Practice Problems 2: Asymmetric Information

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1 Single-Agent Problems

1. Sequential Screening with Different Priors

At time I, a principal signs a contract $\langle q_1, t_1, q_2, t_2 \rangle$ with an agent for trade conducted at time II. At the time of contracting, the principal and agent are both uninformed of the agent's period II utility.

At time II, the state $s \in \{1, 2\}$ is revealed. The agent's utility in state s is $u_s(q) - t$. The cost to the principal of providing quantity q is $c(q)$ in both states. A contract $\langle q_1, t_1, q_2, t_2 \rangle$ then specifies the quantity $q \in \Re_+$ and transfer $t \in \Re$ in both states of the world. Assume that $u_1'(q) > u_2'(q)$ $(\forall q)$. For technical simplicity, also assume that utility functions are increasing and concave, while the cost function is increasing and convex.

The agent and principal have different priors over the state. The principal is experienced and knows that state 1 will occur with probability p. The agent is mistaken, and believes that state 1 will occur with probability θ. Assume that $\theta > p$, so the agent is more confident than the principal.

(a) Suppose that the state s is publicly observable. The principal thus maximises her profit

$$\Pi = p[t_1 - c(q_1)] + (1 - p)[t_2 - c(q_2)]$$

subject to the individual rationality constraint of the agent,

$$\theta[u_1(q_1) - t_1] + (1 - \theta)[u_2(q_2) - t_2] \geq 0$$

Describe the principal's profit–maximising contract.

For the rest of this question, suppose the state s is only observed by the agent.

(b) Show that your optimal contract from (a) is not incentive compatible after the state has been revealed.

(c) Suppose the principal maximises her profit subject to individual rationality and incentive compatibility. Derive the optimal contract. [Hint: you can ignore one of the (IC) constraints and later show that it does not bind at the optimal solution].

2. Theory of A Market Maker

Suppose a risk–neutral agent wishes to trade one unit of a share with a risk–neutral intermediary. That is, the agent can buy one share, sell one share, or choose not to trade. All parties start with a common prior on the value of the share, $\theta \sim g(\theta)$. The game is as follows.

1. The intermediary sets bid price B and ask prices A. Assume the market for intermediaries is competitive, so they make zero profits on each trade.
2. With probability $1 - \alpha \in (0, 1)$ the agent is irrational, buying one share at price A and selling one share at price B.[1] With probability α the agent is rational. In this case, the agent receives a signal $s \in [\underline{s}, \overline{s}]$ with nondegenerate distribution $f(s|\theta)$, and chooses to buy at A or sell at B. Assume $f(s|\theta)$ obeys the MLRP.
3. The value of the share, θ, is revealed. The agent and intermediary receive their payoffs. The rational agent's payoffs are as follows: if he buys, he receives $\theta - A$; if he sells he receives $B - \theta$; and if he does not trade he receives 0. The intermediaries payoffs are the opposite.

(a) Fix prices (A, B). For which signals will the rational agent trade?

(b) Given the zero profit condition for the intermediary, how are equilibrium prices (A, B) determined?

(c) Show that, in equilibrium, $A \geq E[\theta] \geq B$. Show that some rational agents will not trade.

(d) Suppose α increases. Show how this affects (a) the equilibrium prices, and (b) the proportion of rational agents trading.

(e) What happens as $\alpha \to 1$?

[1]This is rather unrealistic, but it makes the maths easier.

3. Screening without Transfers

A principal employs an agent who privately observes the state of the world $\theta \in [\underline{\theta}, \overline{\theta}]$ which is distributed with density $f(\theta)$. The principal first makes a report to the principal who chooses an action $q \in \{1, 2\}$. Consider the following direct–revelation mechanism:

1. The principal commits to a mechanism $q(\hat{\theta}) \in \{1, 2\}$.
2. The agent observes the state θ.
3. The agent then sends a message to the principal $\hat{\theta}$.
4. The principal receives payoff $v(\theta, q)$ and the agent receive payoff $u(\theta, q)$.

(a) Suppose $u(\theta, q)$ is supermodular in that

$$u(\theta_H, q_H) + u(\theta_L, q_L) > u(\theta_H, q_L) + u(\theta_L, q_H)$$

for $\theta_H > \theta_L$ and $q_H > q_L$. Show incentive compatibility implies that $q(\theta)$ is increasing.

(b) Characterise the mechanism, $q(\cdot)$, that maximises the principal's expected payoff.

(c) Intuitively, what happens to the optimal mechanism as the principal's preferences converge to those of the agent's? That is, $v(\theta, q) \to u(\theta, q)$ in L^1.

4. Holdup and Private Information

Suppose a buyer invests b at cost $c(b)$, where $c(\cdot)$ is increasing and convex. Investment b induces a stochastic valuation v for one unit of a good. The valuation is observed by the buyer and is distributed according to $f(v|b)$.

The seller then makes a TIOLI offer to the buyer of a price p. The buyer accepts or rejects.

(a) First suppose the seller observes v. How much will the buyer invest?

For the rest of the question, suppose that the seller observes neither b nor v. Assume that buyer's and seller's optimisation problems are concave.

(b) Assume $f(v|b)$ satisfies the hazard rate order in that

$$\frac{f(v|b)}{1 - F(v|b)} \quad \text{decreases in } b \tag{HR}$$

Derive the seller's optimal price. How does the optimal price vary with b?

(c) Derive the buyer's optimal investment choice. Notice that (HR) implies that $F(v|b)$ decreases in b. How does the optimal investment vary with the expected price, p?

(d) Argue that there will be a unique Nash equilibrium in (b, p) space.

(e) How does the level of investment differ from part (a)? Why?

5. Moral Hazard and Asymmetric Information

A firm employs an agent who is risk–neutral, but has limited liability (i.e. they cannot be paid a negative wage). There is no individual rationality constraint. The agent can choose action $a \in \{L, H\}$ at cost $\{0, c\}$. There are two possible outputs $\{q_L, q_H\}$. The high output occurs with probability p_L or p_H if the agent takes action L or H, respectively. The agent's payoff is

$$w - c(a)$$

where w is the wage and $c(a)$ the cost of the action. The principal's payoff is

$$q - w$$

where q is the output and w is the wage.

(a) Characterise the optimal wages and action.

Suppose there are two types of agents, $i \in \{1, 2\}$. The principal cannot observe an agent's type but believes the probability of either type is 1/2. The agents are identical except for their cost of taking the action: for agent $i \in \{1, 2\}$ the cost of $a \in \{L, H\}$ is $\{0, c^i\}$, where $c^2 > c^1$.

(b) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{L, L\}$?

(c) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{H, H\}$?

(d) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{L, H\}$?

(e) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{H, L\}$?

6. Pricing

Consider the pricing problem of a monopolist who has 300 units to sell and is only allowed to choose a price p per unit (i.e. no first degree price discrimination). There are 100 agents who are identical and have the following demand:

$$\begin{aligned} D(p) &= 0 \quad \text{if} \quad p > 2 \\ &= 1 \quad \text{if} \quad p \in (1, 2] \\ &= 5 \quad \text{if} \quad p \in [0, 1] \end{aligned}$$

(a) Suppose the firm can charge a single price, p, per unit. What is the best they can do?

(b) Suppose the firm can separate the agents into two groups. The first group of N are charged price p_1 per unit. The second are charged p_2 per unit. What is the best they can do?

(c) Agents are identical so, intuitively, how can splitting them into two groups help? Does this relate to anything we covered in class?

7. Pricing

Consider a second degree price discriminating firm facing customers with two possible types $\theta \in \{3, 4\}$ with equal probability. An agent with type θ gains utility $u(\theta) = \theta q - p$ from quality q supplied at price p. If the agent does not purchase they gain utility 0. The cost of quality q is $c(q) = q^2/2$.

(a) Suppose the firm could observe each agents type θ. What quantity would she choose for each type?

For the next two parts assume the firm cannot observe agents' types. She can choose two quantity–price bundles $\{q(\theta), p(\theta)\}$ for $\theta \in \{3, 4\}$.

(b) Suppose there is a single outside good of quality $q^* = 1$ and price $p^* = 1$. What quantity would the firm choose for each type?

(c) Now suppose the outside good has quality $q^* = 6$ and price $p^* = 18$. What quantity would the firm choose for each type?

8. Optimal Taxation

There are two types of agents, $\theta_H > \theta_L$. Proportion β have productivity θ_L. An agent of type θ who exerts effort e produces output $q = \theta e$. The utility of an agent who produces quantity q with effort e is then

$$u(q - t - g(e))$$

where t is the net tax. Assume $g(e)$ is increasing and strictly convex, and $u(\cdot)$ is strictly concave.

Suppose that output is contractible so that a mechanism consists of a pair $(q(\theta), t(\theta))$. The government's problem is to maximise

$$\beta u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) + (1-\beta)u\left(q_H - t_H - g\left(\frac{q_H}{\theta_H}\right)\right)$$

subject to budget balance (BB), $\beta t_L + (1-\beta)t_H \geq 0$. Notice that there are no (IR) constraints here.

(a) First, suppose the government can observe agents' types. Solve for the first–best contract. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type L, for example, is

$$u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) \geq u\left(q_H - t_H - g\left(\frac{q_H}{\theta_L}\right)\right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum $u'_L \geq u'_H$, where u'_i is the marginal utility of type i.

(d) Show that at the optimum (IC_H) binds.

(e) Consider the government's relaxed problem of maximising welfare subject to (BB) and

(IC_H), ignoring (IC_L). Show the optimal contract satisfies:

$$\begin{aligned} 1-\frac{1}{\theta_H}g'\left(\frac{q_H}{\theta_H}\right) &= 0 && (1)\\ 1-\frac{1}{\theta_L}g'\left(\frac{q_L}{\theta_L}\right) &= \frac{u'_L-u'_H}{u'_L}(1-\beta)\left(1-\frac{1}{\theta_H}g'\left(\frac{q_L}{\theta_H}\right)\right) && (2)\end{aligned}$$

(f) Show that (2) implies

$$1-\frac{1}{\theta_L}g'\left(\frac{q_L}{\theta_L}\right)\geq 0 \tag{3}$$

(g) Using equations (1) and (3) show that $q_H \geq q_L$. Use this and the fact that (IC_H) binds, to show that (IC_L) holds.

(h) What does (3) imply about the level of work performed by the low type. Provide an intuition for this distortion.

2 Many-Agent Problems

1. Auctions with Correlated Values

A seller wants to sell a good to one of two symmetric buyers. Buyer i gains utility $v_i x_i - t_i$, where v_i is his valuation, x_i is the probability he gets the good and t_i is his payment to the seller. The seller wishes to maximise expected payments.

A seller designs a mechanism $(x_i(v_1, v_2), t_i(v_1, v_2))$, $i \in \{1, 2\}$, where the allocation probability and payments are a function of the agents' reports. The mechanism must allocate the good to the highest valuation buyer if valuations are different, and to each buyer with probability 1/2 if the valuations are the same. We consider only symmetric mechanisms, where payments depend on the agents' reports and not their identities. Denote $t_{ab} := t_1(v_a, v_b) = t_2(v_b, v_a)$.

Each buyer has one of two valuations, v_l or v_h, where $v_h > v_l$. The probability that the agents have valuations a, b is given by p_{ab}, where $a, b \in \{l, h\}$. We assume $p_{hh} p_{ll} > p_{hl}^2$, so valuations are positively correlated.

(a) The seller wants to design an ex–post individually rational (EPIR) and ex–post incentive compatible (EPIC) mechanism to maximise their expected revenue.[2] Determine the optimal transfers and the expected utility of a high and low type.

(b) The seller now drops the EPIR and EPIC requirement. The mechanism only has to be interim individually rational (IR) and interim incentive compatible (IC). Show that the seller can fully extract from the buyers. [Hint: Choose $t_{hh} = v_h/2$ and $t_{hl} = v_h$.] Intuitively, why can the seller fully extract the buyers' rent?

(c) The seller is concerned the buyers may collude. Suppose that if the buyers collude, they choose a pair of reports that minimises the sum of the transfers they pay. Show that if the buyers collude in the mechanism from part (a), they pay a total of v_l. Show that if the buyers collude in the mechanism from part (b), they pay less than v_l.

(d) Show that any (IR) and (IC) mechanism where buyers pay at least v_l by colluding, gives them at least as much rent as the mechanism from part (a).

[2] That is, every type should be happy to participate and reveal their type truthfully after knowing their opponent's type.

2. All Pay Auction

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[0,1]$. Suppose the good is sold via an all–pay auction.

(a) Derive the symmetric equilibrium bidding strategy directly.

(b) Derive the symmetric equilibrium bidding strategy via revenue equivalence.

3. Negotiations and Auctions

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[\underline{V}, \overline{V}]$. Define marginal revenue as

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

(a) Show that $E[MR(v)] = \underline{V}$.

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be v_0. In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume $\underline{V} \geq v_0$, i.e. all bidders are "serious". How is revenue affected if one bidder is swapped for a reservation price?

4. Asymmetric Auctions

(a) There is one bidder with value $v_1 \sim U[a, a+1]$, where $a \geq 0$. What is the optimal auction? Intuitively, why is the optimal reservation price increasing in a?

(b) Now there is a second bidder with value $v_2 \sim U[0,1]$, where agents' types are independent. What is the optimal auction?

5. Grants

Each of N agents have a project which needs funding. The value they place on funding is $\theta \sim F$ on $[0, 1]$. The NSF wants to fund the most worthwhile project, but cannot observe θ. Agents write proposals which are time consuming: an agent who spends time t on a proposal gains utility $u_i(\theta_i) = P_i \cdot \theta - t_i$, where the project is funded with probability P_i. The NSF can only observe the time t_i each agent spends writing their proposal. Their aim is to maximise welfare which, since writing proposals is wasteful, is the same as maximising $\sum_i u_i$.

(a) Specify the problem as a DRM and write down the agents' utility.

(b) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(c) Suppose $(1 - F(x))/f(x)$ is strictly decreasing in x. Show the NSF's optimal policy is to allocate the grant randomly.

6. Auctions with Hidden Quality

The economics department is trying to procure teaching services from one of N potential assistant professors. Candidate i has an outside option of wage $\theta_i \in [0, 1]$ with distribution function F. This wage is private information and can be thought of as the candidate's type. The department gets value $v(\theta_i)$ from type θ_i.

Consider a direct revelation mechanism consisting of an allocation function $P(\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$ and a transfer function $t(\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$. Suppose candidate i's utility is $u(\theta_i, \tilde{\theta}_i) = E_{-i}[t(\tilde{\theta}) - P(\tilde{\theta})\theta_i]$ and the department's profit is $\pi = E[P(\tilde{\theta})v(\theta_i) - t(\tilde{\theta})]$.

(a) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(b) Using (a), what is the department's profit?

For the rest of the question assume that

$$1 \geq \frac{d}{d\theta_i} \frac{F(\theta_i)}{f(\theta_i)} \geq 0$$

(c) If $v'(\theta_i) \leq 1$ what is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

(d) Suppose $v'(\theta_i) \geq 2$ and $E[v(\theta_i)] \geq 1$. What is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

7. Double Auction

A seller and buyer participate in a double auction. The seller's cost, $c \in [0,1]$, is distributed according to F_S. The buyer's value, $v \in [0,1]$, is distributed according to F_B. The seller names a price s and the buyer a price b. If $b \geq s$ the agents trade at price $p = (s+b)/2$, the seller gains $p - c$ and the buyer gains $v - p$. If $s < b$ there is no trade and both gain 0.

(a)Write down the utilities of buyer and seller. Derive the FOCs for the optimal bidding strategies.

For the rest of the question assume $c \sim U[0,1]$ and $v \sim U[0,1]$.

(b) Show that $S(c) = \frac{2}{3}c + \frac{1}{4}$ and $B(v) = \frac{2}{3}v + \frac{1}{12}$ satisfy the FOCs.

(c) Under which conditions on (v,c) does trade occur?

8. Auctions with Endogenous Entry

This question studies optimal auction design with endogenous entry. There are a large number of potential bidders who must pay k in order to enter an auction. After the entry decision, each entering bidder learns their private value θ_i which are distributed independently and identically with positive density $f(\theta)$, distribution function $F(\theta)$ and support $[\underline{\theta}, \overline{\theta}]$. The auctioneer has known valuation θ_0.

Denote the direct mechanism by $\langle N, P_i, t_i \rangle$, which is common knowledge. The auctioneer first allows bidders in the set N to enter. Each entering bidder learns their type θ_i and reports $\tilde{\theta}_i$. If the other bidders report truthfully, bidder i wins the good with probability $P_i(\tilde{\theta}_i, \theta_{-i})$ and pays $t_i(\tilde{\theta}_i, \theta_{-i})$ yielding utility,

$$u_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}} \left[\theta_i P_i(\tilde{\theta}_i, \theta_{-i}) - t_i(\tilde{\theta}_i, \theta_{-i}) \right]$$

where the lowest type gets utility $u_i(\underline{\theta})$.

(a) Show that incentive compatibility (IC) implies that utility obeys an integral equation and a monotonicity constraint.

(b) Write down the ex–ante individual rationality (IR) constraint which ensures that each bidder is happy to pay the entry cost and participate.

(c) Write down the auctioneer's program or maximising revenue, equal to the sum of payments, subject to (IC) and (IR).

(d) Show that the (IR) constraint will bind at the optimum.

(e) Optimal allocation function. Show that the revenue maximising mechanism awards the object to the agent with the highest valuation if that value exceeds θ_0.

(f) Optimal entry policy. Define welfare with n bidders by

$$W(n) := E_\theta \max\{\theta_0, \theta_1, \ldots, \theta_n\}$$

Show that $W(n+1) - W(n)$ decreases in n. Use this to show that the optimal number of bidders, n^*, obeys $W(n^*) - W(n^* - 1) \geq k \geq W(n^* + 1) - W(n^*)$.

(g) Argue that the optimal mechanism can be implemented by a standard auction with reserve price, entry fee and having bidders make their entry decisions sequentially. What are the optimal entry fee and reserve price?

9. Public Goods Provision

A firm is considering building a public good (e.g. a swimming pool). There are n agents in the economy, each with IID private value $\theta_i \in [0, 1]$. Agents' valuations have density $f(\theta)$ and distribution $F(\theta)$. Assume that

$$MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment by each agent $t_i(\theta_1, \ldots, \theta_n) \in \Re$. The mechanism must be individually rational and incentive compatible. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type θ_i, whose utility is given by

$$\theta_i P - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism.

(b) Given an build decision $P(\cdot)$, derive the firm's profits.

(c) What is the firm's optimal build decision?

(d) Show that $E[MR(\theta)] = 0$.

(e) Show that as $n \to \infty$, so the probability of provision goes to zero. [You might wish to use the Chebyshev inequality, which says that $\Pr(|Z - E[Z]| \geq \alpha) \leq \frac{\text{Var}(Z)}{\alpha^2}$ for a random variable Z.]

Next, suppose the firm can exclude agents. A mechanism now consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$, a participation decision for each agent $x_i(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment $t_i(\theta_1, \ldots, \theta_n) \in \Re$. Agent i's utility is now given by

$$\theta_i x_i P - t_i$$

The cost is still given by cn, where n is the number of agents in the population.

(f) Solve for the firm's optimal build decision $P(\cdot)$ and participation rule $x_i(\cdot)$.

(g) Suppose $n \to \infty$. Show there exists a cutoff c^* such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

10. Bilateral Trade

Suppose two agents wish to trade a single good. The seller has privately known cost $c \sim g(\cdot)$ on $[0,1]$. The buyer has privately known value $v \sim f(\cdot)$ on $[0,1]$. These random variables are independent of each other. The agents' payoffs are

$$U_S = t - cp$$
$$U_B = vp - t$$

where $t \in \Re$ is a transfer and $p \in [0,1]$ is the probability of trade. If an agent abstains from trade, they receive 0.

In class, we showed that it is impossible to implement the ex–post efficient allocation. We now wish to find the revenue and welfare maximising mechanisms.

(a) Consider the problem of a middleman who runs mechanism $\langle p(\tilde{v},\tilde{c}), t_B(\tilde{v},\tilde{c}), t_S(\tilde{v},\tilde{c})\rangle$ where t_B and t_S are the transfers from the buyer and to the seller respectively. Show that a middleman can make profit

$$\Pi = E\Big[[MR(v) - MC(c)]p(v,c)\Big] - U_B(\underline{v}) - U_S(\overline{c})$$

where

$$MR(v) = v - \frac{1-F(v)}{f(v)} \quad \text{and} \quad MC(c) = c + \frac{G(c)}{g(c)}$$

(b) Maximise the middleman's expected profits.

(c) Maximise expected welfare subject to $\Pi = 0$. [Note: We have not shown that $\Pi = 0$ implies one can find a common transfer function $t(v,c)$. We leave this for another day.]