

# Homework 1: Basic Moral Hazard

October 10, 2011

## Question 1 (Normal–Linear Model)

The following normal–linear model is regularly used in applied models. Given action  $a \in \mathfrak{R}$ , output is  $q = a + x$ , where  $x \sim N(0, \sigma^2)$ . The cost of effort is  $g(a)$  is increasing and convex. The agent’s utility equals  $u(w(q) - g(a))$ , while the principal’s is  $q - w(q)$ . Suppose the agent’s outside option is  $u(0)$ .

We make two large assumptions. First, the principal uses a linear contract:

$$w(q) = \alpha + \beta q$$

Second, the agent’s utility is CARA, i.e.,  $u(w) = -e^{-w}$ .

(a) Suppose  $w \sim N(\mu, \sigma^2)$ . Denote the certainty equivalent of  $w$  by  $\bar{w}$ , where

$$u(\bar{w}) = E[u(w)]$$

Show that  $\bar{w} = \mu - \sigma^2/2$ .

(b) Suppose effort is unobservable. The principal’s problem is

$$\begin{aligned} & \max_{w(q), a} E[q - w(q)] \\ \text{s.t.} \quad & E[u(w(q) - g(a))|a] \geq u(0) \\ & a \in \operatorname{argmax}_{a' \in \mathfrak{R}} E[u(w(q) - g(a'))|a'] \end{aligned}$$

Using the first order approach, characterise the optimal contract  $(\alpha, \beta, a)$ . [Hint: write utilities in terms of their certainty equivalent.]

(c) How would the solution change if the agent knows  $x$  before choosing his action (but after signing the contract)?

### Question 2 (Signal of Effort)

Consider the same normal-linear model as in Question 1. After  $a$  is chosen, the principal observes output  $q$  and a signal  $y$  that is correlated with  $x$ . For example, if the agent is selling cars, then  $y$  could be the sales of the dealer next door. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$

Suppose the principal uses the linear wage function:

$$w = \alpha + \beta(q + \gamma y)$$

Using the same approach as above, solve for the optimal value of  $\gamma$ . How does  $\gamma$  vary with  $\sigma_{xy}$ ? Provide an intuition.

### Question 3 (Insurance)

An agent has increasing, concave utility  $u(\cdot)$ . They start with wealth  $W_0$  and may have an accident costing  $x$  of their wealth. Assume  $x$  is publicly observable. The agent has access to a perfectly competitive market of risk-neutral insurers who offer payments  $R(x)$  net of any insurance premium. The distribution of  $x$  is as follows

$$f(0, a) = 1 - p(a) \tag{1}$$

$$f(x, a) = p(a)g(x) \quad \text{for } x > 0 \tag{2}$$

where  $\int g(x)dx = 1$ . The agent can affect the probability of an accident through their choice of  $a$ . The cost is given by increasing convex function,  $\psi(a)$ . The function  $p(a)$  is decreasing and convex. Utility is then given by  $u(W_0 - x + R(x)) - \psi(a)$ .

- (a) Suppose there is no insurance market. What action  $\hat{a}$  does the agent take?
- (b) Suppose  $a$  is contractible. Describe the first-best payment schedule  $R(x)$  and the effort choice,  $a^*$ .
- (c) Suppose  $a$  is not contractible. Describe the second-best payment schedule  $R(x)$ .
- (d) Interpret the second-best payment schedule. Would the agent ever have an incentive to

hide an accident? (i.e. report  $x = 0$  when  $x > 0$ ).

#### Question 4 (Private Evaluations with Limited Liability)

A principal employs an agent. The game is as follows.

1. The agent privately chooses an action  $a \in \{L, H\}$ . The cost of this action is  $g(a)$ .
2. The principal *privately* observes output  $q \sim f(q|a)$  on  $[\underline{q}, \bar{q}]$ . Assume this distribution function satisfies strict MLRP. That is,

$$\frac{f(q|H)}{f(q|L)}$$

is strictly increasing in  $q$ .

3. Suppose the principal reports that output is  $\tilde{q}$ . The principal then pays out  $t(\tilde{q})$ , while the agent receives  $w(\tilde{q})$ , where  $w(\tilde{q}) \leq t(\tilde{q})$ . The difference is burned. The payments  $\langle t, q \rangle$  are contractible.

Payoffs are as follows. The principal obtains

$$q - t$$

The agent obtains

$$u(w) - g(a)$$

where  $u(\cdot)$  is strictly increasing and concave, and  $g(\cdot)$  is increasing and convex. The agent has no (IR) constraint, but does have limited liability. That is,  $w(q) \geq 0$  for all  $q$ .

First, assume the principal wishes to implement  $a = L$ .

(a) Characterise the optimal contract.

Second, assume the principal wishes to implement  $a = H$ .

(b) Write down the principal's problem as maximising expected profits subject to the agent's (IC) constraint, the principal's (IC) constraint, the limited liability constraint and the constraint that  $w(q) \leq t(q)$ .

(c) Argue that  $t(q)$  is independent of  $q$ .

(d) Characterise the optimal contract. How does the wage vary with  $q$ ?

### Question 5 (Debt Contracts)

A risk neutral agent seeks funding from a risk neutral principal. The game is as follows:

1. The project requires investment  $I$  from the principal.
2. The agent chooses effort  $a \in \{L, H\}$  at cost  $c(a)$ . Assume  $c(H) > c(L)$ .
3. Output  $q$  is realised. Assume  $q$  takes values  $\{q_1, \dots, q_N\}$ , where  $q_{i+1} > q_i$ . Output is distributed according to  $f(q_i|a)$ .
4. If  $q_i$  is realised, the principal obtains payment  $B_i$  and the agent obtains  $q_i - B_i$ . The agent's utility is  $u = q_i - B_i - c(a)$ ; the principal's profit is  $\pi = B_i - I$ .

A contract specifies the payment to the principal as a function of the output  $\langle B_i \rangle$ . Assume the principal has outside option 0 and the agent makes a TIOLI offer to the principal. We also assume the contract satisfies feasibility (FE):

$$0 \leq B_i \leq q_i$$

and monotonicity (MON):

$$B_i \text{ is increasing in } i$$

Finally assume that  $f(q_i|a)$  satisfies the monotone hazard rate principle (MHRP):

$$\frac{f(q_i|L)}{1 - F(q_i|L)} \geq \frac{f(q_i|H)}{1 - F(q_i|H)} \quad \text{for each } q_i.$$

Assume the agent wishes to implement the high action. Show that a debt contract is optimal.

[Aside: In class we showed that MLRP implies debt contracts are optimal. The key insight is that, if we use the (MON) condition, we can use the weaker MHRP assumption.]

### Question 6 (Bargaining Power)

Suppose a risk neutral principal employs a risk averse agent. The two parties both sign a contract stating wage profile  $w(q)$ . The agent then chooses action  $a \in A$  at cost  $g(a)$ .

Payoffs are as follows. The agent gets

$$u(w - g(a))$$

where  $g(a)$  is increasing and convex. Utility is strictly increasing and strictly concave. The principal gets

$$q - w$$

The principal has reservation profit 0; the agent has reservation utility  $u(0)$ .

First, suppose the principal makes a TIOLI offer to the agent.

- (a) Assume the effort  $a$  is observable. Set up and solve the principal's optimal contract.
- (b) Assume effort  $a$  is not observable. Set up the principal's problem.

Next, suppose the agent makes a TIOLI offer to the principal.

- (c) Assume the effort  $a$  is observable. Show that the optimal contract induces the same effort as when the principal proposes the contract.
- (d) Assume effort  $a$  is not observable. Set up the agent's problem. Next, suppose that utility is CARA, i.e.  $u(w) = -\exp(-w)$ , which implies that  $u(w+x) = u(w)e^{-x}$ . Show that the optimal contract induces the same effort as when the principal proposes the contract (part (b)).