Homework 3

Due: Wed 5th December

0. Nonlinear Pricing with Two Types

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price.

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit $\pi t_1 + (1 - \pi)t_2$

(a) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2}q^2 - t$$

Derive the first–best and profit–maximising qualities.

(b) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Derive the first–best and profit–maximising qualities.

1. Dynamic Mechanism Design

A firm sells to a customer over $T = 2$ periods. There is no discounting.

The consumer's per-period utility is

$$u = \theta q - p$$

where $q \in \Re$ is the quantity of the good, and p is the price. The agent's type $\theta \in \{\theta_L, \theta_H\}$ is privately known. In period 1, $\Pr(\theta = \theta_H) = \mu$. In period 2, the agent's type may change. With probability $\alpha > 1/2$, her type remains the same; with probability $1 - \alpha$ her type switches (so a high type becomes a low type, or a low type becomes a high type).

The firm chooses a mechanism to maximise the sum of its profits. The per-period profit is given by

$$\pi = p - \frac{1}{2}q^2$$

A mechanism consists of period 1 allocations $\langle q_L, q_H \rangle$, period 2 allocations $\langle q_{LL}, q_{LH}, q_{HL}, q_{HH} \rangle$, and corresponding prices, where q_{LH} is the quantity allocated to an agent who declares L in period 1 and H in period 2.

(a) Consider period $t = 2$. Fix the first period type, θ. Assume in period 2 that the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Characterize the second period rents obtained by the agents, $U_{\theta L}$ and $U_{\theta H}$, as a function of $\{q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$

(b) Consider period $t = 1$. Assume the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Derive the lifetime rents obtained by the agents, U_L and U_H, as a function of $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$.

(c) Derive the firm's total expected profits.

(d) Assume the firm does not want to exclude, i.e. that $\Delta := \theta_H - \theta_L$ is sufficiently small. Derive the profit-maximizing allocations $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$. In particular, show that q_{HL} is first-best. Can you provide an intuition for this result?

(Bonus) Suppose T is arbitrary. Can you derive the form of the optimal mechanism?

2. Public Goods Provision

A firm is considering building a public good (e.g. a swimming pool). There are n agents in the economy, each with IID private value $\theta_i \in [0, 1]$. Agents' valuations have density $f(\theta)$ and distribution $F(\theta)$. Assume that

$$MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0, 1]$ and a payment by each agent $t_i(\theta_1, \ldots, \theta_n) \in \Re$. The mechanism must be individually rational and incentive compatible. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type θ_i, whose utility is given by

$$\theta_i P - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism.

(b) Given an build decision $P(\cdot)$, derive the firm's profits.

(c) What is the firm's optimal build decision?

(d) Show that $E[MR(\theta)] = 0$.

(e) Show that as $n \to \infty$, so the probability of provision goes to zero. [You might wish to use the Chebyshev inequality, which says that $\Pr(|Z - E[Z]| \geq \alpha) \leq \frac{\text{Var}(Z)}{\alpha^2}$ for a random variable Z.]

Next, suppose the firm can exclude agents. A mechanism now consists of a build decision $P(\theta_1, \ldots, \theta_n) \in [0,1]$, a participation decision for each agent $x_i(\theta_1, \ldots, \theta_n) \in [0,1]$ and a payment $t_i(\theta_1, \ldots, \theta_n) \in \Re$. Agent i's utility is now given by

$$\theta_i x_i P - t_i$$

The cost is still given by cn, where n is the number of agents in the population.

(f) Solve for the firm's optimal build decision $P(\cdot)$ and participation rule $x_i(\cdot)$.

(g) Suppose $n \to \infty$. Show there exists a cutoff c^* such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

3. Costly State Verification

There is a risk–neutral entrepreneur E who has a project with privately observed return y with density $f(y)$ on $[0, Y]$. The project requires investment $I < E[y]$ from an outside creditor C.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports y they pay $s(y) \le y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies E they pay exogenously given cost c and get to observe E's type.

The game is as follows:

- E chooses $(s(y), B(y))$ to raise I from a competitive financial market.
- Output y is realised.
- E claims the project yields $\hat{y}$. If $B(\hat{y}) = 0$ then E pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then C pays c and observes E's true type. If they are telling the truth they pay $s(y)$; if not, then C can take everything.
- Payoffs. E gets $y - s(y)$, while C gets $s(y) - cB(y) - I$.

(a) Show that a contract is incentive compatible if and only if there exists a D such that $s(y) = D$ when $B(y) = 0$ and $s(y) \le D$ when $B(y) = 1$.

Consider E's problem:

$$\begin{aligned}
\max_{s(y),B(y)} \quad & E[y - s(y)] \\
\text{s.t.} \quad & s(y) \le y \qquad (MAX) \\
& E[s(y) - cB(y) - I] \ge 0 \qquad (IR) \\
& s(y) \le D \quad \forall y \in B^V \qquad (IC1) \\
& s(y) = D \quad \forall y \notin B^V \qquad (IC2)
\end{aligned}$$

where B^V is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now E's problem becomes

$$\begin{aligned}
\min_{s(y),B(y)} \quad & E[cB(y)] \\
\text{s.t.} \quad & (MAX),\ (IC1),\ (IC2) \\
& E[s(y) - cB(y) - I] = 0 \qquad (IR)
\end{aligned}$$

(c) Show that any optimal contract $(s(y), B(y))$ has a verification range of the form $B^V = [0, D]$ for some D. [Hint: Proof by contradiction.]

(d) Show that any optimal contract $(s(y), B(y))$ sets $s(y) = y$ when $B(y) = 1$. [Hint: Proof by contradiction.]

(e) A contract is thus characterised by D. Which D maximises E's utility? Can you give a financial interpretation to this contract?

4. Ironing

Consider the continuous–type price discrimination problem from class, where the principal chooses $q(\theta)$ to maximise

$$E[q(\theta)MR(\theta) - c(q(\theta))]$$

subject to $q(\theta)$ increasing in θ.

For $v \in [0, 1]$, let

$$H(v) = \int_0^v MR(F^{-1}(x))dx$$

be the expected marginal revenue up to $\theta = F^{-1}(v)$. Let $\overline{H}(v)$ be the highest convex function under $H(v)$. Then define $\overline{MR}(\theta)$ by

$$\overline{H}(v) = \int_0^v \overline{MR}(F^{-1}(x))dx$$

Finally, let $\Delta(\theta) = H(F(\theta)) - \overline{H}(F(\theta))$.[1]

(a) Argue that $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)$ is flat. Also argue that $\Delta(\underline{\theta}) = \Delta(\overline{\theta}) = 0$.

(b) Since $q(\theta)$ is an increasing function, show that

$$E[q(\theta)MR(\theta) - c(q(\theta))] = E[q(\theta)\overline{MR}(\theta) - c(q(\theta))] - \int_{\underline{\theta}}^{\overline{\theta}} \Delta(\theta)dq(\theta)$$

(c) Derive the profit–maximising allocation $q(\theta)$.

[1] Note, it is important that we take the convex hull in quantile space. If we use θ–space, then $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)f(\theta)$ is flat, which is not particularly useful.

5. Negotiations and Auctions

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[\underline{V}, \overline{V}]$. Define marginal revenue as

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

(a) Show that $E[MR(v)] = \underline{V}$.

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be v_0. In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume $\underline{V} \geq v_0$, i.e. all bidders are "serious". How is revenue affected if one bidder is swapped for a reservation price?

6. Revenue Management

A firm has one unit of a good to sell over T periods. One agent enters each period and has a value drawn IID from $F(\cdot)$ on [0,1]. Agents values are privately known. The discount rate is $\delta \in (0, 1)$.

First, assume there is no recall, so agent t leaves at the end of period t if they do not buy. A mechanism $\langle P_t, Y_t \rangle$ gives the probability of allocating the object in period t as a function of the reports until time t, and the corresponding payment. Agent t's utility is then given by

$$u_t = v_t \delta^t P_t - Y_t$$

and the firm's profits equal $\Pi = \sum_t Y_t$. Assume $MR(v) = v - (1 - F(v))/f(v)$ is increasing.

(a) Argue that the firm's profit is given by

$$\Pi = E_0 \left[\sum_t \delta^t P_t MR(v_t) \right]$$

where E_0 is the expectation at time 0, over all sequences of values. [Note: you don't have to be formal]

(b) What is the optimal allocation in period T? [Hint: it is easy to think in terms of cutoffs v_T^*, where the firm is indifferent between allocating the object and not]

(c) Using backwards induction, characterize the optimal cutoff in period $t < T$? What happens to the cutoffs over time?

Next, suppose there is recall. Hence agent t can buy in any period $\tau \geq t$. A mechanism $\langle P_{t,\tau}, Y_t \rangle$ gives the probability of allocating the object to agent t in period τ as a function of the reports until time τ, and the corresponding payment. Agent t's utility is then given by

$$u_t = \sum_{\tau \geq t} v_t \delta^\tau P_{t,\tau} - Y_t$$

and the firm's profits equal $\Pi = \sum_t Y_t$.

(d) Argue that the firm's profit is given by

$$\Pi = E_0 \left[\sum_t \sum_{\tau \geq t} \delta^\tau P_{t,\tau} MR(v_t) \right]$$

(e) When there is recall, what is the optimal allocation in period T?

(f) Using backwards induction, what is the optimal allocation in period $t < T$? What happens to the cutoffs over time? [Hint: try $t = T - 1$ and $t = T - 2$, and the general case only if you have time]