## Homework 3

November 15, 2010

## 1. Nonlinear Pricing with Three Types

Consider the nonlinear pricing model with three types, $\theta_{3}>\theta_{2}>\theta_{1}$. The utility of agent $\theta_{i}$ is

$$
u\left(\theta_{i}\right)=\theta_{i} q-t
$$

Denote the bundle assigned to agent $\theta_{i}$ by $\left(q_{i}, t_{i}\right)$. We now have six (IC) constraint and three (IR) constraints. For example, $\left(\mathrm{IC}_{1}^{2}\right)$ says that $\theta_{1}$ must not want to copy $\theta_{2}$, i.e.

$$
\begin{equation*}
\theta_{1} q_{1}-t_{1} \geq \theta_{1} q_{2}-t_{2} \tag{1}
\end{equation*}
$$

The firm's profit is

$$
\sum_{i=1}^{3} \pi_{i}\left[t_{i}-c\left(q_{i}\right)\right]
$$

where $\pi_{i}$ is the proportion of type $\theta_{i}$ agents and $c(q)$ is increasing and convex.
(a) Show that $\left(\mathrm{IR}_{2}\right)$ and $\left(\mathrm{IR}_{3}\right)$ can be ignored.
(b) Show that $q_{3} \geq q_{2} \geq q_{1}$.
(c) Using $\left(\mathrm{IC}_{2}^{1}\right)$ and $\left(\mathrm{IC}_{3}^{2}\right)$ show that we can ignore $\left(\mathrm{IC}_{3}^{1}\right)$. Using $\left(\mathrm{IC}_{2}^{3}\right)$ and $\left(\mathrm{IC}_{1}^{2}\right)$ show that we can ignore $\left(\mathrm{IC}_{1}^{3}\right)$.
(d) Show that $\left(\mathrm{IR}_{1}\right)$ will bind.
(e) Show that $\left(\mathrm{IC}_{2}^{1}\right)$ will bind.
(f) Show that $\left(\mathrm{IC}_{3}^{2}\right)$ will bind.
(g) Assume that $q_{3} \geq q_{2} \geq q_{1}$. Show that $\left(\mathrm{IC}_{1}^{2}\right)$ and $\left(\mathrm{IC}_{2}^{3}\right)$ can be ignored.

## 2. Downward Sloping Demand I

Suppose a seller of wine faces two types of customers, $\theta_{1}$ and $\theta_{2}$, where $\theta_{2}>\theta_{1}$. The proportion of type $\theta_{1}$ agents is $\pi \in[0,1]$. Let $q$ be the quality of the wine and $t$ the price. Agent $\theta_{i}$ has utility

$$
u\left(\theta_{i}\right)=\theta_{i} q-\frac{1}{2} q^{2}-t
$$

Let type $\theta_{1}$ buy contract $\left(q_{1}, t_{1}\right)$ and type $\theta_{2}$ buy $\left(q_{2}, t_{2}\right)$. The cost of production is zero, $c(q)=0$, and the seller maximises profit

$$
\pi t_{1}+(1-\pi) t_{2}
$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit-maximising qualities.

## 3. Downward Sloping Demand II

Suppose a seller of wine faces two types of customers, $\theta_{1}$ and $\theta_{2}$, where $\theta_{2}>\theta_{1}$. The proportion of type $\theta_{1}$ agents is $\pi \in[0,1]$. Let $q$ be the quality of the wine and $t$ the price. Agent $\theta_{i}$ has utility

$$
u\left(\theta_{i}\right)=\theta_{i}\left(q-\frac{1}{2} q^{2}\right)-t
$$

Let type $\theta_{1}$ buy contract $\left(q_{1}, t_{1}\right)$ and type $\theta_{2}$ buy $\left(q_{2}, t_{2}\right)$. The cost of production is zero, $c(q)=0$, and the seller maximises profit

$$
\pi t_{1}+(1-\pi) t_{2}
$$

(a) Suppose the seller observes the agent's types. Solve for the first best qualities and prices.
(b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
(c) Derive the profit-maximising qualities.

## 4. Dynamic Mechanism Design

A firm sells to a customer over $T=2$ periods. There is no discounting.
The consumer's per-period utility is

$$
u=\theta q-p
$$

where $q \in \Re$ is the quantity of the good, and $p$ is the price. The agent's type $\theta \in\left\{\theta_{L}, \theta_{H}\right\}$ is privately known. In period $1, \operatorname{Pr}\left(\theta=\theta_{H}\right)=\mu$. In period 2 , the agent's type may change. With
probability $\alpha>1 / 2$, her type remains the same; with probability $1-\alpha$ her type switches (so a high type becomes a low type, or a low type becomes a high type).

The firm chooses a mechanism to maximise the sum of its profits. The per-period profit is given by

$$
\pi=p-\frac{1}{2} q^{2}
$$

A mechanism consists of period 1 allocations $\left\langle q_{L}, q_{H}\right\rangle$, period 2 allocations $\left\langle q_{L L}, q_{L H}, q_{H L}, q_{H H}\right\rangle$, and corresponding prices, where $q_{L H}$ is the quantity allocated to an agent who declares $L$ in period 1 and $H$ in period 2.
(a) Consider period $t=2$. Fix the first period type, $\theta$. Assume in period 2 that the lowtype's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Characterise the second period rents obtained by the agents, $U_{\theta L}$ and $U_{\theta H}$, as a function of $\left\{q_{L L}, q_{L H}, q_{H L}, q_{H H}\right\}$
(b) Consider period $t=1$. Assume the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Derive the lifetime rents obtained by the agents, $U_{L}$ and $U_{H}$, as a function of $\left\{q_{L}, q_{H}, q_{L L}, q_{L H}, q_{H L}, q_{H H}\right\}$.
(c) Derive the firm's total expected profits.
(d) Assume the firm does not want to exclude, i.e. that $\Delta:=\theta_{H}-\theta_{L}$ is sufficiently small. Derive the profit-maximising allocations $\left\{q_{L}, q_{H}, q_{L L}, q_{L H}, q_{H L}, q_{H H}\right\}$. In particular, show that $q_{H L}$ is first-best. Can you provide an intuition for this result?
(Bonus) Suppose $T$ is arbitrary. Can you derive the form of the optimal mechanism?

## 5. Costly State Verification

There is a risk-neutral entrepreneur $E$ who has a project with privately observed return $y$ with density $f(y)$ on $[0, Y]$. The project requires investment $I<E[y]$ from an outside creditor $C$.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports $y$ they pay $s(y) \leq y$ and are verified if $B(y)=1$ and not verified if $B(y)=0$. If the creditor verifies $E$ they pay cost $c(y)$ and get to observe $E$ 's type.

The game is as follows:

- $E$ chooses $(s(y), B(y))$ to raise $I$ from a competitive financial market.
- Output $y$ is realised.
- $E$ claims the project yields $\hat{y}$. If $B(\hat{y})=0$ then $E$ pays $s(\hat{y})$ and is not verified. If $B(\hat{y})=1$ then $C$ pays $c(y)$ and observes $E$ 's true type. If they are telling the truth they pay $s(y)$; if not, then $C$ can take everything.
- Payoffs. $E$ gets $y-s(y)$, while $C$ gets $s(y)-c(y) B(y)-I$.
(a) Show that a contract is incentive compatible if and only if there exists a $D$ such that $s(y)=D$ when $B(y)=0$ and $s(y) \leq D$ when $B(y)=1$.

Consider E's problem:

$$
\begin{array}{cl}
\max _{s(y), B(y)} & E[y-s(y)] \\
\text { s.t. } & s(y) \leq y \quad(M A X) \\
& E[s(y)-c(y) B(y)-I] \geq 0  \tag{IR}\\
& s(y) \leq D \quad \forall y \in B^{V} \quad(I C 1) \\
& s(y)=D \quad \forall y \notin B^{V} \quad(I C 2)
\end{array}
$$

where $B^{V}$ is the verification region (where $B(y)=1$ ).
(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now E's problem becomes

$$
\begin{array}{cl}
\min _{s(y), B(y)} & E[c(y) B(y)] \\
\text { s.t. } & (M A X),(I C 1),(I C 2) \\
& E[s(y)-c(y) B(y)-I]=0 \tag{IR}
\end{array}
$$

(c) Show that any optimal contract $(s(y), B(y))$ has a verification range of the form $B^{V}=[0, D]$ for some $D$. [Hint: Proof by contradiction.]
(d) Show that any optimal contract $(s(y), B(y))$ sets $s(y)=y$ when $B(y)=1$. [Hint: Proof by contradiction.]
(e) A contract is thus characterised by $D$. Which $D$ maximises $E$ 's utility? Can you give a financial interpretation to this contract?

## 6. Ironing

Consider the continuous-type price discrimination problem from class, where the principal chooses $q(\theta)$ to maximise

$$
E[q(\theta) M R(\theta)-c(q(\theta))]
$$

subject to $q(\theta)$ increasing in $\theta$.
For $v \in[0,1]$, let

$$
H(v)=\int_{0}^{v} M R\left(F^{-1}(x)\right) d x
$$

be the expected marginal revenue up to $\theta=F^{-1}(v)$. Let $\bar{H}(v)$ be the highest convex function under $H(v)$. Then define $\overline{M R}(\theta)$ by

$$
\bar{H}(v)=\int_{0}^{v} \overline{M R}\left(F^{-1}(x)\right) d x
$$

Finally, let $\Delta(\theta)=H(F(\theta))-\bar{H}(F(\theta)) .{ }^{1}$
(a) Argue that $\Delta(\theta)>0$ implies $\overline{M R}(\theta)$ is flat. Also argue that $\Delta(\underline{\theta})=\Delta(\bar{\theta})=0$.
(b) Since $q(\theta)$ is an increasing function, show that

$$
E[q(\theta) M R(\theta)-c(q(\theta))]=E[q(\theta) \overline{M R}(\theta)-c(q(\theta))]-\int_{\underline{\theta}}^{\bar{\theta}} \Delta(\theta) d q(\theta)
$$

(c) Derive the profit-maximising allocation $q(\theta)$.

[^0]
[^0]:    ${ }^{1}$ Note, it is important that we take the convex hull in quantile space. If we use $\theta$-space, then $\Delta(\theta)>0$ implies $\overline{M R}(\theta) f(\theta)$ is flat, which is not particularly useful.

