_1, q_2)) - \varepsilon \end{aligned}$$

- Does not affect agent's IC or IR constraint
- By first-order Taylor approximation

$$\begin{aligned} \widehat{w}_1(q_1) &= w_1(q_1) + \frac{\varepsilon}{u'(w_1(q_1))} \\ \widehat{w}_2(q_1, q_2) &= w_2(q_1, q_2) - \frac{\varepsilon}{u'(w_2(q_1, q_2))}, \forall q_2 \end{aligned}$$

- Thus, effect on Revenue

$$\widehat{R} - R = -\varepsilon \left(\frac{1}{u'(w_1(q_1))} - \mathbb{E}\left[\frac{1}{u'(w_2(q_1, q_2))} | q_1, a \right] \right)$$

- As ε can be chosen positive or negative, optimality requires that the term in parantheses vanishes

$\blacksquare$

1.4 Discussion

- Optimal long-term contract has memory
  - Unless w_1 independent of q_1, LHS depends on q_1
  - So does RHS, in particular $w_2 \neq w_2(q_2)$
- Optimal long-term contract is complex
- Agent would like to save - not borrow
  - Apply Jensen's inequality to (*)

– $f(x) = 1/x$ is a convex function

– Thus $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$

$$u'(w_1(q_1)) = 1/\mathbb{E}\left[\frac{1}{u'(w_2(q_1, q_2))}\right] \leq \mathbb{E}\left[u'(w_2(q_1, q_2))\right]$$

– Intuition?

2 Asymptotic Efficiency

2.1 Setup

- ∞ periods, common discount factor δ
- Output $q_t \in \left[\underline{q}, \overline{q}\right]$
- Actions $a_t \in A$
- First best action a^* and quantity $q^* = \mathbb{E}[q|a^*]$
- Time-separable and stationary

2.2 Result

Proposition 2 *If everybody is patient, first-best is almost achievable: $\forall \varepsilon, \exists \overline{\delta}, \forall \delta \geq \overline{\delta}$ there is a contract generating agent utility greater than $u(q^*) - g(a^*) - \varepsilon$ (and yielding at least 0 to the principal).*

- Statement assumes that agent proposes contract and has to satisfy principal's IR constraing
- If principal proposes, can also get first-best

Idea:

- Make agent residual claimant
- He can build up savings and then smooth his consumption

Proof.

- Agent's wealth w_t
- If wealth is high, $w_t \geq \left(q^* - \underline{q}\right)/\delta$, consume

$$q_t = q^* + (1-\delta) w_t - \widetilde{\varepsilon}$$

- Earnings q^*
- Interest $(1-\delta)\, w_t$
- save a little $\widetilde{\varepsilon} \in \left(0; (1-\delta)\left(q^* - \underline{q}\right)\right)$

- If wealth is low, $w_t \leq (q^* - q)/\delta$, consume

$$q_t = \underline{q} + (1-\delta)\, w_t$$

- Minimal earning q^*
- Interest $(1-\delta)\, w_t$

- This is pretty arbitrary. The point is that wealth grows

$$\mathbb{E}\left[w_{t+1}\right] - w_t \geq \min\left\{\left(q^* - \underline{q}\right)/\delta, \widetilde{\varepsilon}\right\} > 0$$

- Thus, wealth is a submartingale with bounded increments and thus the probability that it exceeds any threshold x, e.g. $x = \left(q^* - \underline{q}\right)/\delta$, after time t approaches 1 as $t \to \infty$

$$\begin{aligned} \lim_{t\to\infty} p_t &= 1 \text{ where} \\ p_t &= \Pr\left(w_\tau \geq x \text{ for all } \tau \geq t\right) \end{aligned}$$

- Omitting non-negative terms gives lower bound on agent's utility

$$\begin{aligned} (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau \left(u\left(q_t\right) - g\left(a^*\right)\right) &\geq \delta^t p_t u\left(q^*\right) - g\left(a^*\right) \\ &\geq u\left(q^*\right) - g\left(a^*\right) - \varepsilon \end{aligned}$$

when we choose δ and p_t close enough to 1

■

3 Short-term Contracts

3.1 Setup

- 2 periods, no discounting
- Time separable technology and preferences
- Agent can save, but principal can monitor this

- Funny assumption, but necessary for tractability and result
  - Maybe reasonable in third world when saving is through landlord

- Outside utility $\overline{u} = u(\overline{q}) - g(\overline{a})$

3.2 Principal's Problem

- Principal chooses $a^*, s^*(q_1), w_1^*(q_1), w_2^*(q_1, q_2)$ to maximize

$$\begin{aligned}
\mathbb{E}\left[(q_1 - w_1(q_1) + q_2 - w_2(q_1,q_2))\,|a\right] \text{ subject to } &: \\
\mathbb{E}\left[u\left(w_1^*(q_1) - s^*(q_1)\right) - g(a_1^*) + u\left(w_2^*(q_1,q_2) + s^*(q_1)\right) - g(a_2^*)\,|a^*\right] &\geq \mathbb{E}\left[...(\widetilde{a},\widetilde{s})\right] && \text{(IC)} \\
\mathbb{E}\left[u\left(w_1^*(q_1) - s^*(q_1)\right) - g(a_1^*) + u\left(w_2^*(q_1,q_2) + s^*(q_1)\right) - g(a_2^*)\,|a^*\right] &\geq 2\overline{u} && \text{(IR)}
\end{aligned}$$

- Can choose $s^*(q_1) = 0$ because principal can save for the agent by adjusting w

3.3 Renegotiation and Spot Contracts

- After period 1, the principal could offer the agent to change the contract

- Optimally, he offers contract $\widehat{a}_2, \widehat{w}_2(q_2)$ to maximize

$$\begin{aligned}
\mathbb{E}\left[q_2 - w_2(q_2)\,|a_2\right] \text{ subject to } &: && \text{(Seq-Eff)} \\
\mathbb{E}\left[u\left(\widehat{w}_2(q_2)\right) - g(\widehat{a}_2)\,|\widehat{a}_2\right] &\geq \mathbb{E}\left[...(\widetilde{a})\right] && \text{(IC')} \\
\mathbb{E}\left[u\left(\widehat{w}_2(q_2)\right) - g(\widehat{a}_2)\,|\widehat{a}_2\right] &\geq \mathbb{E}\left[u\left(w_2^*(q_1,q_2)\right) - g(a_2^*)\,|a_2^*\right] && \text{(IR')}
\end{aligned}$$

  where the last line captures the idea that the agent can insist on the original long-term contract

- Of course, $\widehat{a}_2, \widehat{w}_2(q_2)$ implicitly depend on q_1 through (IR')

- Call contract sequentially efficient, or renegotiation-proof if there is no such mutually beneficial deviation after any realization of q_1, and thus $\widehat{a}_2 = a_2^*$ and $\widehat{w}_2(q_2) = w_2^*(q_1, q_2)$.

- The long-term contract a^*, w^* can be implemented via spot contracts if there is a saving strategy $s(q_1)$ for the agent such that the second period spot contract $\overline{a}_2, \overline{w}_2(q_2)$ that maximizes

$$\begin{aligned}
\mathbb{E}\left[q_2 - w_2(q_2)\,|a_2\right] \text{ subject to } &: && \text{(Spot)} \\
\mathbb{E}\left[u\left(\overline{w}_2(q_2) + s(q_1)\right) - g(\overline{a}_2)\,|\overline{a}_2\right] &\geq \mathbb{E}\left[...(\widetilde{a})\right] && \text{(IC-spot)} \\
\mathbb{E}\left[u\left(\overline{w}_2(q_2) + s(q_1)\right) - g(\overline{a}_2)\,|\overline{a}_2\right] &\geq u\left(\overline{q} + s(q_1)\right) - g(\overline{a}_2) && \text{(IR-spot)}
\end{aligned}$$

  yields the same actions $\overline{a}_2 = a_2^*$ and wages $\overline{w}_2(q_2) + s(q_1) = w_2^*(q_1, q_2)$ as the original contract.

3.4 Result

Proposition 3 *1. The optimal long-term contract is renegotiation-proof.*

2. A renegotiation-proof contract can be implemented by spot contracts.

- If there was a profitable deviation after q_1, there is a weakly more profitable deviation where IR' is binding
- The original contract could then be improved by substituting the deviation into the original contract. This proves 1.
- For 2, set $s(q_1)$ so that $u(\overline{q} + s(q_1)) = \mathbb{E}[u(w_2(q_1, q_2)) | a_2]$
- Then if $\widehat{a}_2, \widehat{w}_2(q_2)$ solves (Seq-Eff), $\overline{a}_2 = \widehat{a}_2, \overline{w}_2(q_2) = \widehat{w}_2(q_2) - s(q_1)$ solves (Spot)

3.5 Discussion

- Rationale for Short-Term Contracting
- Separates incentive-provision from consumption smoothing
- Yields recursive structure of optimal long-term contract - Memory of contract can be captured by one state variable: savings
- Generalizes to
  - T periods
  - Preferences where a_1 does not affect trade-off between a_2 and c_2

4 Optimal Linear Contracts (Holmstrom, Milgrom '87)

4.1 Setup

- 2 periods, no discounting
- Time separable technology and preferences
- Funny utility function

$$u(w_1, w_2, a_1, a_2) = -\exp(-(w_1 + w_2 - g(a_1) - g(a_2)))$$

  - Consumption at the end (-> no role for savings)
  - Monetary costs of effort

  - CARA - no wealth effects

- Outside wage w per period
- Optimal static contract a^s, w^s

4.2 Result

Proposition 4 *1. The optimal long-term contract repeats the optimal static contract:*

$$w_1^*(q_1) = w^s(q_1) \text{ and } w_2^*(q_1, q_2) = w^s(q_2)$$

2. If q is binary, or Brownian, the optimal contract is linear in output: $w^*(q_1, q_2) = \alpha + \beta(q_1 + q_2)$

Idea: CARA makes everything separable

Proof.

- Principal chooses a^*, w^* to maximize

$$\begin{aligned}
&\mathbb{E}[q_1 - w_1(q_1) + q_2 - w_2(q_1, q_2)|a] \text{ subject to } &:& \\
&\mathbb{E}[-\exp(-(w_1^*(q_1) + w_2^*(q_1, q_2) - g(a_1^*) - g(a_2^*)))|a^*] &\geq& \mathbb{E}[-\exp(...)|\widetilde{a}] \qquad \text{(IC)} \\
&\mathbb{E}[-\exp(-(w_1^*(q_1) + w_2^*(q_1, q_2) - g(a_1^*) - g(a_2^*)))|a^*] &\geq& u(2w) \qquad \text{(IR)}
\end{aligned}$$

- Can choose $w_2^*(q_1, q_2)$ so that $\mathbb{E}[-\exp(-(w_2^*(q_1, q_2) - g(a_2^*)))|a_2^*] = u(w)$ for all q_1
  - Add $\Delta(q_1)$ to all $w_2^*(q_1, q_2)$
  - Subtract $\Delta(q_1)$ from $w_1^*(q_1)$
  - Does not affect $w_1^*(q_1) + w_2^*(q_1, q_2)$ for any realization (q_1, q_2)
  - Principal and agent only care about this sum

- Sequential efficiency implies that in the second period after realization of q_1, principal chooses $\widehat{a}_2, \widehat{w}_2$ to maximize

$$\begin{aligned}
&\mathbb{E}[q_2 - \widehat{w}_2(q_2)|a_2] \text{ subject to } &:& \\
-\exp(-(w_1^*(q_1) - g(a_1^*)))&\mathbb{E}[\exp(-(\widehat{w}_2(q_2) - g(\widehat{a}_2)))|\widehat{a}_2] &\geq& -\exp(...)\mathbb{E}[\exp(...)|\widetilde{a}_2] \qquad \text{(IC 2)} \\
-\exp(...)&\mathbb{E}[\exp(-(\widehat{w}_2(q_2) - g(\widehat{a}_2)))|\widehat{a}_2] &\geq& -\exp(...)u(w) \qquad \text{(IR 2)}
\end{aligned}$$

- As period one factors out (this is because there are no wealth effects), the optimal second period contract $\widehat{a}_2, \widehat{w}_2$ is the optimal short-term contract $\widehat{w}_2(q_2) = w^s(q_2)$ - independent of q_1

- Taken $\widehat{a}_2, \widehat{w}_2$ as given, the principal chooses $\widehat{a}_1, \widehat{w}_1$ to maximize

$$\begin{aligned}
\text{maximize } \mathbb{E}\left[q_1 - w_1(q_1) \mid a_1\right] \text{ subject to } &: \\
-\mathbb{E}\left[\exp\left(-\left(\widehat{w}_1(q_1) - g(\widehat{a}_1)\right)\right) \mid \widehat{a}_1\right] \mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g(\widehat{a}_2)\right)\right) \mid \widehat{a}_2\right] &\geq -\mathbb{E}\left[... \mid \widetilde{a}_1\right] \mathbb{E}\left[... \mid \widehat{a}_2\right] \quad \text{(IC 1)} \\
-\mathbb{E}\left[\exp\left(-\left(\widehat{w}_1(q_1) - g(\widehat{a}_1)\right)\right) \mid \widehat{a}_1\right] \mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g(\widehat{a}_2)\right)\right) \mid \widehat{a}_2\right] &\geq u(2w) \quad \text{(IR 1)}
\end{aligned}$$

- This is again the static problem, proving (1)
- (2) follows because every function of q binary is linear, and a Brownian motion is approximated by a binary process

■

4.3 Discussion

- Not very general, but extends to any number of periods
- Stationarity not so suprising:
  - technology independent
  - no consumption-smoothing
  - no wealth-effects
  - no benefits from long-term contracting
- Agent benefits ability to adjust his actions according to realized output
  - Consider generalization with $t \in [0; T]$ and $dq_t = adt + \sigma dW_t$, so that $q_T \sim N\left(a, \sigma^2 T\right)$
  - If agent cannot adjust his action, principal can implement first-best via tail-test and appropriate surplus
  - Tail-test does not work if agent can adjust effort
    - ∗ Can slack at first...
    - ∗ ... and only start working if q_t drifts down to far
  - More generally with any concave, say, reward function $w(q_T)$, agent will
    - ∗ work in steep region, after bad realization
    - ∗ shirk in flat region, after good realization
  - Providing stationary incentives to always induce the static optimal a^* is better

5 Continuous Time (Sannikov 2008)

5.1 Setup

- Continuous time $t \in [0; \infty)$, discount rate r
- Think about time as tiny discrete increments dt and remember $rdt \approx 1 - e^{-rdt}$
- Time separable technology

$$dX_t = a_t dt + dZ_t$$

- Brownian Motion Z_t (also called Wiener process) characterized by
  - Sample paths Z_t continuous almost surely
  - Increments independent and stationary with distribution $Z_{t+\Delta} - Z_t \sim \mathcal{N}(0, \Delta)$
- Wealth of agent

$$w = r \int_{t=0}^{\infty} e^{-rt} \left(u\left(c_t\right) - g\left(a_t\right) \right) dt$$

(the "r" annuitizes the value of the agent and renders it comparable to u and g)

- Cost function g with $g(0) = 0$, $g' > 0$, $g'' > 0$
- Consumption utility with $u(0) = 0, u' > 0, \lim_{x \to \infty} u'(x) = 0$
- Consumption = wage; no hidden savings
- Revenue of firm

$$\begin{aligned} \Pi &= r\mathbb{E}\left[\int e^{-rt} dX_t\right] - r\int e^{-rt} c_t dt \\ &= r\int e^{-rt}\left(a_t - c_t\right) dt \end{aligned}$$

5.2 Firm's problem

- Choose a_t, c_t as function of $X_{s \leq t}$ to maximize Π subject to (IC) and (IR)
- Recursive approach: Let w_t be the continuation wealth of the agent (in utils)

$$w_t = r \int_{s=t}^{\infty} e^{-r(s-t)} \left(u\left(c_s\right) - g\left(a_s\right) \right) ds$$

- Principal chooses c_t, a_t, w_{t+dt} to maximize Π_t subject to (IC), (IR) and

$$w_t = rdt\left(u\left(c_t\right) - g\left(a_t\right)\right) + (1 - rdt)\,\mathbb{E}\left[w_{t+dt} | a_t\right] \tag{PK}$$

- The "Promise Keeping" constraint is not a proper constraint, but just an accounting identity that ensures that w_t is actually the continuation value of the agent
  - Current payoff $u(c_t) - g(a_t)$
  - Continuation wealth $\mathbb{E}[w_{t+dt}]$
- Example: Retiring agent with wealth w_t
  - Instruct agent not to take any effort $a_t = 0$
  - Pay out wealth as annuity $u(c_t) = w_t$
  - Firm profit from this contract $\Pi_0(u(c)) = -c$
- Draw picture of Π_0 and Π
- Decompose principal's NPV into current profits and continuation value

$$\begin{aligned}\Pi_t &= r(a_t - c_t)\,dt + (1 - rdt)\,\mathbb{E}[\Pi_{t+dt}] \\ r\Pi_t dt &= r(a_t - c_t)\,dt + \mathbb{E}[d\Pi_t]\end{aligned}$$

- Principal's expected profit Π_t is function of state variable, i.e. of agent's wealth $\Pi(w_t)$
- The expected value of the increment $\mathbb{E}[d\Pi_t]$ can be calculated with Ito's Lemma

Lemma 5 (Ito) *Consider the stochastic process w_t governed by*

$$dw_t = \gamma(w_t)\,dt + \sigma dZ_t$$

and a process $\Pi_t = \Pi(w_t)$ that is a function of this original process. Then the expected increment of Π_t is given by

$$\mathbb{E}[d\Pi(w)] = \left[\gamma(w)\,\Pi'(w) + \frac{1}{2}\sigma^2\Pi''(w)\right]dt$$

Proof. By Taylor expansion

$$\begin{aligned}\Pi(w_{t+dt}) - \Pi(w_t) &= \Pi(w_t + \gamma(w_t)\,dt + \sigma dZ_t) - \Pi(w_t) \\ &= (\gamma(w_t)\,dt + \sigma dZ_t)\,\Pi'(w_t) + \frac{1}{2}(\gamma(w_t)\,dt + \sigma dZ_t)^2\,\Pi''(w_t) + o(dt) \\ &= (\gamma(w_t)\,dt + \sigma dZ_t)\,\Pi'(w_t) + \frac{1}{2}\sigma^2 dZ_t^2\Pi''(w_t) + o(dt) \\ \mathbb{E}[\Pi(w_{t+dt}) - \Pi(w_t)] &= \gamma(w_t)\,dt\Pi'(w_t) + \frac{1}{2}\sigma^2 dt\Pi''(w_t) + o(dt)\end{aligned}$$

- The reason, the Ito term $\frac{1}{2}\sigma^2\Pi''(w_t)$ comes in is that w_t is oscillating so strongly, with stdv. $\sqrt{dt}$ in every dt.

■

5.3 Solving the agent's problem

5.3.1 Evolution of Wealth

- Subtracting $(1 - rdt)\, w_t$ from (PK)

$$\begin{aligned} rdtw_t &= rdt\,(u(c_t) - g(a_t)) + (1 - rdt)\,(\mathbb{E}[w_{t+dt}] - w_t) \\ &= rdt\,(u(c_t) - g(a_t)) + \mathbb{E}[dw_t] \end{aligned}$$

  the agent's value is a function of

  - his current consumption $u(c_t)$
  - his current effort $-g(a_t)$
  - the expected drift of his value

- Reversely

$$\mathbb{E}[dw_t] = r\,(w_t - (u(c_t) - g(a_t)))\,dt$$

- To get at the actual dynamics of w_t, assume that value increments dw_t are linear in output increments dX_t with wealth dependent sensitivity $rb(w_t)$

$$\begin{aligned} dw_t &= rb(w_t)\,dX_t \\ \mathbb{E}[dw_t] &= rb(w_t)\,\mathbb{E}[dX_t] \\ &= rb(w_t)\,\mathbb{E}[a_t dt + dZ_t] \\ &= rb(w_t)\,a_t dt \end{aligned}$$

- Therefore, the actual increment of wealth is governed by

$$\begin{aligned} dw_t &= \mathbb{E}[dw_t] + (dw_t - \mathbb{E}[dw_t]) \\ &= r\,(w_t - (u(c_t) - g(a_t)))\,dt + rb(w_t)\,(dX_t - a_t dt) \\ &= r\,(w_t - (u(c_t) - g(a_t)))\,dt + rb(w_t)\,dZ_t \end{aligned}$$

  - Drifting

    * up when wealth and interest rw_t are high
    * down when consumption c_t is high
    * up when effort a_t is high
  - Wiggling
    * up, when production exceeds expectations $dX_t > a_t dt$
    * down, when production falls short of expectations $dX_t < a_t dt$

5.3.2 Agent IC

- Agent's effort a_t affects value rw_t through
  - current marginal cost $rg'(a)$
  - marginal continuation value $b(w_t)$
- Then, if agent with wealth w is instructed to exert effort $a(w)$ his FOC becomes

$$rg'(a(w)) = rb(w) \tag{IC}$$

- So, if the principal wants to incentivize effort $a = a(w)$ he needs to link the evolution of wealth w_t to output dX_t via $b(w)$
- By (IC) we can write b as a function of a, $\beta(a) = g'(a)$

5.4 The Firm's Problem - continued

- In the case at hand we have

$$\begin{aligned} \gamma(w_t) &= r(w_t - (u(c_t) - g(a_t))) \\ \sigma &= r\beta(a_t) \end{aligned}$$

  and we get

$$r\Pi(w) = r(a-c) + r(w - (u(c) - g(a)))\Pi'(w) + \frac{1}{2}r^2\beta(a)^2\Pi''(w) \tag{(*)}$$

- So the principal chooses plans $a = a(w) > 0$ and $c = c(w)$ to maximize the RHS of (*)
- The agent has to retire at some point w_r
  - Marginal utility of consumption $u'(c) \to 0$
  - Marginal cost of effort $g'(a) \geq \varepsilon > 0$

- Boundary conditions
  - $\Pi(0) = 0$: If the agent's wealth is 0, he can achieve this by setting future effort $a_t = 0$, yielding 0 to the firm
  - $\Pi(w_r) = \Pi_0(w_r) = -u^{-1}(w_r)$: At some retirement wealth w_r, the agent retires
  - $\Pi'(w_r) = \Pi_0'(w_r)$: Smooth pasting: The profit function is smooth and equals Π_0 above w_r

Theorem 6 *There is a unique concave function $\Pi \geq \Pi_0$, maximizing (*) under the above boundary conditions. The action and consumption profiles $a(w), c(w)$ constitute an optimal contract.*

5.5 Properties of Solution

5.5.1 Properties of Π

- $\Pi(0) = 0$
- $\Pi'(0) = 0$: terminating the contract at $w = 0$ is inefficient, and $w > 0$ serves as insurance against termination
- $\Pi(w) < 0$ for large w, e.g. $w = w_r$, because agent has been promised a lot of continuation utility

5.5.2 Properties of optimal effort a^*

- Optimal effort maximizes

$$ra + rg(a)\Pi'(w) + \frac{1}{2}r^2\beta(a)^2\Pi''(w)$$

  - Increased output ra
  - Compensating agent for effort through continuation wealth $rg(a)\Pi'(w)$ - yes, this is positive for $w \approx 0$
  - Compensating agent for income risk $\frac{1}{2}r^2\beta(a)^2\Pi''(w)$ - how so?
- Monotonicity of $a^*(w)$ unclear
  - $\Pi'(w)$ decreasing
  - $\Pi''(w)$ could be increasing or decreasing
- As $r \to 0$, $a^*(w)$ decreasing

5.5.3 Properties of optimal consumption c^*

- Optimal consumption maximizes
$$-rc - ru(c)\,\Pi'(w)$$
and thus
$$-\frac{1}{u'(c^*)} = \Pi'(w) \text{ or } c^* = 0$$
  - $-\frac{1}{u'(c^*)}$: cost of current consumption utility
  - $\Pi'(w)$: cost of continuation utility
- Thus agent does not consume as long as w small

5.6 Extensions: Career Paths

- Performance-based compensation $c(w_t)$ serves as short-term incentive
- Now incorporate long-term incentives into model
- In baseline model principal's outside option was retirement $\Pi_0(u(c)) = -c$
- Can model, quitting, replacement or promotion by different outside options $\widetilde{\Pi}_0$
- This only changes the boundary conditions but not the differential equation determining Π
  - If agent can quit at any time with outside utility $\widetilde{w}$, then $\widetilde{\Pi}_0(u(c)) = \begin{cases} -c & \text{if } u(c) > \widetilde{w} \\ 0 & \text{if } u(c) = \widetilde{w} \end{cases}$
  - If agent can be replaced at profit D to firm, then $\widetilde{\Pi}_0(u(c)) = D - c$
  - If agent can be promoted at cost K, resulting into new value function Π_p, then $\widetilde{\Pi}_0(w) = \max\{\Pi_0(w); \Pi_p(w) - K\}$
- Find that
$$\text{quitting} < \text{benchmark} < \text{replacement, promotion}$$

5.7 Stuff

- Change in agent's value

$$
\begin{aligned}
\mathbb{E}\left[dw_t\right] &= \mathbb{E}\left[w_{t+dt}\right] - w_t \\
&=
\end{aligned}
$$

$$
\begin{aligned}
dw_t &= w_{t+dt} - w_t \\
&= rdt\left(w_{t+dt} - \left[u\left(c_t\right) - g\left(a_t\right)\right]\right) + \left(1 - rdt\right)\left(w_{t+dt} - \mathbb{E}\left[w_{t+dt}\right]\right) \\
&= rdt\left(w_t - \left[u\left(c_t\right) - g\left(a_t\right)\right]\right) + \left(w_{t+dt} - \mathbb{E}\left[w_{t+dt}\right]\right)
\end{aligned}
$$

- Drift in agent's value
  - Increasing via interest at rate r on current wealth w_t
  - Decreasing in current consumption $-u\left(c_t\right)$ (if agent eats today, he has less tomorrow)
  - Increasing in current effort $g\left(a_t\right)$
- Assume that value increments are linear in output increments

$$
\begin{aligned}
dw_t &= b\left(w_t\right) dX_t \\
&= b\left(w_t\right) a_t dt + b\left(w_t\right) dZ_t \\
\mathbb{E}\left[dw_t\right] &= b\left(w_t\right) \mathbb{E}\left[dX_t\right] = b\left(w_t\right) a_t dt
\end{aligned}
$$

- Then,

$$
\begin{aligned}
w_{t+dt} - \mathbb{E}\left[w_{t+dt}\right] &= dw_t - \mathbb{E}\left[dw_t\right] \\
&= b\left(w_t\right) dX_t - \mathbb{E}\left[b\left(w_t\right) dX_t\right]
\end{aligned}
$$

  Random shocks

  - through in annuity of accumulated wealth w_t, minus
  - Current utility $u\left(c_t\right) - g\left(a_t\right)$ (wealth will increase, $dw_t > 0$, if agent eats less / works more today, i.e. c_t lower or a_t higher), plus
  - Random shocks