Economics 211A: Final

9:00am–12:00pm, 9th December, 2009

1. Public Goods Provision

A firm is considering building a public good (e.g. a swimming pool). There are n agents in the economy, each with IID private value $\theta_i \in [0,1]$. Agents' valuations have density $f(\theta)$ and distribution $F(\theta)$. Assume that

$$\mathrm{MR}(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment by each agent $t_i(\theta_1, \ldots, \theta_n) \in \Re$. The mechanism must be incentive compatible and individually rational. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type θ_i, whose utility is given by

$$\theta_i P - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism.

(b) Given an build decision $P(\cdot)$, derive the firm's profits.

(c) What is the firm's optimal build decision?

(d) Show that $E[\mathrm{MR}(\theta)] = 0$.

(e) Show that as $n \to \infty$, so the probability of provision goes to zero. [You might wish to use the Chebyshev inequality, which says that $\Pr(|Z - E[Z]| \geq \alpha) \leq \frac{\mathrm{Var}(Z)}{\alpha^2}$ for a random variable Z.]

Next, suppose the firm can exclude agents. A mechanism now consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$, a participation decision for each agent $x_i(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment $t_i(\theta_1, \ldots, \theta_n) \in \Re$. Agent i's utility is now given by

$$\theta_i x_i P - t_i$$

The cost is still given by cn, where n is the number of agents in the population.

(f) Solve for the firm's optimal build decision $P(\cdot)$ and participation rule $x_i(\cdot)$.

(g) Suppose $n \to \infty$. Show there exists a cutoff c^* such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

2. Dynamic Mechanism Design

A firm sells to a customer over $T = 2$ periods. There is no discounting.

The consumer's per-period utility is

$$u = \theta q - p$$

where $q \in \Re$ is the quantity of the good, and p is the price. The agent's type $\theta \in \{\theta_L, \theta_H\}$ is privately known. In period 1, $\Pr(\theta = \theta_H) = \mu$. In period 2, the agent's type may change. With probability $\alpha > 1/2$, her type remains the same; with probability $1 - \alpha$ her type switches (so a high type becomes a low type, or a low type becomes a high type).

The firm chooses a mechanism to maximise the sum of it's profits. The per-period profit is given by

$$\pi = p - \frac{1}{2} q^2$$

A mechanism consists of period 1 allocations $\langle q_L, q_H \rangle$, period 2 allocations $\langle q_{LL}, q_{LH}, q_{HL}, q_{HH} \rangle$, and corresponding prices, where q_{LH} is the quantity allocated to an agent who declares L in period 1 and H in period 2.

(a) Consider period $t = 2$. Fix the first period type, θ. Assume in period 2 that the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Characterise the second period rents obtained by the agents, $U_{\theta L}$ and $U_{\theta H}$, as a function of $\{q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$

(b) Consider period $t = 1$. Assume the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Derive the rents obtained by the agents, U_L and U_H, as a function of $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$.

(c) Derive the firm's total expected profits.

(d) Assume the firm cannot exclude, i.e. that $\Delta := \theta_H - \theta_L$ is sufficiently small. Derive the profit-maximising allocations $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$. Can you provide an intuition for this result?

(Bonus) Suppose T is arbitrary. Can you derive the form of the optimal mechanism?

3. Dynamic Contracts with Hidden Wage Offers

A risk neutral firm employs a risk avers worker. There are infinite periods, with discount rate $\delta \in (0, 1)$.

In period t, the firm's payoff is

$$\pi = q - w_t$$

where q is some fixed output, and w_t is the wage. The worker obtains

$$u(w_t).$$

Each period the worker obtains a wage offer $\overline{w}_t$ with a strictly positive density $f(\cdot)$, distribution $F(\cdot)$ and support $[0, 1]$. These wage offers are IID and are *not observed* by the firm. Denote $\overline{V} = E[\overline{w}]/(1 - \delta)$. Assume $q > 1$.

The firm offers the worker a contract $\{w_t\}$ that consists of a series of wages. These do not depend on the outside offers.[1]

Each period proceeds as follows. First, the worker sees the outside wage offer $\overline{w}_t$. Second, the worker chooses whether to quit or stay. If he quits, he never works for the firms again and obtains $u(\overline{w_t}) + \delta\overline{V}$. If he stays, he's paid according to the contract and the game proceeds to the next period.

(a) Suppose the agent has promised utility V. The worker quits if his outside wage offer exceeds a threshold, w^*. How is w^* determined?

[1] One might allow the agent to make reports to the firm. We do not allow this here.

(b) Write down the firm's profit $\Pi(V)$ as a function of the wage w and future promised utility V_+.

(c) Write down the promise keeping constraint. [The PK constraint says that the principal delivers the utility it promises, V].

(d) The firm maximises profit subject to (i) the promise keeping constraint, (ii) w^* being determined by the equation in (a). Assume V is sufficiently large so that $\Pi(V)$ is decreasing. Also assume that $\Pi(V)$ is concave. Show that the optimal choices of w and V_+ are related by the equation

$$-\Pi'(V_+) = \frac{1}{u'(w)}$$

(e) Suppose we are in a steady state, so $V_+ = V$ and wages are constant. Show that the probability of quitting is zero, i.e., $w^* \geq 1$. You can either do this via a the FOC from part (d) and the envelope theorem, or from a direct argument.